American Spread Option Pricing

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Declaration

I confirm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

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6.9 Square of di�erence between UNM and BS prices (lambda =

Notations

Throughout this report we follow the usual notations given below:

K - strike price

T - time to expiry / residual maturity

S - price of the stock i

σ - volatility of the stock i

ρ - c11426(t)-325.33(t)S(c)2(½)Tjjj[(-)-30Td[(i)-0.218509]TJ-13434343cc½a-0.147034(t).147034(i)0

In addition to this, the early-exercise feature of American options makes it even worse. Although there has been extensive research in this field, no efficient and accurate pricing model has yet been developed.

Some of the work done so far for pricing American spread options can be spread options.

We then move on to price American spread options in chapter 4, by introducing the 3-D binomial tree model. How these models mentioned so far are extended to price American spread options, consistent with the market model [2] is explained in sec. 4.2.

The implementation of these approaches using C++ and the various modules in the program are explained in chapter. 5.2. The last two chapters deal with the output and analysis of the program results where we do a comparative study of the performance of the code and the market models.

[2] A od con n d nd o

for a call or put respectively. Since the payoff depends only on the stock price for a particular strike, tracking the stock price movement would be a useful tool in option pricing. Hence we have volatility

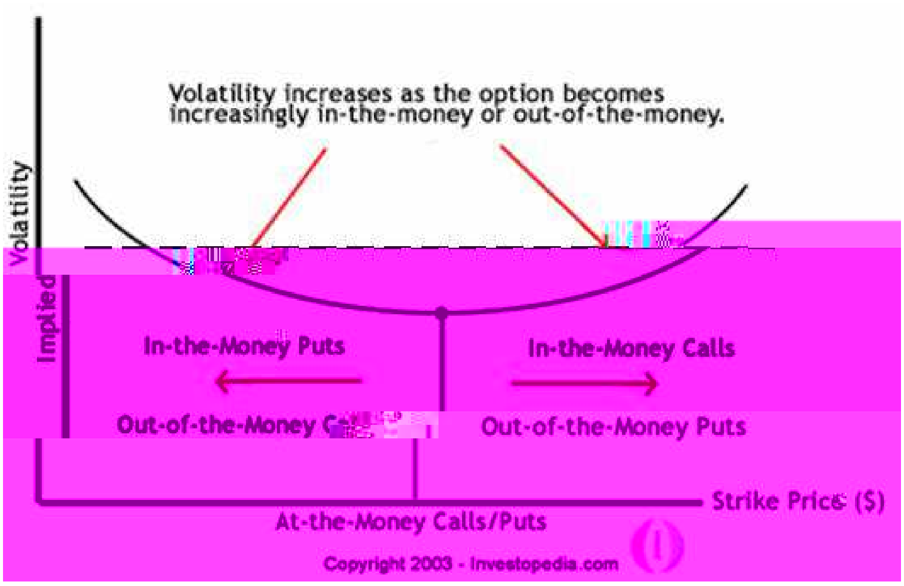

Figure 2.1: Volatility smile (source: www.investopedia.com)

maturity. Although the Black-Scholes(BS) model performs well the assumption is proved to be flawed. When the volatility is computed (implied) by a model for a set of market prices for different strikes, the volatility is not observed to be a constant rather it is skewed. In the case of currency option markets the implied volatility of in-the-money and out-of-the-money options is greater than the at-the-money options as shown in the figure. Hence the volatility smiles in this case! This is explained by the fact that traders speculate a larger price movement than is assumed in the BS model. Since every other parameter is a constant in the BS model the disparity in the computed and market prices can be explained only by increasing the volatility.

2.3 Spread options

Spread options are derivative products on two or more assets. Most often they are referred to those written on the difference between the values of two indexes. For example, a European call spread on two underlying assets with prices S_1 and S_2 will have a pay off function $[S_1 - S_2 - K]^+$. The + superscript denotes that payoff can only be positive, for any negative value it equals zero.

implied correlation. It is similar to the former in all respects except that it frowns and does not smile! That is, the implied correlation is lesser for in-

Chapter 3

Option Pricing

detailed illustration the reader is advised to refer to the works suggested therein.

