Some Problems in Vectorial Calculus of Variations in L^1

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Declaration

I con rm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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Dedication

This thesis is dedicated to:

My parents,

My wife and son.

Abstract

that the image of a solution u is piecewise affine if either the rank of Du is equal to one or $n = 2$ and u has additively separated form. As a consequence we obtain corresponding flatness results for p-Harmonic maps for $p \in [2, \infty]$.

The aim of the Chapter 4 is to derive new explicit solutions to the ∞-Laplace equation, the fundamental PDE arising in Calculus of Variations in the space L^∞. These solutions obey certain symmetry conditions and are derived in arbitrary dimensions, containing as particular sub-cases the already known classes of two-dimensional infinity-harmonic functions.

Chapter 5 is the joint paper with N.Katzourakis. We discuss two distinct minimality principles for general supremal first order functionals for maps and characterise them through solvability of associated second order PDE systems. Specifically, we consider Aronsson's standard notion of absolute minimisers and the concept of ∞-minimal maps introduced more recently by N.Katzourakis. We prove that C^1 absolute minimisers characterise a divergence system with parameters probability measures and that C^2 ∞-minimal maps characterise Aronsson's PDE system. Since in the scalar case these different variational concepts coincide, it follows that the non-divergence Aronsson's equation has an equivalent divergence counterpart.

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Chapter 1

Background and motivations

1.1 Introduction

Minimization problems have been studied by many mathematician for a di er-ent purposes. Most of their e orts were focused in study the relation between minimality conditions and partial di erential equations (PDEs). One of the way to view the minimality as a variational approach, which is the core idea of Calculus of Variations. We introduce some fundamental methods of Calculus of Variations to solve possibly non-linear PDE, which for a simplicity we have in the following form

$$L[u] = 0; \tag{1.1.1}$$

where $L[u]$ is a given di erential operator and u is the unknown. The equation (1.1.1) can be characterised as the minimiser of appropriate energy functional $E[u]$ such that

$$E'[u] = L[u]:$$

The usefulness of this method that now we can proof existence of extremum points of the functional energy E[] and consequently the solution of (1.1.1). One of di culties of described method that in general the minimiser of the functional might not be a classical solution of the PDE and the de nition of

the method mentioned above. Section 1.2 and 1.3 of this chapter give more details on Calculus of Variations in L^1 and organization of the thesis respectively.

1.2 Calculus of Variations in L^1

Calculus of Variations in L^1 has a long history started in the 1960s by G. Aronsson [4–8]. He considered the following variational problem for the supremal functional

$$E_1(u;O) := \operatorname*{ess\,sup}_{x \in O} |Du(x)| \quad (1.5)$$

work for mappings, coe cients of full system are discontinuous and solutions are need to be C^2 to make a classical sense. However there is a method to reduce the regularity of solutions to C^1 using measures as parameters. For more details about this method we refer to Chapter 5.

For more details about explicit classical solutions of (1.2.3) we refer to the introduction of the paper presented in Chapter 4.

1.3 Organisation of thesis

The aim of the thesis is to nd new classical solutions, derive necessary and su cient conditions and describe a geometric properties of absolute minimisers. We have reached our goal by publishing and submitting papers in di erent elds of nonlinear PDEs. Each paper is presented in this thesis as chapter. Every chapter below is explained in the outline below.

Chapter 2 is the joint paper with N.Katzourakis. The estimated contribution is 50%. The paper has been accepted at Proceedings of the Royal Society of Edinburgh A (Mathematics). We study a pointwise characterisation of the PDE system of vectorial calculus of variations in L^1. In this chapter we prove that generalized solution to PDE

$$\begin{cases} H_P(\ ;u;Du)\,D\ H(\ ;u;Du) = 0; \\ H(\ ;u;Du)\,[\![H_P(\ ;u;Du)]\!]^{?}\ \mathrm{Div}\ H_P(\ ;u;Du) \quad H\ (\ ;u;Du) = 0; \end{cases}$$

can be characterized as local minimisers for appropriate classes of a ne variations to the following energy

$$E_1(u;O) := \operatorname*{ess\,sup}_{O} H(\ ;u;Du); \quad u \in W^{1;1}_{\mathrm{loc}}(\ ;R^N); \quad O \ b\ ;$$

