Bragg Resonance

by Ripple Beds

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Abstract

The overall aim of this project is to gain an understanding of how asymptotic methods can be used in linear water-wave theory. Furthermore these methods will be used to develop a solution to a problem posed in Porter and Chamberlain (1997).

Porter and Chamberlain (1997) show the unsuccessful application of a regular series expansion to demonstrate the motion of plane harmonic waves. Here we have investigated how the method of multiple scales can be used to improve approximations made with a regular series expansions, exposing the phenomenon of Bragg resonance.

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Declaration

I con rm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

Zoe Gumm

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Acknowledgments

I would like to acknowledge the nancial assistance of the NERC and my Boohbah for the tireless hours it has spent with me in front of my computer. Also to Dr Biggs for designing my project and Dr Chamberlain for allowing me to work on his problem.

1 Introduction

Asymptotic expansions can provide us with relatively simple examples for studying certain systems in linear water-wave theory. In particular, given certain circumstances, different approaches within the region of asymptotics can be used where non-uniformities are apparent.

In this report, I have attempted to recreate and thus solve a problem given in Porter and Chamberlain (1997). Not only do I list the results and consequences, I have also tried to make connections to the original paper, Asfar and Nayfeh (1983), which gives the method to the problem posed. I assume a background knowledge in elementary asymptotics, as I should have explained anything more complicated in detail.

Firstly, in section two, I work through the problem posed in Porter and Chamberlain (1997). This will be done, omitting only details from the paper that are not relevant to the problem we are focusing on, in a purely mathematical context.

In section three, we rigorously go through the example given in Asfar and Nayfeh (1983) for electro-magnetics. We do this by rst applying a regular series expansion to the governing equations set, and upon showing that this fails, we then show that by

2 A Regular Asymptotic Expansion

2.1 General Problem

Porter and Chamberlain (1997) begin by considering a fairly general problem in linear water-wave theory. As a base for the problem, they consider a fluid that is incompressible, irrotational, inviscid and time harmonic. In order to set up this problem, we can suppose that x, y and z are cartesian coordinates with $z = 0$ being the position of the fluids free-surface. Our velocity potential $\phi(x; y; z)$ satisfies

$$\left.\begin{array}{lll} \nabla^2\phi = 0 & & (-h < z < 0) \\ \phi_z - \nu\phi = 0 & & (z = 0) \\ \phi_z + \nabla_h h \cdot \nabla_h\phi = 0 & & (z = -h) \end{array}\right\};$$

where $\nabla_h = (\partial/\partial x; \partial/\partial y)$ and $\nu = \omega^2/g$, ω being the given angular frequency and g the acceleration due to gravity. The condition at $z = 0$, (the free-surface), actually satisfies a rectangular shape, therefore we know that the free surface we are modelling is flat with no perturbations. Also the bed condition at $z = -h$ tells us that there is no normal flow at the bottom.

If $h(x; y)$ is constant then

$$\begin{aligned} \nabla_h h &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) h; \\ &= 0; \end{aligned}$$

rf$_z$ 93.34 0 Td h Tf 25.096 $= 0$ $(z =$

2.2 Separation Solutions

Here we suppose that h is a constant and seek separation solutions. By using separation of variables, we can find a general solution. Begin by letting $\phi(x,y,z) = X(x,y)Z(z)$. For $-h < z < 0$, our substitution transforms our equation into

$$Z(z)\,\nabla_h^2 X(x) + Z''(z)X(x) = 0,$$

and therefore

$$\frac{\nabla_h^2 X(x)}{X(x)} = -\frac{Z''(z)}{Z(z)} = \lambda,$$

where λ is the separation constant and prime denotes differentiation with respect to z.

If we first consider the case for the function $Z(z)$, we have

$$Z''(z) + \lambda Z(z) = 0. \tag{2}$$

There are now three separate cases to consider, according to the sign of λ.

When $\lambda = 0$, equation (2) becomes

$$Z''(z) = 0$$

and for this we obtain the general solution

$$Z(z) = Az + B,$$

where A and B are constants.

We can evaluate A and B using the boundary conditions. The boundary condition at $z = -h$ is

$$Z'(-h) = 0,$$

so we have

$$Z'(-h) = A.$$

Therefore A must be zero for the boundary condition to be satisfied at $z = -h$. Now we use the boundary condition at $z = 0$,

$$Z'(0) - Z(0) = 0.$$

So we have

$$\begin{aligned} Z'(0) - Z(0) &= A - (A(0) + B), \\ &= -B, \end{aligned}$$

therefore B must also equal zero for the condition at $z = 0$ to be satisfied. We have found $A = B = 0$. When putting these values into our general solution we will obtain

$$\begin{aligned} Z(z) &= (0)z + (0), \\ &= 0. \end{aligned}$$

We can therefore deduce that $\lambda \neq 0$ as it does not give us any nontrivial solutions.

Now we can consider when $\lambda > 0$. For this we will let $\lambda = k^2$, where $k \in \mathbb{R}$. Equation (2) becomes

$$Z''(z) + k^2 Z(z) = 0,$$

and the general solution for this equation is

$$Z(z) = A\cos kz + B\sin kz.$$

This time, we will manipulate the general solution to satisfy the boundary conditions. So if we use the addition formula, we can then express our general solution as

$$Z(z) = A\cos k(z + h),$$

which now also satisfies our boundary condition at $z = -h$. Next we use the boundary condition for $z = 0$ to get

$$Z'(0) - Z(0) = 0.$$

So we have

$$Z'(0) - Z(0) = -Ak\sin k(0 + h) - A\cos k(0 + h),$$

which gives us

$Ak \sin$

Note that because is positive and real in this case, then the terms are evanescent and can be neglected.

Now consider when < 0, for this condition we will let $=$ k^2, where k 2 R. Equation (2) becomes

$$Z^{00}(z) \quad k^2 Z(z) = 0:$$

For this equation we have the general solution

$$Z(z) = A \cosh kz + B \sinh kz:$$

We can repeat the process of manipulating the solution to satisfy the boundary condition. Using the addition formula, we get

$$Z(z) = A \cosh k(z + h): \tag{4}$$

This satis es the boundary condition at $z =$ h. Next we consider the boundary condition for $z = 0$,

$$Z^{0}(0) \qquad Z(0) = 0:$$

So we have,

$$Z^{0}(0) \qquad Z(0) = Ak \sinh k(0 + h) \qquad A \cosh k(0 + h);$$

which gives us

$$Ak \sinh kh \qquad A \cosh kh = 0;$$

and therefore

$$= \frac{k \sinh kh}{\cosh kh};$$

which gives us

$$= k \tanh kh: \tag{5}$$

This equation is illustrated below.

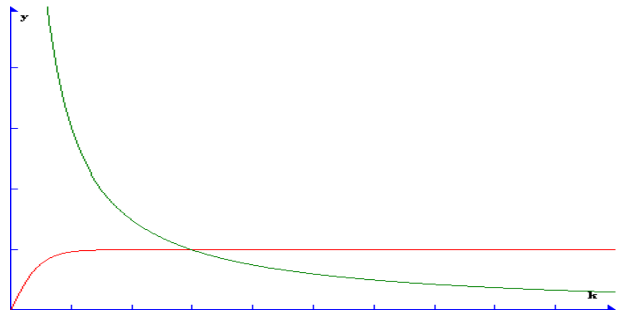


Figure 2: Graph of dispersion relation for < 0 where $= 3$, $h = 2$.

As we can see from this graph, there is only one positive root. This tells us that there is one wave propagating for $k > 0$. Because this function is symmetric, we also have the same root for k, $k < 0$. This tells us that there is the one wave, moving both to the left and to the right. To gain any information about how this wave is moving, we should consider the horizontal component of the wave. For the general solution concerning the $X(x)$ component we have

$$X(x) = Ae^{ikx} + Be^{\ ikx}:$$

Because in this case < 0 and the terms are imaginary, we obtain two propagating modes. This accounts for the wave moving parallel to the x axis in both positive and negative directions.