3.2 Vanilla Option pricing

3.2.1 Black-Scholes(BS) model

The most earliest and powerful tool to compute the price of European options was discovered by Black and Scholes(1973). Even thirty years later it remains to be one of the most preferred model and serves as the basis for many others in the world of options theory. It states that the price of a call option at a time t is given by the solution of the backward parabolic partial differential equation

$$\frac{\delta p_0}{\delta T} = (r \quad q)S_0 \frac{\delta p_0}{\delta S} \quad \pm$$

where r is the short rate of interest. We then have,

$$p = S(0)\Phi(d_1) \quad Ke^{-} \quad \Phi(d_2) \tag{3.3}$$

where

$$d_1 = \frac{\ln\left(\frac{(0)\quad rT}{\quad}\right)}{\sigma \quad \overline{T}} + \frac{1}{2}\sigma \quad \overline{T} \qquad \text{and} \qquad d_2 = d_1 \quad \sigma \quad \overline{T} \tag{3.4}$$

Here $\Phi(x)$ represents the cumulative distribution function of the standard normal $N(0,1)$ distribution, i.e.,

$$\Phi(x) = \frac{1}{\overline{2\pi}} \int_{-\infty} e^{\frac{-u^2}{2}} du \tag{3.5}$$

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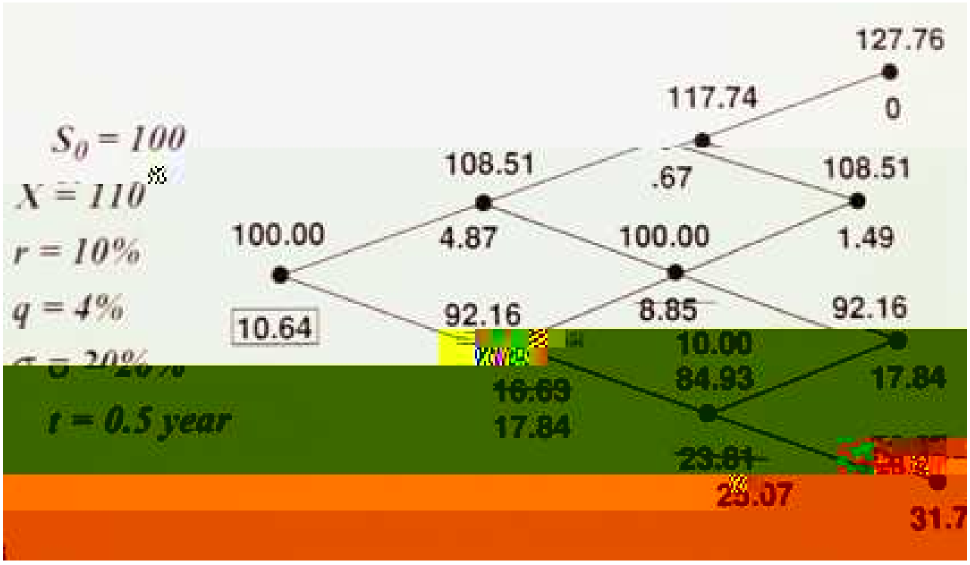


Figure 3.1: 2-D Binomial tree

the latter and continue calculating the option price in the preceding level.

- The price thus obtained at the initial node corresponding to time t = 0, is the required American option price.

Figure 3.1 shows how the American option price is calculated.

3.3 Spread option pricing

was based on the Black-Scholes price for spread options expressed as an expectation of the payo function, as in 3.6. The formula is as follows:

$$\hat{p} = x_2\Phi\left(\frac{\ln\left(\frac{x_2}{x \quad Ke^{-rT}}\right)}{K} + \frac{K}{2}\right) \quad (x_1 + Ke^{-} \quad)\Phi\left(\frac{\ln\left(\frac{x_2}{x \quad Ke^{-rT}}\right)}{K} \quad \frac{K}{2}\right)$$

where

$$\sigma = \sqrt{\sigma_2^2 \quad 2\rho\sigma_1\sigma_2\frac{}{+ \quad \exp^{-rT}} + \sigma_1^2 \quad \frac{}{+ \quad \exp^{-rT}}^{2}} \tag{3.10}$$

Carmona and Durrleman performs a comparative study of how this model performs against other models. A more refined approach can be found in Eydeland and Wolyniec(2003).