where $n;N \ 2\ N$ with R^n open, $H 2\ C^2(\ R^N\ R^{Nn})$ is a given, $H \dot{P};$

open set and$n; N \quad 1$ and $u \; 2 \; C^2(\; ; R^N)$ be a solution to the nonlinear system $[\![Du]\!]^? \quad u = 0$ in , satisfying that the rank of its gradient matrix is at most one:

$$rk(D u) \quad 1 \quad \text{in} \quad :$$

Then, its image$u( )$ is contained in a polygonal line in R^N, consisting of an at most countable union of a ne straight line segments (possibly with self-intersections). After we show that this theorem is optimal by giving an example that system can not have a ne image but only piecewise a ne. Then we have next theorem as

following

$$\ln|t| + c = \begin{cases} \frac{1}{2}\ln\left|\frac{{}^2(t)+B^2}{{}^2(t)\quad(t)+B^2}\right| \quad \frac{1}{2}p\frac{1}{B^2\quad\frac{1}{4}}\arctan p\frac{(t)\quad\frac{1}{2}}{B^2\quad\frac{1}{4}}; & \text{if } B^2\quad\frac{1}{4} > 0 \\ \frac{1}{2}\ln\left|\frac{{}^2(t)+B^2}{{}^2(t)\quad(t)+B^2}\right| + \frac{1}{2}\frac{1}{(t)\quad\frac{1}{2}}; & \text{if } B^2\quad\frac{1}{4} = 0 \\ \frac{1}{2}\ln\left|\frac{{}^2(t)+B^2}{{}^2(t)\quad(t)+B^2}\right| \quad \frac{1}{4}p\frac{1}{\frac{1}{4}\quad B^2}\ln\left|\frac{(t)\quad\frac{1}{2}p\frac{1}{4}\quad B^2}{(t)\quad\frac{1}{2}+p\frac{1}{4}\quad B^2}\right|; & \text{if } B^2\quad\frac{1}{4} < 0; \end{cases}$$

where c is any constant, provided RHS is well de ned.

Finally second result, let $n \quad 2$ and $u : \quad R^n \; ! \quad R$ be a $C^2(\,)$ separated 1 -harmonic function of the 1 -Laplace equation

$$\sum_{i;j=1} D_i u\, D_j u\, D^2_{ij} u \;=\; 0:$$

Then

$$|f_i(x_i)| = |f_i(x_i^0)|\, e^{A_i(x_i \quad x}$$

The converse statement is true if in addition for any $x \in \Omega$, $H(x,\cdot,\cdot)$ is convex on $\mathbb{R}^n \times \mathbb{R}^{N\times n}$.

(III) If u is $\mathcal{A}_1$-minimal map for (1.1.1) on Ω, then it solves the (reduced) Aronsson system

$$\begin{aligned}\mathcal{A}_1 u := {} & H_P(x,u,Du)\cdot D H(x,u,Du) \\ & + H(x,u,Du)\left[\left[H_P(x,u,Du)\right]\right]^{\star} \operatorname{Div} H_P(x,u,Du) - H(x,u,Du) = 0.\end{aligned}$$

The converse statement is true if in addition H does not depend on $x \in \mathbb{R}^N$, $H_P(x,Du)$ has full rank on Ω and for any $x \in \Omega$ $H(x,\cdot)$ is convex in $\mathbb{R}^{N\times n}$.

Second result "Divergence PDE characterisation of Absolute minimiser" says, let $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ be a map in $C^1(\Omega;\mathbb{R}^N)$. Fix also $O \subset \Omega$ and consider the following statements:

(I) u is a vectorial minimiser of $E_1(\cdot,O)$ in $C^1_u(\overline{O};\mathbb{R}^N)$[1].

(II) We have

$$\max_{\operatorname{Argmax}\{H(x,u,Du):\overline{O}\}} \left[H_P(x,u,Du) : D + H(x,u,Du)\right] \geq 0,$$

for any $\in C^1_0(\overline{O};\mathbb{R}^N)$[1].