2.3 Localized Bed Perturbations

Now we want to approximate the bed condition in (1) to $O(\;)$. We will concentrate on the equation at $z = \quad h$, because it will be the most a ected by the new perturbation. Note that because we are rede ning h to be a function of x, we need to revert back to our previous boundary condition. To focus on the boundary condition at $z = \quad h(x)$, we have

$$_z + \; r_h h \quad r_h \quad = 0;$$

that is

$$_z(x; \quad h(x)) + h_x \;\; _x(x; \quad h(x)) = 0:$$

Now substitute in the bed perturbation, $h(x) = h_0 \qquad (x)$, to get where we have used the fact that $_{xx} = \quad _{zz}$. In order to progress, we must consider a particular incident wave, a wave which will propagate and result in a transmitted and re ected wave. We can use the expansion $= {}_0 + \quad {}_1 + O(\ ^2)$ to substitute into equation (1) to get a set of equations of $O(\ )$. To help, we can de ne the incident wave as

$$_0(x; z) = e^{ikx} Z_0(h_0; z); \qquad (9)$$

where $Z_0(h_0; z)$ is the equivalent to the boundary condition found previously in the form $Z(h; z) = A \cosh k(z + h)$ in Porter and Chamberlain (1997). In $\quad h_0 < z < 0$, we have

$$r^2 \quad = 0;$$

which will give us

$$_{0xx} + \quad _{0zz} + \ (\ _{1xx} + \quad _{1zz}) + O(\ ^2) = 0:$$

The terms at order $O(\ )$ are

$$_{1xx} + \quad _{1zz} = 0: \qquad (10)$$

For the boundary at $z = 0$, we have

$$_z \qquad = 0;$$

and after substituting in our asymptotic expansion, we obtain

$$_{0z} + \quad _{1z} \qquad _0 \qquad _1 + O(\ ^2) = 0:$$

So the $O(\ ^1)$ terms are

$$_{1z} \qquad _1 = 0: \qquad (11)$$

It is more complicated to determine the boundary condition satis ed by $_1$ at the bed, since it proves to be inhomogeneous and includes terms invoking $_0$. By substituting our regular expansion into the boundary condition

$$_z + r_h h \quad r_h \quad = 0;$$

we have

$$(\ _0 + \quad _1 + O(\ ^2))_z + r_h h \quad r_h(\ _0 + \quad _1 + O(\ ^2)) = 0;$$

evaluated at $z = \quad h(x)$. We can be more explicit and say

$$_{0z}(x; \quad h_0 + \quad) + \quad _{1z}(x; \quad h_0 + \quad) \qquad _x \; _{0x}(x; \quad h_0 + \quad) + O(\; ^2) = 0;$$

and from using Taylor series and taking the only terms necessary, this becomes

$$_{0z}(x; \quad h_0) + \qquad _{0zz}(x; \quad h_0) + \quad _{1z}(x; \quad h_0) \qquad _x \; _{0x}(x; \quad h_0) + O(\; ^2) = 0:$$

The $O(\; ^1)$ terms are

$$_{1z} = \; _x \; _{0x} \qquad _{0zz} = \; _x \; _{0x} + \quad _{0xx}: \tag{12}$$

Because equation (12) contains $\; _0$ we must use the incident wave, equation (9), to complete our expression. From equation (9) we know that

$$_{0x} = ike^{ikx}Z_0(h_0; z);$$

and

$$_{0xx} = \quad k^2e^{ikx}Z_0(h_0; z):$$

If we substitute the above expressions into equation (12) we obtain

$$_{1z} = \; _xZ_0(h_0; z)ike^{ikx} \qquad Z_0(h_0; z)k^2e^{ikx}:$$

From equation (4), we can work out that at $z = \quad h_0$, we have

$$Z_0(h_0; \quad h_0) = A_0 \cosh k(0) = A_0:$$

We can use this to obtain

$$\begin{aligned} _{1z} &= \quad _x A_0ike^{ikx} \qquad A_0k^2e^{ikx}; \\ &= e^{ikx}A_0(\; _xik \qquad k^2); \\ &= ike^{ikx}A_0(\; _x + ik \;): \end{aligned} \tag{13}$$

Now we have a new set of governing equations at $O(\; ^1)$,

$$\left.\begin{array}{ll} _{1xx} + \; _{1zz} = 0 & (\quad h_0 < z < 0) \\ _{1z} \qquad _1 = 0 & (z = 0) \\ _{1z} = ikA_0(h_0)(\; _x + ik \;)e^{ikx} & (z = \quad h_0) \end{array}\right\}: \tag{14}$$

To gain an explicit expression for ϕ_1 we must now go a step further and integrate the new boundary condition at $z = -h_0$. We do this using Green's function for $-h_0 < z < 0$ which satisfies

$$
\left.
\begin{array}{ll}
\nabla^2 G = \delta(x - x_0)\,\delta(z - z_0) & (-h_0 < z < 0;\ -\infty < x < \infty) \\
G_z - \nu G = 0 & (z = 0) \\
G_z = 0 & (z = -h_0)
\end{array}
\right\}, \qquad (15)
$$

as $x \to \pm\infty$. The solution to Green's function, $G(x; z; x_0; z_0)$, would be

$$
G(x; z; x_0; z_0) = \frac{i}{2k} Z_0(h_0; z) Z_0(h_0; z_0) e^{ik|x - x_0|} + \sum_{n=1}^{\infty} \frac{1}{2k_n} Z_n(h_0; z) Z_n(h_0; z_0) e^{-k_n|x - x_0|}.
$$

If we consider this wave as it goes to $\pm\infty$, then for $n \geq 1$, this equation contributes for 0 x l and where l_0 is the length of one ripple. Now we can consider what would happen for this example, by substituting the expressions for and $_x$ into the integral and seeing what happens as x_0 ! 1 .

As x_0 ! 1

Expression (19) can then be substituted into equation (18) to give us

$$I_n = \frac{e^{\ ikx_0}}{4k^2 l_0{}^2 + 4\ \ {}^2}\ l_0 ik$$

only valid for $n \qquad ^{1}$. Now we know that the regular series expansion is not uniform and therefore there is a known non-uniformity for $k \qquad = l_0$

3 Two-Dimensional Electromagnetic Waveguides

3.1 Regular Series Expansion

To nd an improved approximation to the problem posed in Porter and Chamberlain (1997), we shall consult the paper by Asfar and Nayfeh (1983) which deals with several types of cases of wave propagation within ducts with periodic wall perturbations. The example we are interested in is an electromagnetic case with multiple transverse magnetic propagating waves, which can be applied equally well to a water-wave case. Note that for Porter and Chamberlain (1997) we have only the one wave which travels in opposite directions to consider, whilst in Asfar and Nayfeh (1983) there will be n-many. The setup for the problem in this paper is to have two parallel wall distortion functions: one on the right, $x = 1 + \ \sin(k_w z + \ )$ and one on the left, $x = \ \sin(k_w z)$ where is the phase shift and is the dimensionless amplitude of the wall undulations, assumed to be much smaller than unity.

The governing equation for the ow in this problem is the Helmholtz equation in the form of

$$r^2 \ + k^2 \ = 0; \qquad (22)$$

where $r = @=@x + @=@z$, k is the free-space dimensionless wavenumber and is the

As we had shown in Porter and Chamberlain (1997), an asymptotic expansion may give invalid results. Asfar and Nayfeh (1983) recreates this and shows how to solve the problem.

First we substitute our asymptotic expansion into equation (22), to get

$$r^2({}_0 + \quad {}_1 + :::) + k^2({}_0 + \quad {}_1 + :::) = 0:$$

Therefore at leading order we obtain

$$r^2 \ {}_0 + k^2 \ {}_0 = 0; \tag{25}$$

and at $O(\ ^1)$ we obtain

$$r^2 \ {}_1 + k^2 \ {}_1 = 0: \tag{26}$$

Now we should do the same thing to the left and right plates. To do this we must linearise the boundaries at $x = 0$ and $x = 1$ by developing and its derivatives using Taylor series. For the right plate, at $x = 1 + \ \sin(k_w z + \ )$, we substitute the asymptotic expansion into equation (24), to obtain

$$F(1 + \ \sin(k_w z + \ ); z) \quad \frac{@^2}{@z^2} + k^2 \ ({}_0 + \quad {}_1 + :::)$$
$$+ \quad k_w \cos(k_w z + \ ) \frac{@^2}{@x@z}({}_0 + \quad {}_1 + :::) = 0:$$

To linearise this equation, we use

$$F(1 + \ \sin(k_w z + \ ); z) = F(1; z) + \ \sin(k_w z + \ ) F_x(1; z) + O(\ ^2):$$

Because there is no dependence on x, $F(1; z)$ is equivalent to $F(1 + \ \sin(k_w z + \ ); z)$. Therefore we can work out $F_x(1; z)$ in a similar way.

$$F_x(1; z) = F_x(1 + \ \sin(k_w z + \ ); z)$$
$$= \frac{@^2}{@z^2} + k^2 \ \frac{@}{@x}({}_0 + \quad {}_1 + :::) + \ k_w \cos(k_w z + \ ) \frac{@^3}{@x^2 @z}({}_0 + \quad {}_1 + :::):$$

If we substitute this into our expression for $F(1 + \ \sin(k_w z + \ ); z)$, we get

$$F(1 + \ \sin(k_w z + \ ); z) \quad \frac{@^2}{@z^2} + k^2 \quad (\ _0 + \quad _1 + :::)$$

$$+ \quad k_w \cos(k_w z + \ ) \ @^2$$

where is the separation constant. For $X(x)$ and $Z(z)$ we have

$$X''(x) + \quad X(x) = 0 \tag{31}$$

and

$$Z''(z) + (k^2 \qquad)Z(z) = 0. \tag{32}$$

There are now three separate cases to consider,

1. $= 0$
2. < 0
3. > 0.