3.3.2 Bivariate normal mixture model

σ , σ are the volatilities of core and tail normal densities. P_2 is the price of 2-Geometric Brownian motion model(2GBM). The 2GBM models assume two correlated log-normal di□usions to model European spread options (Ravindran 1993, Shimko 1994, Kirk 1995, James 2002 and others).

The di□erence here is that the terminal risk neutral density will be a bivariate normal mixture instead of bivariate normal, but the transition probabilities still remains normal. An interesting fact is that although the option price is a linear combination at time t = 0 and T (bivariate normal mixture), at time t = υ one can uniquely identify the price (P_2'

Chapter 4

Pricing American Spread Options

4.1 3-D tree model

The three dimensional binomial tree model for two asset options is shown in figure 4.1. The space variables used are $x = \ln S^{(1)}/S_0^{(1)}$ and $y = \ln S^{(2)}/S_0^{(2)}$ instead of the stock prices themselves. This means that the step sizes are of constant sizes, rather than proportional to the stock prices, hence making it simpler. The first node in the tree has value zero. If the risk-neutral drift of $S_1^{(1)}$ is $r \quad q_1$, then the drift of x is $r \quad q_1 \quad \frac{1}{2}\sigma_1^2 = m$, and y is $r \quad q_2 \quad \frac{1}{2}\sigma_2^2 = m$

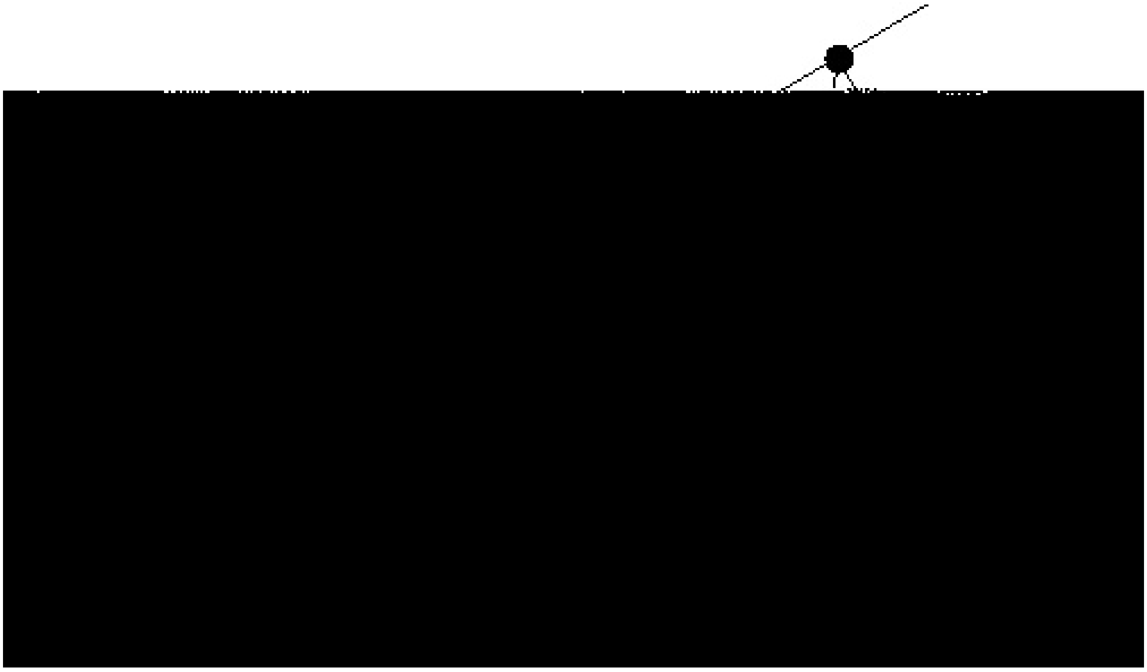


Figure 4.1: 3-D tree structure

The Wiener processes for the two space variables can be written as

$$\delta x \;=\; m\,\delta t \;+\; \sigma\quad \overline{\delta t} z_1$$

$$\delta y \;=\; m\,\delta t \;+\; \sigma\ \mathrm{sqrt}\delta t z_2 \;=\; m\,\delta t \;+\; \sigma\quad \overline{\delta t}\ \rho z_1 \;+\; \sqrt{1 \quad \rho^2} z_3\}$$