(III) For any $\in C^1_0(\overline{O};\mathbb{R}^N)$, there exists a non-empty compact set

$$K \subset \operatorname{Argmax}\{H(x,u,Du) : \overline{O}\} \tag{1.3.3}$$

such that,

$$H_P(x,u,Du) : D + H(x,u,Du)\big|_K = 0. \tag{1.3.4}$$

Then, (I) $\Rightarrow$ (II) $\Rightarrow$ (III). If additionally $H(x,\cdot,\cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N\times n}$ for any fixed $x \in \Omega$, then (III) $\Rightarrow$ (I) and all three statements are equivalent. Further, any of the statements above are deducible from the statement:

(IV) For any Radon probability measure[2] $\in \mathcal{P}(\overline{O})$ satisfying

$$\operatorname{supp}(\ ) \subset \operatorname{Argmax}\{H(x,u,Du) : \overline{O}\}; \tag{1.3.5}$$

we have

$$\operatorname{div} H_P(x,u,Du) + H(x,u,Du) = 0; \tag{1.3.6}$$

in the dual space $(C^1_0(\overline{O};\mathbb{R}^N))^*$.

---

[1] We say $u \in C^1_g(\overline{O};\mathbb{R}^N)$ if $u - g \in C^1_0(\overline{O};\mathbb{R}^N)$, where $C^1_0(\overline{O};\mathbb{R}^N) := \{ \in C^1(\mathbb{R}^n;\mathbb{R}^N) : = 0 \text{ on } \partial O\}$.

[2] A Radon measure is a Borel measure that is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets. See [42] for precise definition.

Finally, all statement are equivalent if K = Argmax $H(\ ;u;Du) : \overline{O}$ in (III) (this happens for instance when the argmax is a singleton set).

The result above provides an interesting characterisation of Aronsson's concept of Absolute minimisers in terms of divergence PDE systems with measures as parameters. The exact distributional meaning of (1.3.6) is

$$\int_{\overline{O}} H_P(\ ;u;Du) : D \quad + H\ (\ ;u;Du) \qquad d\ = 0$$

for all $2\ C_0^1(\overline{O};R$

Chapter 2

A Pointwise Characterisation of the PDE System of Vectorial Calculus of Variations in L^1

2.1 Introduction

Calculus of Variations

prescribed boundary condition on $\partial\Omega$ to avoid trivial minimisers. This means that any putative minimiser $u \in \mathscr{C}$, if it exists, should satisfy

$$E(u;\ ) \qquad E(v;\ )\ ; \quad \text{for all } v \in \mathscr{C} \text{ with } u = v \text{ on } \partial\Omega :$$

If such a minimiser exists, then the real function $t \mapsto E(tv + (1 \quad t)u)$ has a minimum at $t = 0$ and should satisfy

$$\frac{d}{}$$

In index form, F_1 reads

$$F_1(x;\ ;P;X)\ :=\ \sum_i H_{P_i}(x;\ ;P)\ \Big(\sum_{;j} H_{P_j}(x;\ ;P)X_{ij} + \sum H\ (x;\ ;P)P_i\Big)$$
$$+\ H_{x_i}(x;\ ;P)\ +\ H(x;\ ;P)\ \sum [\![H_P(x;\ ;P)]\!]^?$$

$$\sum_{i;j} H_{P_iP_j}(x;\ ;P)X_{ij} + \sum_i H_{P_i}\ (x;\ ;P)P_i\Big)$$
$$+\ \sum_i H_{P_ix_i}(x;\ ;P)\ \ H\ (x;\ ;P)\ ;$$

where $= 1;:::;N$. Note that, although H is C^2, the coe cient $[\![H_P(\ ;u;Du)]\!]^?$ is discontinuous at points where the rank of $H_P(\ ;u;Du)$ changes. Further, because of the perpendicularity of H_P and $[\![H_P]\!]^?$ (that is $[\![H_P]\!]^?H_P = 0$), the system can be decoupled into the two independent systems

$$\begin{cases} H_P(\ ;u;Du)\,D\ H(\ ;u;Du)\ = 0; \\ H(\ ;u;Du)[\![H_P(\ ;u;Du)]\!]^?\ \mathrm{Div}\ H_P(\ ;u;Du)\ \ H\ (\ ;u;Du)\ = 0: \end{cases}$$

When $H(x;\ ;P) = jPj^2$ (the Euclidean norm on R^{Nn} squared), the system (2.1.2)-(2.1.4) simpli es to the so-called $_1$ -Laplacian:

$$_1u\ :=\ \ Du\ \ Du + jDuj^2[\![Du]\!]^?\ \ I\ \ :D^2u\ =\ 0: \qquad (2.1.5)$$

solutions of (2.1.5) to local minimisers of $u \mapsto \|Du\|_{L^1(\Omega)}$ in terms of certain classes of local affine variations. This result offered new insights to the difficult problem of establishing connections of (2.1.1) to (2.1.2)-(2.1.4).