When $= 0$, equation (31) becomes

$$X''(x) = 0$$

and for this we obtain the general solution

$$X(x) = Ax + B;$$

where A and B are constants. Also for equation (32), we obtain

$$Z''(z) + k^2 Z(z) = 0$$

which has a general solution of the form

$$Z(z) = Ce^{ikz} + De^{\ ikz};$$

where C and D are also constants. This then tells us that is of the form

$${}_0 = X(x)Z(z) = (Ax + B)(Ce^{ikz} + De^{\ ikz}).$$

We can evaluate A, B, C and D using the boundary conditions. The boundary condition at $x = 0$ is

$$\frac{\partial^2}{\partial z^2} + k^2 \qquad {}_0 = 0;$$

so we have

$$\begin{aligned} \frac{\partial^2}{\partial z^2} + k^2 \qquad (0; z) &= \quad (A(0) + B)k^2(Ce^{ikz} + De^{\ ikz}) \\ &+ \quad k^2(A(0) + B)(Ce^{ikz} + De^{\ ikz}). \end{aligned}$$

Both k and m cannot equal zero, therefore $_0 = 0$, so we have

$$A(C\cosh(\sqrt{k^2 + m^2}z) + D\sinh(\sqrt{k^2 + m^2}z)) = 0.$$

For the expression above to be satisfied, $A = 0$, then the boundary condition at $x = 0$ will be satisfied. Now if we consider when $x = 1$, we have

$$B\sinh m(1)(C\cosh(\sqrt{k^2 + m^2}z) + D\sinh(\sqrt{}$$

This tells us that $\psi_0(0; z) = 0$, $\forall x \in \mathbb{R}$ since $m \neq 0$. This then gives us

$$\begin{aligned}\psi_0(0; z) &= (A \sin m(0) + B \cos m(0))(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}); \\ &= B(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}).\end{aligned}$$

Therefore $B = 0$ for the boundary condition to remain valid at $x = 0$. Now we consider when $x = 1$. We begin with

$$\psi_0(1; z) = 0;$$

so we get

$$\psi_0(1; z) = A \sin m(1)(Ce^{i\sqrt{k^2-m^2}z} + De^{-i\sqrt{k^2-m^2}z}).$$

For this boundary condition to be satisfied, $m = n\pi$, for $n \in \mathbb{Z}$. For this we obtain a general solution of the form

$$\psi_0 = \sum_{n=0}^{\infty} \sin(n\pi x)\left\{C_n e^{i\sqrt{k^2-(n\pi)^2}z} + D_n e^{-i\sqrt{k^2-(n\pi)^2}z}\right\}. \tag{33}$$

Asfar and Nayfeh (1983) does not have a general solution that accounts for the waves that are moving to the left, instead the general solution is

$$\psi_0(x; z) = \sum_{n=-\infty}^{\infty} A_n \sin(n\pi x)e^{ik_n z}. \tag{34}$$

We can only assume that k_n is defined in the following way,

$$k_n = \begin{cases} \sqrt{k^2-(n\pi)^2}; & n > 0 \text{ and } k^2 > (n\pi)^2 \\ i\sqrt{(n\pi)^2-k^2}; & n > 0 \text{ and } k^2 < (n\pi)^2 \\ -\sqrt{k^2-(n\pi)^2}; & n < 0 \text{ and } k^2 > (n\pi)^2 \\ -i\sqrt{(n\pi)^2-k^2}; & n < 0 \text{ and } k^2 < (n\pi)^2. \end{cases} \tag{35}$$

The linearised position of the right plate is $x = 1$, so using the result $\cos(n\ \ ) = (\ \ 1)^n$, we have

@^{2}

By substituting equation (38) into equation (26) and our recently derived boundary conditions, equations (36) and (37), we yield a governing equation and boundary conditions for both $\psi_1(x)$ and $\psi_2(x)$. Equation (26) will become

$$\frac{i}{2}\sum_{n=-\infty}^{\infty} n\, A_n \Big\{ ((n\pi)^2 - k_n k_w)\frac{\partial^2}{\partial x^2}\psi_1(x)e^{i(k_n+k_w)z} - ((n\pi)^2 + k_n k_w)\frac{\partial^2}{\partial x^2}\psi_2(x)e^{i(k_n-k_w)z}$$
$$-(k_n + k_w)^2((n\pi)^2 - k_n k_w)\psi_1(x)e^{i(k_n+k_w)z} + (k_n - k_w)^2((n\pi)^2 + k_n k_w)\psi_2(x)e^{i(k_n-k_w)z}$$
$$+ k^2\Big[((n\pi)^2 - k_n k_w)\psi_1(x)e^{i(k_n+k_w)z} - ((n\pi)^2 + k_n k_w)\psi_2(x)e^{i(k_n-k_w)z}\Big]\Big\} = 0.$$

Now we collect coefficients of the exponential terms to obtain

$$((n\pi)^2 - k_n k_w)\left(\frac{\partial^2}{\partial x^2}\psi_1(x) + \psi_1(x)(k^2 - (k_n + k_w)^2)\right) = 0$$

and

$$((n\pi)^2 + k_n k_w)\left(\frac{\partial^2}{\partial x^2}\psi_2(x) + \psi_2(x)(k^2 - (k_n - k_w)^2)\right) = 0;$$

which is true for all $n \in \mathbb{R}$. This tells us that

$$\frac{\partial^2}{\partial x^2}\psi_j(x) + {}_j^2\,\psi_j(x) = 0; \qquad (39)$$

where $j \in \{1, 2\}$ and ${}_j^2 = k^2 - (k_n - k_w)^2$.

We can find ψ_1 and ψ_2 for the right plate, at $x = 1$, by first using the same method as we had for equation (26) to (28). We have

$$\frac{i}{2}\sum_{n=-\infty}^{\infty} n\, A_n \Big[(k_n - k_w)^2((n\pi)^2 + k_n k_w)\psi_2(1)e^{i(k_n-k_w)z}$$
$$-(k_n + k_w)^2((n\pi)^2 - k_n k_w)\psi_1(1)e^{i(k_n+k_w)z} + k^2\{((n\pi)^2 - k_n k_w)\psi_1(1)e^{i(k_n+k_w)z}$$
$$-((n\pi)^2 + k_n k_w)\psi_2(1)e^{i(k_n-k_w)z}\}\Big] \; i$$

By allowing $j \in \{1, 2\}$ we can write this equation more succinctly as

$$_j(1)(k^2 \quad (k_n \quad k_w)^2) = (\quad 1)^n e^{\ \ i} \ ;$$

and by again letting $\ _j{}^2 = k^2 \quad (k_n \quad k_w)^2$ we get

$$_j(1) \ \ _j{}^2 = (\quad 1)^n e^{\ \ i} \ ;$$

where the plus (minus) sign corresponds to $\ _1$ and $\ _1$, ($\ _2$ and $\ _2$). By dividing through by $\ _j{}^2$ we obtain

$$_j(1) = (\quad 1)^n \ \ _j \ \ {}^2 e^{\ \ i} \ : \tag{40}$$

The boundary condition at the left plate can be found using the same method. We have

$$(k_n \quad k_w)^2((n \ \ )^2 + k_n k_w) \ \ _2(0) e^{i(k_n \quad k_w)z} \quad (k_n + k_w)^2((n \ \ ) \ \ ) \ \ ) \ \ )$$

so we get

$$
\begin{aligned}
{}_j(0) &= A \sin \ {}_j(0) + B \cos \ {}_j(0); \\
&= B:
\end{aligned}
$$