where z_1 and z_3 are uncorrelated standard normal variates.

Hence the following equations:

$$\delta y \;=\; m\,\delta t \;+\; \sigma\quad \overline{\delta t}\ \rho \;+\; \sqrt{1 \quad \rho^2}\}$$

$$\delta y \;=\; m\,\delta t \quad \sigma\quad \overline{\delta t}\ \rho \quad \sqrt{1 \quad \rho^2}\}$$

$$\delta y \;=\; m\,\delta t \quad \sigma\quad \overline{\delta t}\ \rho \;+\; \sqrt{1 \quad \rho^2}\}$$

$$\delta y \;=\; m\,\delta t \;+\; \sigma\quad \overline{\delta t}\ \rho \quad \sqrt{1 \quad \rho^2}\} \qquad (4.1)$$

For a detailed discussion on 3D tree models see James(2002). A more advanced 3-D tree approach can be found in Boyle(1988).

4.2 Extension of BNM model

We aim to extend the frown consistent[1] bivariate normal mixture model introduced by Alexander and Scourse (2004) for pricing European spread options to American Spread Options. The need for American option prices that are consistent with the market prices of European options requires us to use prices obtained from a smile consistent model[2]. We assume that the marginal distribution of each correlated asset return is a mixture of normal distributions.

Leaning upon the existing volatility models a substantial time would be dedicated in extending the Bivariate normal mixture(BNM) model (Alexander and Scourse, 2004) for pricing American spread options calibrated to both volatility smiles and the correlation frown. Alexander and Scourse assume that each asset return density is a mixture of two normal densities and that their joint density is a bivariate normal mixture.

Firstly we calibrate the univariate normal mixture(UNM) model and then the Bivariate normal mixture model to the market prices of European options. These calibrated models will be used to find the American option price using a 3-D binomial tree approach as described in James(2002). Since the BNM model is smile and frown consistent and also the univariate normal mixture model is smile consistent, the American spread option price obtained

[1] o d c con n co on o n

[2] A o on c n d con n o y

normal mixture model prices are analytic, we do not actually need to do

of pairs (S_1, S_2) for which S_1 S_2 = K. In order to overcome this it was assumed that the strike convention used to calculate the implied volatility was $K_1 = S_1$ (K $S_1 + S_2)/2$ and $K_2 = S_2$ (K $S_1 + S_2)/2$. When the strike is zero they give rise to exchange options, which are more easier to handle. An analytic pricing formula for exchange options was first derived by Margrabe(1978).

For the sake of simplicity, we assume that $\sigma_1 > \sigma$ and $\sigma_2 < \sigma$. One would expect σ_2 σ as that addresses the core volatility of the normal mixture. Without loss of generality we assume that $0 < \lambda < 0.5$. This implies that the higher volatility makes lower contribution and the lower volatility makes higher contribution to the overall volatility.

A similar argument applies for the correlation as well, where ρ < ρ and $\rho < \rho$. We assume that ρ = ρ and that ρ takes values close to twice as that of ρ.

Then, three dimensional binomial trees are constructed using each of the above correlations ρ , ρ , ρ , and ρ and the corresponding volatilities. That is, each of the four covariance matrices, $V_1, V_2, V_3,$ and V_4

option on a single underlying asset. The tolerance value was assumed to be 0.05% of the Black-Scholes price for a call option.

The module involves a straight-forward implementation of equation (3.7). The BS function calculates the price as in equation (3.3) by calling the function phi to calculate the cumulative density. Phi in turn uses Simpson's rule to evaluate the line integral. The lower limit of integration is restricted to -25 instead of without any significant contribution to the error.