In this paper we generalise the results of [56], characterising general D-solutions to (2.1.2)-(2.1.4) in terms of local affine variations of (2.1.1). Our main result is Theorem 2.3.2 that follows and asserts that D-solutions to (2.1.2)-(2.1.4) in $C^1(\Omega;\mathbb{R}^N)$ can be characterised variationally in terms of (2.1.1). The a priori C^1 regularity assumed for our putative solutions is slightly higher than the generic membership in the space $W^{1,1}(\Omega;\mathbb{R}^N)$, but as a compensation we impose no convexity of any kind for the hamiltonian H for the derivation of the system.

In special case of classical solutions, our result reduces to the following corollary which shows the geometric nature of our characterisation[1]

2.1.1 Corollary [C^2 solutions of $F_1 = 0$]

Let $\Omega \subset \mathbb{R}^n$ be open, $u \in C^2(\Omega;\mathbb{R}^N)$ and $H \in C^2(\Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn})$. Then,

$$F_1(\cdot;u;Du;D^2u) = 0 \text{ in } \Omega \iff \Big(E_1(u;O) \leq E_1(u+A;O);\ \forall\, O \subset\subset \Omega;\ \forall\, A \in A_O^{k;1}(u) \cup A_O^{?;1}(u).$$

Here $A_O^{k;1}(u), A_O^{?;1}(u)$ are sets of affine maps given by

$$A_O^{k;1}(u) = \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} D^2A \equiv 0;\ A(x) = 0 \text{ and exist } \xi \in \mathbb{R}^N \text{ and} \\ x \in O(u) \text{ s.t. the image of } A \text{ is parallel} \\ \text{to the tangent map of } H(\cdot;u;Du) \text{ at } x \end{array} \right\};$$

$$A_O^{?;1}(u) = \left\{ A : \mathbb{R}^n \to \mathbb{R}^N \;\middle|\; \begin{array}{l} D^2A \equiv 0 \text{ and there exists } x \in O(u) \text{ s.t. the} \\ \text{image of } A \text{ is normal to } H_P(\cdot;u;Du) \text{ at } x \\ \text{and } A^{\triangleright} H_P(\cdot;u;Du) \text{ is divergenceless at } x \end{array} \right\};$$

and

$$O(u) := \operatorname{Argmax}\{ H(\cdot;u;Du) : \overline{O} \}.$$

This paper is organised as follows. In Section 2.2 that follows we record all the basic facts needed regarding the concept of our D-solutions, namely our notion of generalised solution required to make rigorous sense of (2.1.2)-(2.1.4). We also include a quick introduction to the analytic setup of so-called Young measures, on which D-solutions are based. We also give two simple auxiliary results which are utilised in the proof of our variational characterisation. Finally, in Section 2.3 we state and prove our main result.

[1] We caution the reader that the statement of Corollary 2.1.1 sacrifices precision for the sake of clarity. The fully precise statement is that given in the main result, Theorem 2.3.2.

2.2 Young measures, D-solutions and auxiliary results

2.2.1 Young Measures

Let $\Omega \subset \mathbb{R}^n$ be open and K a compact subset of some Euclidean space $\mathbb{R}^{N\times n^2}$. The set of Young measures $\mathcal{Y}(\Omega; K)$ forms a subset of the unit sphere of a certain L^1_w space of measure-valued maps and this provides its useful properties, including sequential weak* compactness. More precisely, $\mathcal{Y}(\Omega; K)$ is defined as

$$\mathcal{Y}(\Omega; K) := \Big\{ \nu : \Omega \to \mathcal{P}(K) \;\big|\; [\nu(\cdot)](U) \in L^1(\Omega) \text{ for any open } U \subset K \Big\},$$