This implies that $B = \ {}_j \ ^2$ for the boundary condition at $x = 0$ to be satis ed. Now we should continue by considering the boundary condition at $x = 1$,

$$
{}_j(1) = (\ 1)^n \ {}_j \ ^2 e^{\ i \ } ;
$$

so we get

$$
\begin{aligned}
{}_j(1) &= A \sin \ {}_j(1) + B \cos \ {}_j(1); \\
&= A \sin \ {}_j + \ {}_j \ ^2 \cos \ {}_j:
\end{aligned}
$$

We can obtain A by rearrangement, to obtain

$$
\begin{aligned}
A \sin \ {}_j &= (\ 1)^n \ {}_j \ ^2 e^{\ i \ } \ \ {}_j \ ^2 \cos \ {}_j; \\
&= (\ 1)^n e^{\ i \ } \ \cos \ {}_j \ \ {}_j \ ^2;
\end{aligned}
$$

and by dividing both sides of the equation by sin ${}_j$ we obtain

$$
A = \frac{(\ 1)^n e^{\ i \ } \ \cos \ {}_j}{{}_j{}^2 \sin \ {}_j}:
$$

Now we can substitute the two constants A and B into the general solution, equation (42), to obtain

$$
{}_j(x) = \ \frac{(\ 1)^n e^{\ i \ } \ \cos \ {}_j}{{}_j{}^2 \sin \ {}_j} \ \sin \ {}_j x + \ {}_j \ ^2 \cos \ {}_j x;
$$

which can be expressed as

$$
{}_j(x) = \frac{[(\ 1)^n e^{\ i \ \cos(j \ )} \ \cos \ {}_j] \sin \ {}_j x + \sin \ {}_j \cos \ {}_j x}{{}_j{}^2 \sin \ {}_j}: \qquad (43)
$$

Now we want to show that equation (43) is not valid for certain parameter combinations, as we had for Porter and Chamberlain (1997). We do this by considering what would happen when we let ${}_j \ ! \ m \ $, where $m \ 2$ R. We would obtain

$$
\lim_{j \ ! \ m} \ {}_j(x) = \frac{[(\ 1)^n e^{\ i \ \cos(j \ )} \ \cos(m \ )] \sin(m \ x) + \sin(m \ ) \cos(m \ x)}{(m \ )^2 \sin(m \ )}:
$$

Since sin(m

Now we have a new expression for ψ_0 we can substitute this into our equations of $O(\epsilon^1)$. First for our governing equation (46), we see that

$$\frac{\partial^2 \psi_0}{\partial z_0 \partial z_1} = A_n'(z_1) i k_n \sin(n\pi x) e^{ik_n z_0} + A_m'(z_1) i k_m \sin(m\pi x) e^{ik_m z_0};$$

giving

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial {z_0}^2} + k^2 \psi_1 = -2i \sum_{j=m,n} A_j'(z_1) k_j \sin(j\pi x) e^{ik_j z_0}. \tag{48}$$

We shall now do the same thing for the boundary condition for the right plate at $x = 1$. Firstly

$$\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 = -\epsilon \sin(k_w z_0 + \phi)\Big[-{k_n}^2 A_n(z_1) n\pi(-1)^n e^{ik_n z_0} - {k_m}^2 A_m(z_1) m\pi(-1)^m e^{ik_m z_0} + k^2\big(A_n(z_1) n\pi(-1)^n e^{ik_n z_0} + A_m(z_1) m\pi(-1)^m e^{ik_m z_0}\big)\Big] - \epsilon k_w \cos(k_w z_0 + \phi)\Big[ik_n A_n(z_1) n\pi(-1)^n e^{ik_n z_0} + ik_m A_m(z_1) m\pi(-1)^m e^{ik_m z_0}\Big];$$

and by using identities for sine and cosine, we can obtain

$$\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 = -\epsilon\left(\frac{e^{i(k_w z_0+\phi)} - e^{-i(k_w z_0+\phi)}}{2i}\right)\Big[-{k_n}^2 A_n(z_1) n\pi(-1)^n e^{ik_n z_0} - {k_m}^2 A_m(z_1) m\pi(-1)^m e^{ik_m z_0} + k^2\big(A_n(z_1) n\pi(-1)^n e^{ik_n z_0} + A_m(z_1) m\pi(-1)^m e^{ik_m z_0}\big)\Big] - \epsilon\left(\frac{e^{i(k_w z_0+\phi)} + e^{-i(k_w z_0+\phi)}}{2}\right) k_w \Big[ik_n A_n(z_1) n\pi(-1)^n e^{ik_n z_0} + ik_m A_m(z_1) m\pi(-1)^m e^{ik_m z_0}\Big].$$

By collecting like terms, we can proceed to get

$$\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 = \frac{i\epsilon}{2}\Big\{(e^{i(k_w z_0+\phi)} - e^{-i(k_w z_0+\phi)})\Big[-{k_n}^2 A_n(z_1) n\pi(-1)^n e^{ik_n z_0} - {k_m}^2 A_m(z_1) m\pi(-1)^m e^{ik_m z_0} + k^2\big(A_n(z_1) n\pi(-1)^n e^{ik_n z_0} + A_m(z_1) m\pi(-1)^m e^{ik_m z_0}\big)\Big] - k_w(e^{i(k_w z_0+\phi)} + e^{-i(k_w z_0+\phi)})\Big[k_n A_n(z_1) n\pi(-1)^n e^{ik_n z_0} + k_m A_m(z_1) m\pi(-1)^m e^{ik_m z_0}\Big]\Big\}.$$

Again, by collecting the coefficients of the exponential terms and summing over j, we can obtain

$$\left(\frac{\partial^2}{\partial {z_0}^2} + k^2\right)\psi_1 = \frac{i\epsilon}{2}\left\{\sum_{j=m,n} e^{i(k_j+k_w)z_0+i\phi}\Big(-{k_j}^2 A_j(z_1) j\pi(-1)^j + k^2 A_j(z_1) j\pi(-1)^j - k_w k_j A_j(z_1 j\pi(-1)^j\Big) - \sum_{j=m,n} e^{i(k_j-k_w)z_0-i\phi}\Big(-{k_j}^2 A_j(z_1) j\pi(-1)^j + k^2 A_j(z_1) j\pi(-1)^j + k_w k_j A_j(z_1 j\pi(-1)^j\Big)\right\};$$

This implies that

$$(m\ )^2 = k^2 \quad (k_n \quad (\ \ 1)^j k_w)^2$$

and from rearrangement we can obtain

$$(k_n \quad (\ \ 1)^j k_w)^2 = k_m{}^2;$$

where

$$k_m{}^2 = k^2 \quad (m\ )^2:$$

Therefore we know that

$$\begin{aligned} k_m &= \ \ k_n \quad (\ \ 1)^j k_w; \\ &= \ \ k_n \quad k_w: \end{aligned}$$

Unfortunately this equation, as shown above, has a non-uniformity where $k_m = k_n \quad k_w$, therefore we need to add a detuning parameter, $\ = O(1)$, so we have

$$k_m = k_n \quad k_w + \quad :$$

This means we can consider the case close to the perfectly tuned case which corresponds to the Bragg condition.

As in Asfar and Nayfeh (1983), we focus on the case $k_m = k_n \quad k_w + \quad$, (the other follows in a similar way). From this relationship we can deduce that

$$e^{i(k_n \ \ k_w)z_0} = e^{i(k_m \qquad)z_0} = e^{ik_m z_0 \ \ i \ \ z_1}; \tag{53}$$

and

$$e^{i(k_m + k_w)z_0} \quad = \quad e^{i(k_n + \quad)z_0} = e^{ik_n z_0 + i \ \ z_1}: \tag{54}$$

Now, as before, we need to nd the governing equation and boundary conditions (at $x = 0$ and $x = 1$) for $\ _j(x; z_1)$. We begin by substituting equation (52) and its derivatives into (46) which yields

$$\begin{aligned} & i\frac{@^2}{@x^2} \ \ _m(x; z_1)e^{ik_m z_0} + i\frac{@^2}{@x^2} \ \ _n(x; z_1)e^{ik_n z_0} \quad ik_m \ \ _m(x; z_1)e^{ik_m z_0} \\ & ik_n \ \ _n(x; z_1)e^{ik_n z_0} + k^2[i \ \ _m(x; z_1)e^{ik_m z_0} i \ \ _n(x; z_1)e^{ik_n z_0}] \\ = \quad & 2i[k_m A^0_m(z_1)\sin(m \ \ x)e^{ik_m z_0} + k_n A^0_n(z_1)\sin(n \ \ x)e^{ik_n z_0}]: \end{aligned}$$