5.1.2 Bivariate normal mixture module

This implements the model described by Alexander and Scourse (2004) to price European spread options. It uses the functions g(), dg(), kirk() and phi(). The calibrated values of the volatilities and lambdas from the UNM module serve as the input for this module. This module is executed third chronologically after UNM and calibration module.

This module too involves a straight-forward implementation of equations 3.6 and 3.10. The kirk() function calculates the price using (3.10) by calling the function phi() given prices of two stocks.

The output of this module is the square of the difference between the bivariate normal mixture model price and the Kirk's price.

5.1.3 Calibration module

This is the most important of all the modules. It finds the optimum values for volatility and correlation for a set of inputs. Moreover, its output can significantly alter the final output of the program due to the cascading effect of errors. This module is run twice separately to optimise the volatilities of

5.1.4 Binomial tree module

This module calculates the American spread option price using the 3-D binomial tree discussed in 4.1. There is no interaction between this module and the rest. The calibrated values of correlation and volatilities, initial stock prices, time-step size and other usual data serve as the input to this module.

This module has an array implementation of a 3-D binomial tree where an array is logically manipulated as a tree with no physical links similar to a be 0.25. The maximum profit condition for American options is taken into consideration by including a conditional statement in the backtracking part. This statement compares the calculated price and the payoff function at that node and stores the greatest among them.

5.2 Working of code

Having explained the different modules we shall see how these are linked and executed as a whole. Figure 5.1 shows the flow of control from one module to the other. Each arrow represents a function call, with the arrow directed to the called function. The program was written in C++ and MATLAB was used for plotting the results. All the modules mentioned below function as discussed in the previous section.

The various functions defined are given in 5.3 along with their input parameters and functionalities. One would notice that there are few functions bearing the same name but have different associated functionalities. This is called as function overloading in C++. Function overloading allows for defining multiple functions under the same name, but with different set of input parameters. When a function is called, C++ automatically chooses the appropriate function by comparing the parameters. In the program the functions that are overloaded are g(), dg(), and newton(). g() calculates the value of function g as in (5.1), dg() calculates the first order differential and newton() performs the Newton's method for both volatility and correlation. This enhances the clarity of the code as name is associated more with the function of the function and not with a block of code!

Going back to figure 5.1 we can see that the function main() forms the

core, calling all the other modules. Since our aim is to obtain American spread option price it is imperative to obtain calibrated values of volatility and correlation at first. Moreover we need to calibrate volatility before calibrating correlation.

In order to calibrate the volatility we run the newton()

dg( )	
r, q, T, K, sigma, x, s, lambda	Finds the first order derivative of the function g with respect to sigma

dg( )	
r, q1, q2, T, K, sigma1, sigma2, sigma11, sigma21, s1, s2, lambda1, lambda2, rho, rhocc, rhott	Finds the first order derivative of the function g with respect to rhocc

newton( )	
r, q, T, K, sigma, x, s, lambda	Finds the value of sigma for which g is a minimum using Newton's method

newton( )	
r, q1, q2, T, K, sigma1, sigma2, sigma11, sigma21, s1, s2, lambda1, lambda2, rho, rho, rhott	Finds the value of rho for which g is a minimum using Newton's method

bintree( )	
r, q1, q2, T, K, sigma1, sigma2, s1, s2, rho, dt	Calculates the value of American Spread Option price using a 3D Binomial tree model for a specified time step dt.