where $\mathcal{P}(K)$ is the set of Borel probability measures on K. To see how it arises, consider the separable space $L^1(\Omega; C(K))$ of Bochner integrable maps. This space contains Carathéodory functions $f : \Omega \times K \to \mathbb{R}$ (namely functions for which $f(\cdot, X)$ is measurable for all $X \in K$ and $f(x, \cdot)$ is continuous for a.e. $x \in \Omega$) which satisfy

$$\|f\|_{L^1(\Omega;C(K))} := \int_\Omega \|f(x,\cdot)\|_{C^0(K)}\, dx < \infty.$$

We refer e.g. to [35, 43, 77] and to [56, 57] for background material on these spaces. The dual space of this space is $L^1_w(\Omega; M(K))$, namely

$$L^1(\Omega; C(K))^* = L^1_w(\Omega; M(K)).$$

This dual Banach space consists of Radon measure-valued maps $x \mapsto \nu_x \in M(K)$ which are weakly* measurable, in the sense that for any open set $U \subset K$, the function $x \mapsto [\nu_x](U)$ is in $L^1(\Omega)$. The norm of the space is given by

$$\|\nu\|_{L^1_w(\Omega;M(K))} := \operatorname*{ess\,sup}_{x\in\Omega} \|\nu_x\|,$$

where "$\|\cdot\|$" denotes the total variation. It thus follows that

$$\mathcal{Y}(\Omega; K) = \Big\{ \nu \in L^1_w(\Omega; M(K)) : \nu_x \in \mathcal{P}(K), \text{ for a.e. } x \in \Omega \Big\}.$$

2.2.2 Remark [Properties of Young Measures]

We note the following facts about the set $\mathcal{Y}(\Omega; K)$ (proofs can be found e.g. in [41]):

i) It is convex and sequentially compact in the weak* topology induced from L^1_w.

ii) The set of measurable maps $V : \Omega \to K$ can be identified with a subset of it via the embedding $V \mapsto \nu_V$, $\nu_V(x) := \delta_{V(x)}$.

iii) Let $V^i; V^1 : R^n \quad ! \quad K$ be measurable maps $i, 2 \quad N$. Then, up the passage to subsequences, the following equivalence holds true $i \quad s1 \quad : V^i \quad ! \quad V^1$ a.e. on if and only if $_{V^i} \quad * \quad _{V^1}$ in $\mathscr{Y} \quad ; K$.

2.2.3 D-solutions

We now give some rudimentary facts about generalised solutions which are required for the main result in this paper. For simplicity we will restrict the discussion to n = 1 for maps $u : R \quad ! \quad R^N$ with an interval. The notion of D-solutions is based on the probabilistic interpretation of limits of di erence quotients by using Young measures. Unlike standard PDE approaches which

The above means for any $2\ L^1 \quad ; C(K)$, we have

$$\int_K \int (x;X)\, d[D^{1;h}{}_k Du](X)\, dx \ ! \int_K \int (x;X)\, d[D^2u](X)\, dx; \quad \text{as } k\ !\ 1 \quad :$$

Note that the set of Young measures is sequentially weakly* compact hence every map as above possesses di use 2nd derivatives

2.2.3.2 De nition [D-solutions to 2nd order systems]

Let R^n be an open set and $F : R^N \quad R^{Nn} \quad R_s^{Nn^2} \ ! \ R^N$ a Borel measurable map which is continuous with respect to the last argument. Consider the PDE system

$$F \quad ; u; Du; D^2u \quad = 0 \quad \text{on} \quad : \qquad (2.2.4)$$

We say that the locally Lipschitz continuous map $u : R^n \quad ! \quad R^N$ is a D-solution of (2.2.4) when for any di use hessian D^2u of u, we have

$$\sup_{\mathbf{X}_x 2 \operatorname{supp}(D^2u(x))} F \quad x; u(x); Du(x); X_x \quad = 0; \quad \text{a.e. } x\ 2 \quad : \qquad (2.2.5)$$

Here \supp " symbolises the reduced support of a probability measure excluding in nity, namely supp (#) := supp(#) n f1g when # 2 P $\overline{R}_s^{Nn^2}$.