We now equate the coe cients of $e^{ik_n z_0}$ to obtain

$$\frac{\partial^2 \ _n}{\partial x^2} + (n\ \ )^2 \ \ _n = \ \ 2k_n A_n^{\prime}(z_1)\sin(n\ \ x); \tag{55}$$

where we have used the relationship $k_n{}^2 = k^2 \ \ \ (n\ \ )^2$. Similarly

$$\frac{\partial^2 \ _m}{\partial x^2} + (m\ \ )^2 \ \ _m = \ \ 2k_m A_m^{\prime}(z_1)\sin(m\ \ x): \tag{56}$$

Now we nd the right plate boundary conditions for $\ _m$ and $\ _n$. To do this we are going to substitute equations (52), (53) and (54) into equation (51). After di erentiating $\ _1$ twice with respect to z_0, we get

$$\begin{aligned}
\frac{\partial^2 \ _1}{\partial z_0{}^2} &= \ \ i \sum_{j=m;n} \ _j(1;z_1)k_j{}^2 e^{ik_j z_0}; \\
&= \ \ i \ \ _m(x;z_1)k_m{}^2 e^{ik_m z_0} \ \ \ i \ \ _n(1;z_1)k_n{}^2 e^{ik_n z_0};
\end{aligned}$$

so equation (51) becomes

$$i \ \ _m(1;z_1)k_m{}^2 e^{ik_m z_0} \ \ \ i \ \ _n$$

and

$$\psi_m(0;z_1) = \frac{1}{2}\frac{n}{m^2}A_n((n\pi)^2 - k_n k_w)e^{i\ z_1}. \qquad (60)$$

Now we want to look at equation (55), this is to couple the two boundary conditions for $x = 0$ and $x = 1$.

$$\frac{\partial^2\psi_n}{\partial x^2} + (n\pi)^2\psi_n = 2k_n A_n'(z_1)\sin(n\pi x).$$

Multiplying this equation by $\sin(n\pi x)$ and integrating between $x = 0$ and 1 yields the relationship

$$\psi_n(0;z_1) - \psi_n(1;z_1)(-1)^n = \frac{A_n'(z_1)k_n}{(n\pi)}. \qquad (61)$$

Now we have expressions for $\psi_n(0;z_1)$ and $\psi_n(1;z_1)$, we can now move forward in finding equations for the amplitudes $A_n(z_1)$ and $A_m(z_1)$. We have, from equations (57), (59) and (61), that

$$\frac{k_n}{n}A_n'(z_1) = \frac{1}{2}\frac{m}{n^2}A_m((m\pi)^2 - k_m k_w)e^{i(\ z_1+\ )}$$
$$-(-1)^n\frac{1}{2}\frac{m}{n^2}(-1)^m A_m((m\pi)^2 - k_m k_w)e^{i(\ z_1+\ )};$$

from which

$$A_n'(z_1) = \frac{1}{2}\frac{m}{k_n n}A_m(k_m k_w - (m\pi)^2)\left(1 - (-1)^{m+n}e^{i\ }\right)e^{i\ z_1}. \qquad (62)$$

We can also find $A_m'(z_1)$ using the same method. We obtain

$$A_m'(z_1) = \frac{1}{2}\frac{n}{k_m m}A_n(k_n k_w - (n\pi)^2)\left(1 - (-1)^{m+n}e^{-i\ }\right)e^{-i\ z}. \qquad (63)$$

We now want the solutions for equations (62) and (63). To do this we define

$$A_m = a_m e^{sz_1}, \qquad (64)$$

and

$$A_n = a_n e^{(s+i\ )z_1}, \qquad (65)$$

where a_m, a_n and s are constants. We now want to substitute equations (64) and (65) into equations (62) and (63).

Note that

$$A_n'(z_1) = \frac{dA_n}{dz_1} = a_n(s+i\ )e^{(s+i\ )z_1},$$

and

$$A'_m(z_1) = \frac{dA_m}{dz_1} = a_m s e^{sz_1}.$$

So equation (62) and (63) become

$$a_n(s + i\ )e^{(s+i\ )z_1} = \frac{1}{2} \quad \frac{m}{k_n n} \quad a_m e^{sz_1}(k_m k_w \quad (m\ )^2)\ 1 \quad (\ 1)^{m+n} e^{i} \quad e^{i\ z}; \qquad (66)$$

and also

$$a_m e^{sz_1} = \frac{1}{2} \quad \frac{n}{k_m m} \quad a_n\ e^{(s+i\ )z_1}(k_n k_w \quad (n\ )^2)\ 1 \quad (\ 1)^{m+n} e^{\ i} \quad e^{\ i\ z}. \qquad (67)$$

If we rearrange equation (66) to get

$$a_n = \frac{1}{2} \quad \frac{m}{nk_n} \quad \frac{(k_m k_w \quad (m\ )^2)a_m e^{sz_1}}{(s + i\ )e^{(s+i\ )z_1}}\ 1 \quad (\ 1)^{m+n} e^{i} \quad e^{i\ z_1};$$

then we can substitute this into equation (67) to obtain

$$\begin{aligned} a_m s e^{sz_1} &= \frac{1}{2} \quad \frac{n}{mk_m} \quad (k_n k_w + (n\ )^2)\ \frac{1}{2} \quad \frac{m}{nk_n} \quad \frac{(k_m k_w \quad (m\ )^2)}{(s + i\ )e^{(s+i\ )z_1}} \\ &\quad [1 \quad (\ 1)^{m+n} e^{i}\ ]a_m e^{sz_1} e^{i\ z_1} \quad e^{(s+i\ )z_1}[1 \quad (\ 1)^{m+n} e^{\ i}\ ]e^{\ i\ z_1}. \end{aligned}$$

This can be simplified by cancelling constants a_m and a_n to become

$$\begin{aligned} s(s + i\ ) &= \frac{1}{2}(k_n k_w + (n\ )^2)(k_m k_w + (m\ )^2)\frac{1}{2}[1 \quad (\ 1)^{m+n}(e^{i} \ + e^{\ i}\ )]; \\ &= \frac{1}{2k_m k_n}(k_n k_w + (n\ )^2)(k_m k_w + (m\ )^2)[1 \quad (\ 1)^{m+n} \cos\ ]. \end{aligned}$$

So we have

$$s(s + i\ ) = \quad ; \qquad (68)$$

where

$$= \frac{1}{2k_m k_n}(k_n k_w + (n\ )^2)(k_m k_w + (m\ )^2)[1 \quad (\ 1)^{m+n} \cos\ ]. \qquad (69)$$

So we have gained a quadratic in s,

$$s^2 + si \qquad = 0;$$

which we can solve for, giving us

$$s = \frac{i}{2}[\qquad (\sqrt{\ ^2 \quad 4\ })]. \qquad (70)$$

We can begin analysing this equation by rst considering which way the modes are moving. Firstly consider when 4 $>$ ^{2}, this tells us that the modes are moving in opposite directions. By using equation (70) we can see that *s* is complex and of the form ($a + bi$), where *a*, *b 2* R.

By allowing *s* to be complex, we can determine that equations (64) and (65) are telling us that the modes are quickly decaying. As the modes are propagating along the two wave guides, they are reducing in strength. This is called a 'stop band'.

Now we can consider when the modes are moving in the same direction, this occurs when 4 $<$ ^{2} and *s* is completely imaginary with no real parts. Because of this, equations (64) and (65) are bounded. Therefore we know that as the waves propagate they cannot grow in size beyond their bounded limit, i.e. not shoot o to in nity. Also from considering the exponentials in equations (64) and (65), if *s* is imaginary then the exponential terms will not decay. Therefore we know that the modes are propagating in the same direction without loss of strength, meaning that energy in the waves and between the waves is conserved. This is called a passband interaction.