Chapter 6

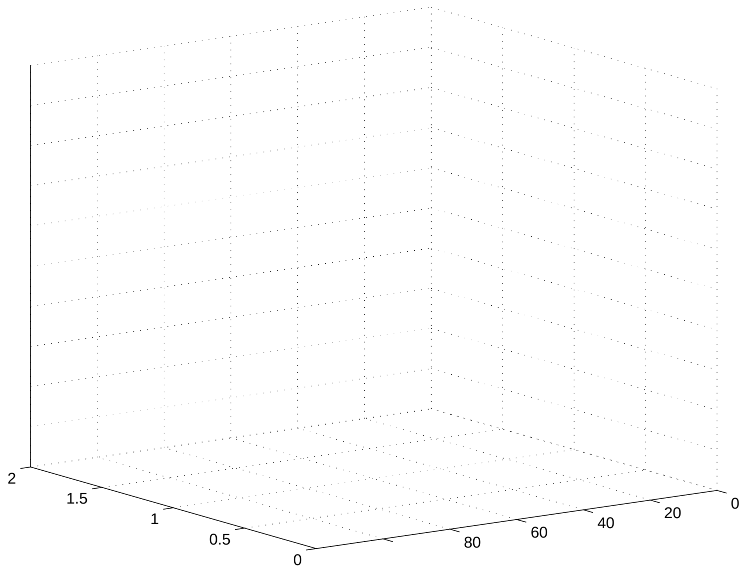
Analysis

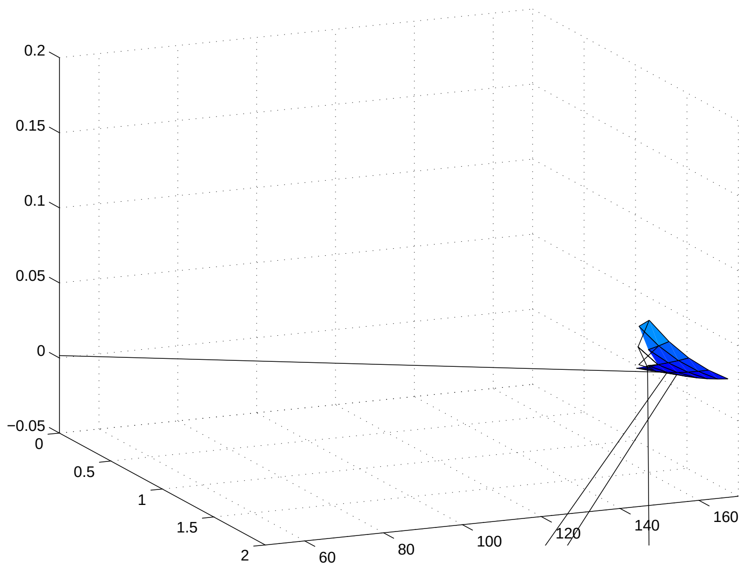
In this section we shall look at the results of the program discussed in 5.2 and discuss its performance. We compare the output of the calibrated Univariate normal mixture(UNM) model with Black-Scholes' and that of Bivariate normal mixture (BNM) model with Kirk's. The behaviour of the prices obtained from each of these models are shown in the figures that follow. Unless specified the stock prices of assets 1 and 2 are taken to be 100. The volatility of stock 1 is 25% and stock 2 is 40%. The correlation between the stocks is -0.5.

Fig. 6.1 shows the Black-Scholes price (p) as a function of strike and maturity. The Black-Scholes price increases linearly with strike as shown and tries to imitate the actual pay off function.

When $K < S$, the Black-Scholes price is comparatively low and when K S, the price increases linearly with strike as shown in fig. 6.1. The change of price with respect to time to maturity (T) is lesser. Fig. 6.2 shows how the price curve shifts away from the actual payoff as T increases.

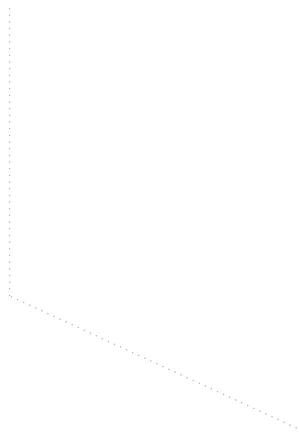
Fig. 6.3 shows how the UNM price function behaves with respect to strike and maturity for the calibrated values of volatility. As discussed in sec. 4.2, since the UNM model was calibrated to the BS model (with different volatil-


2
1.5
1
0.5
0
80
60
40
20
0


0.2
0.15
0.1
0.05
0
−0.05
0
0.5
1
1.5
2
60
80
100
120
140
160

maturity.