a) If we have $E_1(u; O) \leq E_1(u + tA; O)$ for all $t > 0$, it follows that

$$\max_{z \in \overline{O}} \left\{ H_P(z; u(z); Du(z)) \cdot DA(z) + H(z; u(z); Du(z)) A(z) \right\} \geq 0.$$

In the above "$\cdot$" and "$|\ |$" denote the inner products in R^{Nn} and R^N respectively.

b) Let $x \in O$ and $0 < \varepsilon < \operatorname{dist}(x; \partial O)$. The set

$$O_\varepsilon(x) := \left\{ y \in O : H(y; u(y); Du(y)) \leq H(x; u(x); Du(x)) \right\} \setminus B_\varepsilon(x)$$

(where "$(\ )$" denotes the interior) is open and compactly contained in

Let also $O(u)$ be as in Lemma 2.2.4.1. Then r is convex, $r(0) = 0$ and also it satisfies

$$\underline{D}r(0^+) \quad \max_{O(u)} \left\{ H_P(\ ; u; Du) : DA \ + \ H \ (\ ; u; Du) \ \ A \ ; \right.$$

where $\underline{D}r(a^+) := \liminf_{\ ! \ 0^+} \frac{r(a+\ ) \ \ r(a)}{\ }$ is the lower right Dini derivative of r at a.

2.2.4.4 Proof of Lemma 2.2.4.3

The result is deducible from Danskin's theorem (see [34]) but we prove it directly since the 1-sided version above is not given explicitly in the paper. By setting

$$R(\ ; y\ ) := H \ \ y\ ; u(y) + \ \ A\ (y)\ ; Du(y) + \ \ DA\ (y)$$

we have $r(\ \ ) = \max_{y2\overline{O}} R(\ ; y\ ) \quad \max_{y2^{-}}$

and the desired inequality has been established. Finally by convexity of H we have for any $x; y > 0$ and any $t \in [0;1]$

$$\begin{aligned} r(tx + (1-t)y) &:= E_1(t(u + xA) + (1-t)(u + yA); O) - E_1(u; O) \\ &\le tE_1(u + xA; O) + (1-t)E_1(u + yA; O) - E_1(u; O) \\ &= tr(x) + (1-t)r(y) \end{aligned}$$

□

Let us record the next simple inequality which follows from the de nitions of lower right Dini derivative, in the case that $H(x; \cdot; \cdot)$ is jointly convex for any $x \in$. This is

$$r(\ ) \quad r(0) \quad \underline{D}r(0^+)\ ; \tag{2.2.6}$$

for all 0.

2.3 The main result of the Chapter 2

Now we proceed to the main result of the paper, the variational characterisation of D-solutions to the PDE system (2.1.2) in terms of appropriate variations of the energy functional (2.1.1). We recall that the Borel mapping F_1 : R^N R^{Nn} $R^{Nn^2}_s$! R^N is given by (2.1.3)-(2.1.4) and R^n is a xed open set.

2.3.1 Notational simpli cations and perpendicularity considerations.

We begin by rewriting $F_1(\ ; u; Du; D^2u) = 0$ in a more malleable fashion (see (2.1.3)). We de ne the maps

$$F^{?}_1(x; \ ; P; X) := H_{PP}(x; \ ; P) : X + H_P(x; \ ; P) : P + H_{Px}(x; \ ; P) : I; \tag{2.3.1}$$

$$F^{k}_1(x; \ ; P; X) := H_P(x; \ ; P) : X + H(x; \ ; P)^{>}P + H_x(x; \ ; P) \tag{2.3.2}$$

and these are abbreviations of

$$\begin{aligned} F^{?}_1(x; \ ; P; X) &= \sum_{;i;j} H_{P_iP_j}(x; \ ; P)X_{ij} + \sum_{;i} H_{P_i}(x; \ ; P)P_i \\ &\quad + \sum H_{P_ix_i}(x; \ ; P)\ ; \\ F^{k}_1(x; \ ; P; X)_i &= \sum^{i}_{;j} H_{P_j}(x; \ ; P)X_{ij} + \sum H(x; \ ; P)P_i + H_{x_i}(x; \ ; P): \end{aligned}$$