If 4 $=$ ^{2}, then this implies that the movement of the modes is changing from one example to the other. This is called a 'transition frequency'.

progress, we now want the $O(\epsilon^1)$ terms. The equations of $O(\epsilon^1)$ are

$$\left.\begin{array}{ll} \phi_{1x_0x_0} + \phi_{1zz} = -2\phi_{0x_0x_1} & (-h_0 < z < 0) \\ \phi_{1z} - \phi_1 = 0 & (z = 0) \\ \phi_{1z} = \frac{2\pi}{l_0}\cos\left(\frac{2\pi x}{l_0}\right)\phi_{0x_0} - \sin\left(\frac{2\pi x}{l_0}\right)\phi_{0zz} & (z = -h_0) \end{array}\right\} . \qquad (73)$$

Now we can focus on obtaining a solution for ϕ_1. We do this by substituting equation (72) into (73). First we differentiate ϕ_0 with respect to both x_0 and x_1 to yield

$$\frac{\partial^2\phi_0}{\partial x_0 \partial x_1} = ikA_R'(x_1)\cosh k(z+h_0)e^{ikx_0} - ikA_L'(x_1)\cosh k(z+h_0)e^{-ikx_0}$$

which gives

$$\frac{\partial^2\phi_1}{\partial x_0^2} + \frac{\partial^2\phi_1}{\partial z^2} = -2\left[ikA_R'(x_1)\cosh k(z+h_0)e^{ikx_0} - ikA_L'(x_1)\cosh k(z+h_0)e^{-ikx_0}\right]; \qquad (74)$$

for $(-h_0 < z < 0)$. The condition at the free surface will not change at this point, since it has no dependence on ϕ_0. So focusing on (73) for $z = -h_0$, we do the same as we did for $(-h_0 < z < 0)$. From (72), we have

$$\frac{\partial\phi_0}{\partial x_0} = ikA_R(x_1)\cosh k(z+h_0)e^{ikx_0} - ikA_L(x_1)\cosh k(z+h_0)e^{-ikx_0};$$

and

$$\frac{\partial^2\phi_0}{\partial z^2} = k^2A_R(x_1)\cosh k(z+h_0)e^{ikx_0} + k^2A_L(x_1)\cosh k(z+h_0)e^{-ikx_0};$$

so the bed condition becomes

$$\frac{\partial\phi_1}{\partial z} = \frac{2\pi}{l_0}\cos\left(\frac{2\pi x_0}{l_0}\right)\left[A_R(x_1)ik\cosh k(z+h_0)e^{ikx_0} - A_L(x_1)ik\cosh(z+h_0)e^{-ikx_0}\right]$$
$$-\sin\left(\frac{2\pi x_0}{l_0}\right)\left[k^2A_R(x_1)\cosh k(z+h_0)e^{ikx_0} + k^2A_L(x_1)\cosh k(z+h_0)e^{-ikx_0}\right];$$

i.e.,

$$\frac{\partial\phi_1}{\partial z} = \frac{2\pi}{l_0}\cos\left(\frac{2\pi x_0}{l_0}\right)\left[A_R(x_1)ike^{ikx_0} - A_L(x_1)ike^{-ikx_0}\right]$$
$$-\sin\left(\frac{2\pi x_0}{l_0}\right)\left[k^2A_R(x_1)e^{ikx_0} + k^2A_L(x_1)e^{-ikx_0}\right];$$

since $z = -h_0$. To be consistent with Asfar and Nayfeh (1983) we write $2\pi/l_0 = k_w$, the wall undulation parameter. Using the identities

$$\cos(k_w x_0) = \frac{e^{ik_w x_0} + e^{-ik_w x_0}}{2};$$

and

$$\sin(k_w x_0) = \frac{e^{ik_w x_0} \quad e^{\ \ ik_w x_0}}{2i};$$

we obtain

$$\frac{@ \ \ _1}{@z} = \frac{i}{2}\Big\{ e^{i(k_w+k)x_0} \ \ kk_w A_R(x_1) + k^2 A_R(x_1) \quad e^{i(k_w \ \ k)x_0} \ \ k_w k A_L(x_1) \quad k^2 A_L(x_1)$$
$$+ \quad e^{i(k \ \ k_w)x_0} \ \ k_w k A_R(x_1) \quad k^2 A_R(x_1) \quad e^{\ \ i(k_w+k)x_0} \ \ k_w k A_L(x_1) + k^2 A_L(x_1) \Big\} :$$

We are now looking for a particular solution for $\ _1$ of the form

$$_1 = i \ \ _+(x_1; z)e^{ikx_0} + i \quad (x_1; z)e^{\ \ ikx_0}: \qquad (75)$$

This equation includes the exponential terms to demonstrate the one wave propagating both to the left and to the right, that we had shown to be the case in section (2.2).

We can now form a relationship between the wave number and the wall undulation. We have shown how to produce this relationship in section (3.2) noting that k represents the wave propagating to the right as k_n did and similarly $\ \ k$ represents the wave k_m propagating to the left. We have

$$k = \quad k \quad k_w;$$

but this equation, as shown before, has a non-uniformity where $2k = \quad k_w$, therefore we need to add a detuning parameter, $\ \ = O(1)$, so we have

$$k = \quad k \quad k_w + \quad :$$

Note that we will be taking the 'minus case' as in Asfar and Nayfeh (1983), to give us the relationship

$$k = \quad k \quad k_w + \quad : \qquad (76)$$

Now from equation (76) we gain the two equalities

$$k + k_w = \quad k \quad 39000\,k_0 \quad k+$$

We can then express these in an exponential form, giving us the two equations

$$e^{i(k+k_w)x_0} = e^{i(-k+\epsilon\delta)x_0} = e^{-ikx_0+i\delta x_1}, \tag{77}$$

and

$$e^{-i(k+k_w)x_0} = e^{i(k-\epsilon\delta)x_0} = e^{ikx_0-i\delta x_1}. \tag{78}$$

To progress we need to substitute our form of the solution, equation (75), into equation (74), which will give us

$$-ik^2\phi_+(x_1,z)e^{ikx_0} - ik^2\phi_-(x_1,z)e^{-ikx_0} + i\frac{\partial^2}{\partial z^2}\phi_+(x_1,z)e^{ikx_0} + i\frac{\partial^2}{\partial z^2}\phi_-(x_1,z)$$
$$= -2\left[ikA_R'(x_1)\cosh k(z+h_0)e^{ikx_0} - ikA_L'(x_1)\cosh k(z+h_0)e^{-ikx_0}\right].$$

To analyse the two different waves we firstly consider the coefficients of e^{ikx_0} by equating them, we obtain

$$\frac{\partial^2}{\partial z^2}\phi_+(x_1,z) - k^2\phi_+(x_1,z) = -2kA_R'(x_1)\cosh k(z+h_0), \tag{79}$$

and the coefficients of e^{-ikx_0} are

$$\frac{\partial^2}{\partial z^2}\phi_-(x_1,z) - k^2\phi_-(x_1,z) = 2kA_L'(x_1)\cosh k(z+h_0). \tag{80}$$

We can also use this method, for the boundary condition at $z = 0$. From substituting equation (75) into the boundary condition at $z = 0$, we have

$$\frac{\partial\phi_1}{\partial z} = ie^{ikx_0}\frac{\partial}{\partial z}\phi_+(x_1,0) + ie^{-ikx_0}\frac{\partial}{\partial z}\phi_-(x_1,0)$$

and hence the free-surface boundary condition becomes

$$ie^{ikx_0}\frac{\partial}{\partial z}\phi_+(x_1,0) + ie^{-ikx_0}\frac{\partial}{\partial z}\phi_-(x_1,0) - \nu(i\phi_+(x_1,0)e^{ikx_0} + i\phi_-(x_1,0)e^{-ikx_0}) = 0.$$

As before we equate the coefficients of the exponential terms, yielding

$$\frac{\partial}{\partial z}\phi_+(x_1,0) - \nu\phi_+(x_1,0) = 0, \tag{81}$$

and

$$\frac{\partial}{\partial z}\phi_-(x_1,0) - \nu\phi_-(x_1,0) = 0. \tag{82}$$

Finally we can use the same method for the boundary condition at $z = -h_0$. For this calculation, we must plug equation (75) into the boundary condition at $z = -h_0$

and also use both equation (77) and (78), then, (omitting the details), we equate the coefficients of e^{ikx_0} to get

$$\frac{\partial}{\partial z} {}_{+}(x_1; \quad h_0) = \quad \frac{1}{2} e^{\quad i \quad x_1} A_L(x_1)[k_w k + k^2]; \tag{83}$$

and for $e^{\quad ikx_0}$ we obtain

$$\frac{\partial}{\partial z} \quad (x_1; \quad h_0) = \frac{1}{2} e^{i \quad x_1} A_R(x_1)[k_w k + k^2]. \tag{84}$$

For a short time, we shall concentrate solely on equation (79). This is so we can combine the equations for ${}_{+}$ and . To repeat the calculations in Asfar and Nayfeh (1983), we multiply both sides of equation (79) by $\cosh k(z + h_0)$ and integrate them from $z = \quad h_0$ to $z = 0$, eventually yielding

$$\cosh kh_0 \frac{\partial}{\partial z} {}_{+}(x_1; 0) \quad \frac{\partial}{\partial z} {}_{+}(x_1; \quad h) \quad k \sinh kh_0 \; {}_{+}(x_1; 0) = \quad \frac{A'_R(x_1) \sinh 2kh_0}{2} \quad A'_R(x_1) kh_0.$$