Since the Kirks formula was derived based on the Black-Scholes model by expressing the option price as an expectation of the payo function, it is natural to expect Kirks spread option price to behave on the lines of the BS price. As in fig. 6.13 we can see that the price increases smoothly as the strike increases. The plot of the surface of the BNM price against sigma1



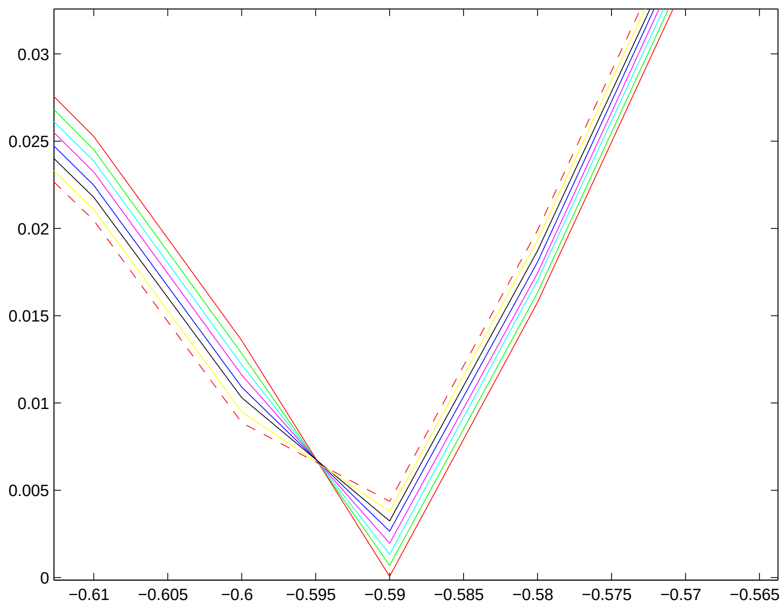


Figure 6.16: Square of difference between BNM and Kirk's prices versus rhoCC - (zoomed near minimum)

between the calibrated BNM price and the Kirks price is plotted as a function of strike and maturity in 6.17. Unlike fig. 6.11 which dips when the strike is near the stock price, we see here that the dip occurs at a different point. This is explained by the complex relation between the stock prices of the two assets and the strike of the spread option. One might observe a different pattern if calibration was done using a different strike convention as explained in Alexander and Scourse(2004).

Alexander and Scourse show that the BNM price is lesser than or equal to 2GBM/Kirk's price due to uncertainty over correlation and greater than or less than 2GBM/Kirk's price due to uncertainty over volatility. Fig. 6.17 also shows that the BNM prices can be greater than or lesser than the Kirk's. Since our calibrated BNM model conforms to this it is frown consistent!