Note that $F_1^?(x;\ ;P;X) \in R^N$, whilst $F_1^k(x;\ ;P;X) \in R^n$. By utilising (2.3.1)-(2.3.2), we can now express (2.1.3) as

$$F_1(x;\ ;P;X) := H_P(x;\ ;P)F_1^k(x;\ ;P;X) + H(x;\ ;P)$$
$$[\![H_P(x;\ ;P)]\!]^? \quad F_1^?(x;\ ;P;X) \quad H(x;\ ;P):$$

Further, recall that in view of (2.1.4), $[\![H_P(x;\ ;P)]\!]^?$ is the projection on the orthogonal complement of $R(H_P(x;\ ;P))$. Hence, by the orthogonality of $[\![H_P(x;\ ;P)]\!]^?$ $F_1^?(x;\ ;P;X)$ $H(x;\ ;P)$ and $H_P(x;\ ;P)F_1^k(x;\ ;P;X)$, we have

$$F_1(x;\ ;P;X) = 0; \quad \text{for some } (x;\ ;P;X) \in \quad R^N \quad R^{Nn} \quad R_s^{Nn^2};$$

if and only if

$$\begin{cases} H_P(x;\ ;P)F_1^k(x;\ ;P;X) = 0; \\ H(x;\ ;P)[\![H_P(x;\ ;P)]\!]^? \quad F_1^?(x;\ ;P;X) \quad H(x;\ ;P) = 0: \end{cases}$$

Finally, for the sake of clarity we state and prove our characterisation below only in the case of C^1 solutions, but due to its pointwise nature, the result holds true for piecewise C^1 solutions with obvious adaptations which we refrain from providing. We will assume that the Hamiltonian H satisfies

$$H_P(x;\ ;\ ) = 0 \qquad H(x;\ ;\ ) = 0 \ ;\ (x;\ ) \in \quad R^N: \qquad (2.3.3)$$

We will also suppose that the next set has vanishing measure

$$\{x \in \ : B_{r_x}(x) \quad h > h(x) \text{ is dense in } B_{r_x}(x)\} = 0; \qquad (2.3.4)$$

where $r_x \quad \text{dist}(x;@)$ and $h \quad H(\ ;u;Du)$. This assumption is natural, in the sense that it is satisfied by all know examples of explicit solutions (see [55, 63–66]). It is trivially satisfied if h has no strict local minima in the domain.

Lets examine three examples for conditions (2.3.3) and (2.3.4). For all examples $H(x;\ ;P) = |P|^2$. Clearly (2.3.3) is satisfied. Remains to show that actually condition (2.3.4) holds for our three examples.

Example 2.3.1. $u(x;y) = |x|^{\frac{4}{3}} \quad |y|^{\frac{4}{3}}$ is well-known explicit solution and let $= [\ 1;1]^2$. The function $h \quad |Du|^2 = \frac{16}{9}|x|^{\frac{2}{3}} + |y|^{\frac{2}{3}}$ has only one point of local minimum at origin which means set $\{h > h(0)\} \setminus B_{r_0}(0)$ is the dense in the ball $B_{r_0}(0)$. Let a point (x,y) be different from origin then it easy to check that set $\{h > h(x;y)\} \setminus B_{r_{x;y}}(x;y)$ is not a dense in the $B_{r_{x;y}}(x;y)$ and as the result we have (2.3.4).

Example 2.3.2. Let $= [0 :1;1]^2$ and $u(x;y) = \sqrt{x^2 + y^2}$ is the conic solution. The function $h \quad |Du|^2 \quad 1$ for any point of . So clearly (2.3.4) is satisfied.

Example 2.3.3. Using notation $e^{it} = (\cos t; \sin t)$ we have vectorial solution $u(x;y) = e^{ix} \quad e^{iy}$ on $= [\ 1;1]^2$ which is Eikonal, namely $|Du|^2 = |D_x u|^2 + |D_y u|^2 \quad 2$. So clearly (2.3.4) is satisfied.