In order to understand what the integral is telling us we can use equation (83). Firstly consider that equation (81) says that

$$\frac{\partial}{\partial z} {}_{+}(x_1; 0) = \quad {}_{+}(x_1; 0);$$

therefore we know that

$$\cosh kh_0 \frac{\partial}{\partial z} {}_{+}(x_1; 0) = \quad \cosh kh_0 \; {}_{+}(x_1; 0).$$

Note that Porter and Chamberlain (1997) states that $\quad = k \tanh kh_0$, therefore we can conclude that

$$\begin{aligned} \cosh kh_0 \frac{\partial}{\partial z} {}_{+}(x_1; 0) &= k \tanh kh_0 \cosh kh_0 \; {}_{+}(x_1; 0); \\ &= k \frac{\sinh kh_0}{\cosh kh_0} \cosh kh_0 \; {}_{+}(x_1; 0); \\ &= k \sinh kh_0 \; {}_{+}(x_1; 0). \end{aligned}$$

So the entire integral becomes

$$k \sinh kh_0 \; {}_{+}(x_1; 0) \quad \frac{\partial}{\partial z} {}_{+}(x; \quad h) \quad k \sinh kh_0 \; {}_{+}(x_1; 0) = \quad A'_R(x_1) \frac{\sinh 2kh_0}{2} \quad A'_R(x_1) kh_0;$$

i.e.

$$\frac{@}{@z} \quad +(x;$$

so we substitute equation (92) into (91) to obtain

$$s a_R e^{s x_1} = \frac{e^{i\ \ x_1} e^{\ \ i\ \ x_1} a_R e^{s x_1}(k_w k + k^2) e^{(s+i\ \ )x_1}(k_w k + k^2)}{(s + i\ \ ) e^{(s+i\ \ )x_1}(2kh_0 + \sinh 2kh_0)(\sinh 2kh_0 + 2kh_0)}.$$

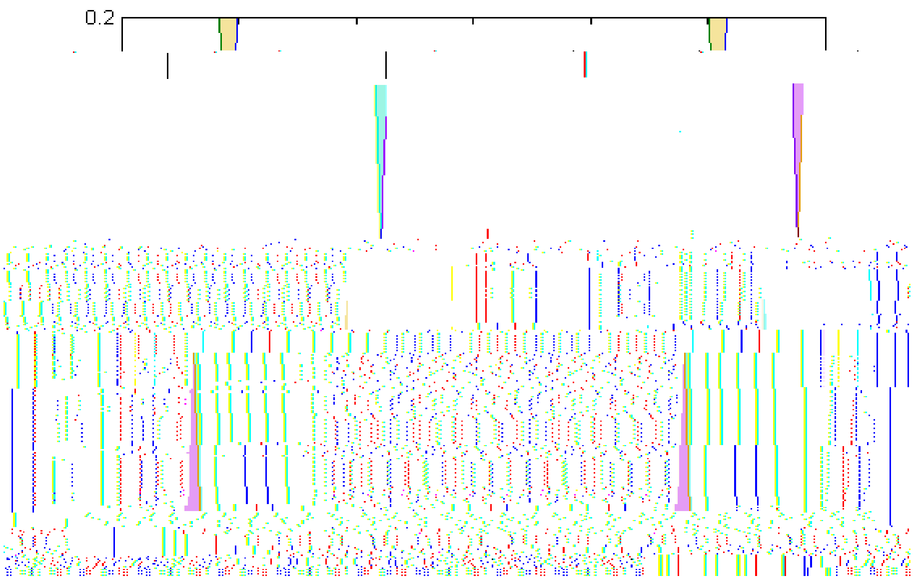
By eliminating the constants a_R and a_L, we obtain

$$(s + i\ \ )s = \frac{(k_w k + k^2)^2}{(\sinh 2kh_0 + 2kh_0)^2} = \ \ ^2. \tag{93}$$

As shown in section (3.2), s can also be expressed as

$$s = \frac{1}{2}\left\{\ \ i\ \ \ \ \sqrt{\ \ ^2 + 4\ \ ^2}\right\}. \tag{94}$$

To demonstrate what equation (94) is telling us, we can consider the plot below.



For this plot we have taken $h_0 = 1$ and to non dimensonalise we let $l_0 = 3h_0$, the length of one ripple. To obtain the plot for $k < 0$, we have plotted the lines $\ \ = \ \ 2\ \ $. We do this by using the relationship between the wavenumber and the wall undulation, we can form a relationship between $\ \ $, $\ \ $ and k, which is

$$\ \ = \ \ \frac{2k + kw}{\ \ }.$$

Thus we can combine $\ \ $ into this equation to obtain

$$\ \ = \ \ \frac{(2k + k_w)(\sinh 2kh_0 + 2kh_0)}{2(k_w k + k^2)}.$$

When $k > 0$, we have considered when equation (76) has taken on the plus sign and becomes

$$k = \quad k + k_w + \quad ;$$

which gives us

$$(s + i \quad)s = \quad ^2; \qquad (95)$$

where

$$^2 = \frac{(k_w k \quad k^2)^2}{(\sinh 2kh_0 + 2kh_0)^2}.$$

Then, again, we can plot k against , where in this case

$$= \quad \frac{(2k \quad k_w)(\sinh 2kh_0 + 2kh_0)}{2(k_w k \quad k^2)}.$$

We can analyse this plot by considering equation (94).

When $4 \ ^2 > \ ^2$ then this implies that s is complex and of the form $(a + bi)$, where $a, b \in \mathbb{R}$. Then after considering equations (89) and (90), we can see that when the $\mathrm{Re}(s) \neq 0$, represented by the shaded yellow regions, that the waves are decaying over time. When the waves are evanescent, the ratio between the two terms $_0$ and $_1$ in the asymptotic expansion describing the velocity potential are unbounded. This definition exhibits Bragg resonance.

Another case to consider would be when $^2 > 4 \ ^2$. s is then imaginary and would display propagating modes. The amplitudes of these modes would be bounded and

5 Multiple Scales for a General Function

5.1 Fourier Series

We can collect the terms of leading order to obtain a new set of equations, which since they are not affected by the second term for $z = -h_0$, will not change from the set of equations (71), which are

$$\left.\begin{array}{ll} \phi_{0x_0x_0} + \phi_{0zz} = 0 & (-h_0 < z < 0) \\ \phi_{0z} - \phi_0 = 0 & (z = 0) \\ \phi_{0z} = 0 & (z = -h_0) \end{array}\right\} .$$

The terms of $O(\epsilon^1)$ will be different for $z = -h_0$, but otherwise remain the same, and become

$$\left.\begin{array}{ll} \phi_{1x_0x_0} + \phi_{1zz} = -2\phi_{0x_0x_1} & (-h_0 < z < 0) \\ \phi_{1z} - \phi_1 = 0 & (z = 0) \\ \phi_{1z} = imk_wC_me^{imk_wx}\phi_{0x_0} - C_me^{imk_wx}\phi_{0zz} & (z = -h_0) \end{array}\right\} . \quad (97)$$

Our next task will be to use ϕ_0, equation (72), in order to find more explicit terms for our ϕ_1 and eventually find a general solution. We need to work through (97), applying ϕ_0 where applicable.