The graph in fig. 6.18 shows the 3-D tree price of an American spread

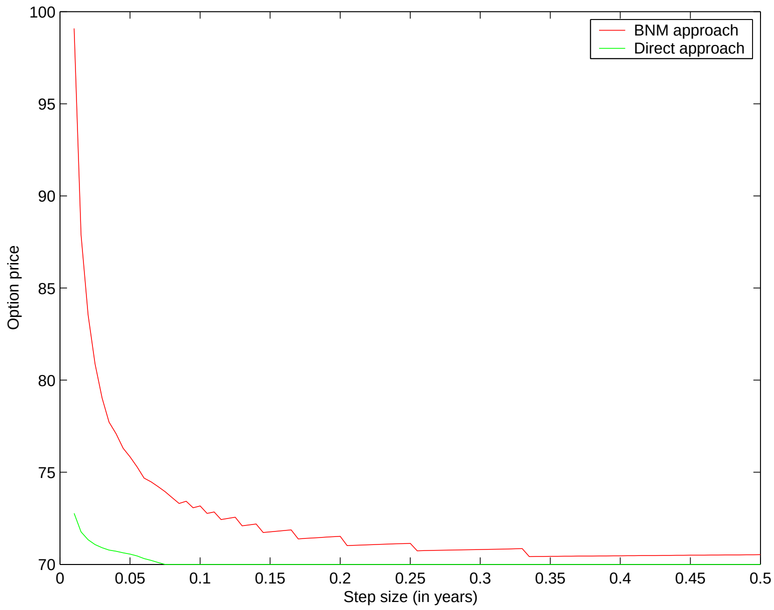
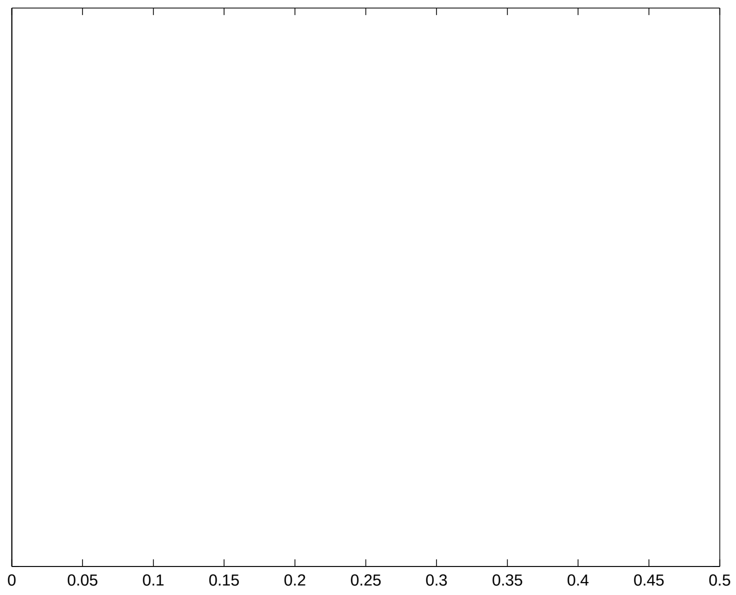


Figure 6.18: American Spread option price using 3-D tree model

option as a function of step size. With decreasing step size the resulting price increases. The price obtained using the BNM approach is found to be greater than that of a direct implementation (by substituting σ_1, σ_2, and ρ) of odel


0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Chapter 7

Summary and Conclusion

This project aims to price American spread options by extending the BNM

The values of the American spread options were found to be greater than that of Europeans' as expected. But this is by no means an effective tool to validate the results obtained. A better conclusion can be arrived by comparing the results with the actual market data.

This project has contributed by using an amalgamation of analytic and numerical approaches to find the American spread option prices. The main advantage of this method, which has never been implemented so far, is its simplicity which is mainly borrowed from the BNM and 3-D binomial tree models. There is a greater scope for further research and one can find innumerable ways of pricing American spread options.

The 3-D binomial tree approach used was a basic approach and the results can be improved if we were to use the model described by Boyle(1988). On the numerical front, since we were interested in the lambda, sigma and rho values only up to two decimal places, the choice of the fixing the lambda and rho values and their step size (see sec. 5.1.3) is justified.

If one were to find more accurate results the univariate approach adopted would not prove a good choice. In that case we can adopt higher dimensional descent methods, like gradient methods, Krylow subspace method and others, for optimisation. Proposing the problem as a linear optimisation problem with a set of constraints would be a more efficient and elegant approach. In brief, by adopting the extended Kirk's formula, advanced 3-D tree approaches and efficient optimisation techniques this new approach can

Appendix A
Newton-Raphson Method

Let f(x) be a continuous smooth monotonically increasing/decreasing or a convex function with only one zero. The Newton-Raphson method allows one to find the zero of the function iteratively considering the function, its derivative, and an arbitrary initial x-value. The value of the iterate depends on the value and derivative of the function at the previous point. It is given by:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \quad \text{where} \quad f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

where, x_n is the current known x-value, $f(x_n)$ represents the value of the function at x_n, and $f'(x_n)$ is the derivative (slope) at x_n. x_{n+1} represents the new x-value that we are trying to find. This method has a quadratic rate of convergence.

The first order derivative in the program was calculated using a Δx of 0.005 which produced a satisfactory approximation to the actual value. This makes a good choice as the values of the Kirk's and Black-Scholes formulas are not much altered for small changes in correlation and volatility, respectively.

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