The governing equation will work out to be the same whilst the boundary condition at $z = 0$ will not change, as it has no dependence on ϕ_0. After applying our expression for ϕ_0 our boundary condition at $z = -h_0$ will become

$$\begin{aligned} \frac{\partial \phi_1}{\partial z} = & -imk_wC_me^{imk_wx_0}\left(ikA_R(x_1)\cosh k(z+h_0)e^{ikx_0} - ikA_L(x_1)\cosh k(z+h_0)e^{-ikx_0}\right) \\ & - C_me^{imk_wx_0}\left(k^2A_R(x_1)\cosh k(z+h_0)e^{ikx_0} + k^2A_L(x_1)\cosh k(z+h_0)e^{-ikx_0}\right) . \end{aligned}$$

After some similar manipulation as applied in section 3, we obtain

$$\frac{\partial \phi_1}{\partial z} = -e^{i(mk_w+k)x_0}A_R(x_1)C_m\left(mk_wk + k^2\right) - e^{i(mk_w-k)x_0}A_L(x_1)C_m\left(k^2 - mk_wk\right) .$$

To develop this equation in terms of the slow variable x_1, we need to recall the relationship we had in section 4 and manipulate it to become

$$k = -k + mk_w + \quad (98)$$

where $m \in \mathbb{Z}$. As we had previously done, we should develop two relationships between the wave number and wall undulation, using equation (98). We get

$$k - mk_w = -k +$$

and

$$mk_w \quad k = k \qquad :$$

This gives us the exponential terms,

$$e^{i(k \quad mk_w)x_0} = e^{i(\quad k+ \quad)x_0} = e^{\quad ikx_0+i \quad x_1}; \tag{99}$$

and

$$e^{i(mk_w \quad k)x_0} = e^{i(k \qquad)x_0} = e^{ikx_0 \quad i \quad x_1}: \tag{100}$$

We can de ne $_1$ in such a way to eventually determine our amplitudes,

$$_1 = i \quad _+(x_1; z)e^{ikx_0} + i \qquad (x_1; z)e^{\quad ikx_0}:$$

We can then work through the new equations for the governing system and boundary conditions. By substituting in the expression for $_0$ and equating the coe cients of the exponential terms, we obtain

$$\frac{@^2}{@z^2} \quad _+(x_1; z) \quad k^2 \quad _+(x_1; z) = \quad 2kA_R^0(x_1)\cosh k(z + h_0); \tag{101}$$

for e^{ikx_0}, and

$$\frac{@^2}{@z^2} \qquad (x_1; z) \quad k^2 \qquad (x_1; z) = 2kA_L^0(x_1)\cosh k(z + h_0): \tag{102}$$

for $e^{\quad ikx_0}$, as we had in section 4. For the boundary condition at $z = 0$, we again obtain the same equations as in the previous section, namely

$$\frac{@}{@z} \quad _+(x_1; 0) \qquad _+(x_1; 0) = 0;$$

for e^{ikx_0} and

$$\frac{@}{@z} \qquad (x_1; 0) \qquad (x_1; 0) = 0;$$

for $e^{\quad ikx_0}$.

The boundary condition at the bed requires a bit more work. We will use the same method as we had in section 4. For the coe cients of e^{ikx_0} we obtain

$$i\frac{@}{@z} \quad _+(x_1; \quad h_0) = \quad e^{\quad i \quad x_1} \quad A_L(x_1)C_m \quad k^2 + mk_wk \quad ; \tag{103}$$

and for the coefficients of $e^{\;ikx_0}$ we obtain

$$i\frac{\partial}{\partial z}\quad(x_1;\ \ h_0) = \ \ e^{i\ \ x_1}\ A_R(x_1)C_m\ \ mk_wk + k^2\ \ : \tag{104}$$

Now as we did in section 4, we shall take equations (101) and (102), multiply through by $\cosh k(z + h_0)$ and integrate them from $z = \ \ h_0$ to $z = 0$, eventually giving us

$$\frac{\partial}{\partial z}\ {}_{+}(x_1;\ \ h_0) = \ \ A_R^0(x_1)kh_0\ \ \ A_R^0(x_1)\frac{\sinh 2kh_0}{2}$$

in terms of the equation for e^{ikx_0} and

$$\frac{\partial}{\partial z}\quad(x_1;\ \ h_0) = A_L^0(x_1)kh_0 + A_L^0(x_1)\frac{\sinh 2kh_0}{2}$$

for $e^{\;ikx_0}$. We can now substitute equations (103) and (104) into the above equations where necessary. By doing this we would obtain,

$$e^{\ \ i\ \ x_1}A_L(x_1)C_m\ \ k^2 + mk_wk\ \ \frac{1}{i} = \ \ A_R^0(x_1)\ \ \frac{2kh_0 + \sinh 2kh_0}{2}\ \ ;$$

which can be rearranged to give

$$A_R^0(x_1) = \ \ \frac{2e^{\ \ i\ \ x_1}A_L(x_1)C_m\ \ k^2 + mk_wk}{i\,(2kh_0 + \sinh 2kh_0)};$$

and also

$$e^{i\ \ x_1}A_R(x_1)C_m\ \ k^2 + mk_wk\ \ \frac{1}{i} = A_L^0(x_1)\ \ \frac{2kh_0 + \sinh 2kh_0}{2}\ \ ;$$

which gives

$$A_L^0(x_1) = \frac{2e^{i\ \ x_1}A_R(x_1)C_m\ \ k^2 + mk_wk}{i\,(2kh_0 + \sinh 2kh_0)}:$$

Now as before we shall define functions for the amplitudes,

$$A_R(x_1) = a_Re^{sx_1}$$

and

$$A_L(x_1) = a_Le^{(s+i\ \ )x_1}:$$

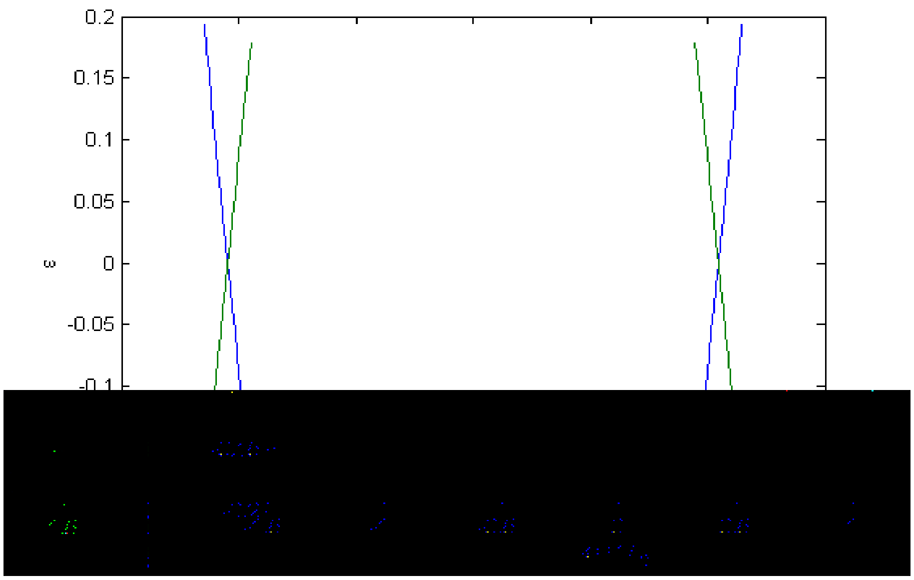
We can use these functions for the amplitudes, to substitute into our equations for $A_L^0(x_1)$ and $A_R^0(x_1)$, and by combining the two we will eventually obtain

$$s(s + i\ \ ) = \frac{4C_m{}^2\ \ k^2 + mk_wk\ \ ^2}{(2kh_0 + \sinh 2kh_0)^2} = \ \ \ ^2:$$

This again gives us a quadratic in s which can be solved to be analysed. We end up with the equation

$$s = \frac{1}{2}\left\{ i \quad \sqrt{\quad ^2 + 4 \quad ^2} \right\}; \qquad (105)$$

As before we can plot the lines = 2 as we did in section 4, to obtain the plot,



For this plot, we had to de ne an l_0-periodic function and work out the Fourier series coe cient for it. The function was chosen as

$$(x) = \cos k_w x;$$

where $k_w = 2 = l_0$. So now we want $\cos k_w$ w wWwWwwWwx wTx 24.242 24215215 Td [(01 Td65d][

Firstly consider when $4 \quad > \quad ^{2}$, this tells us that the modes are reversed and also that s is complex. As seen in sections 3 and 4, if s is complex then there is a stop band. For the four triangles made up of the lines of $\quad = \quad$, this is where the modes are decaying and Bragg resonance is occurring, as the exponential contains a real part, which will dominate the movement of the mode.

Now we can consider when the modes are codirectional, this occurs when $4 \quad < \quad ^{2}$ and Re(s) = 0. Because our exponential terms have no real parts, we know that the modes are propagating and are therefore bounded - never to decay or grow. Within this region, outside of the crosses, is a passband interaction.

When $4 \quad = \quad ^{2}$ a transition frequency is occurring.

This analysis would be true for any general function that can be manipulated into the form of equation (96). As this function had only the two coe cients, we would expect a function of more/ranging coe cients to have more 'crosses' and to change in gradient, depending on the coe cient.

6 Conclusion

From the use of the method of multiple scales we have been able to nd a solution for the problem posed in Porter and Chamberlain (1997). In conclusion we have been able

References

[1] D. Porter and P. G. Chamberlain (1997) *Linear Wave Scattering by Twn 997)*