Our main result is as follows:

2.3.2 Theorem [Variational characterisation of the PDE system arising in L^1]

Let R^n be open, $u \in C^1(; R^N)$ and $H \in C^2(R^n R^{Nn})$ a function satisfying (2.3.3) and suppose that (2.3.4) holds. Then:

(A) We have

$$F_1(; u; Du; D^2$$

in the D-sense, if and only if

$$E_1(u;\ O) \quad E_1(u + A;\ O); \quad 8\ O\ b\ ;\ 8\ A\ 2\ A_O^{k;1}(u)$$

and also

$$H(\ ;u;Du)\ [\![H_P(\ ;u;Du)]\!]^{?}\ F_1^{?}(\ ;u;Du;D^2u) \quad H\ (\ ;u;Du) \quad = 0\ \text{ in } \quad ;$$

in the D-sense, if and only if

$$E_1(u;\ O) \quad E_1(u + A;\ O); \quad 8\ O\ b\ ;\ 8\ A\ 2\ A_O^{?;1}(u):$$

We note that in the special case of C^2 solutions, Corollary 2.1.1 describes the way that classical solutions $u : R^n \quad ! \quad R^N$ to (2.1.2)-(2.1.4) are characterised.

2.3.3 Remark [About pointwise properties of C^1 D-solutions]

Let $u : R^n \quad ! \quad R^N$ be a D-solution to (2.1.2)-(2.1.4) in $C^1(\ ;R^N)$. By De nition 2.2.3.2, this means that for any $D^2u\ 2\ \mathscr{Y}\ ;\overline{R}_s^{Nn^2}$,

$$F_1\ x;u(x);Du(x);X_x\ = 0; \quad \text{a.e. } x\ 2\ ; \ 8\ X_x\ 2\ \text{supp}\ D^2u(x)\ :$$

By De nition 2.2.3.1, every di use hessian of a putative solution is de ned a.e. on as a weakly* measurable probability valued map R^n

and an $X_x \in \operatorname{supp}\ D^2u(x)$. In view of (2.3.1), if $H_p\ x;u(x);Du(x) = 0$, then, by our assumption on the level sets of H, we have $H\ x;u(x);Du(x) = 0$ as well and as a consequence we readily obtain

$$\begin{aligned} H\ x;u(x);Du(x)\ [\![H_p\ x;u(x);Du(x)\ ]\!]^{\perp} \\ F_1^{\perp}\ x;u(x);Du(x);X_x \quad H\ x;u(x);Du(x) = 0 \end{aligned} \tag{2.3.5}$$

is clearly satisfied at x. If $H_p\ x;u(x);Du(x) \neq 0$, then we select any direction normal to the range of $H_p\ x;u(x);Du(x) \in R^{N\,n}$, that is

$$n_x \in R\ H_p\ x;u(x);Du(x)^{\perp} \quad R^N$$

which means $n_x^{>} H_p\ x;u(x);Du(x) = 0$. Of course it may happen that the linear map $H_p\ x;u(x);Du(x) : R^n$

and r is convex, by inequality (2.2.6) we have $r(t) \geq 0$ for all $t \geq 0$. Therefore,

$$E_1(u;O) \leq E_1(u+A;O), \quad \forall\, O \subset\subset \Omega;\ \forall\, A \in \mathcal{A}_O^{k,1}(u).$$

The case of $A \in \mathcal{A}_O^{?,1}$ is completely analogous. Fix $D^2u \in \mathscr{Y}(\Omega;\overline{\mathbb{R}}_s^{Nn^2})$, $O \subset\subset \Omega$, $x \in O(u)$, $X_x \in \operatorname{supp}(D^2u(x))$ and an A with $A(x) \in R\big(H_P(x,u(x),Du(x))\big)$ and $DA \in \mathscr{L}(x,A(x),X_x)$. By applying Lemma 2.2.4.3 again, we have

$$\begin{aligned}\underline{D}r(0^+) &\leq \max_{y\in O(u)}\Big\{H_P(y,u(y),Du(y)):DA(y)+H_x(y,u(y),Du(y))A(y)\Big\}\\ &\leq H_P(x,u(x),Du(x)):DA(x)+H_x(x,u(x),Du(x))A(x).\end{aligned}$$

If $H_P(x,u(x),Du(x)) \neq 0$, then by the definition of $\mathscr{L}(x,A(x),X_x)$ we have

$$\begin{aligned}\underline{D}r(0^+) &\leq H_P(x,u(x),Du(x)):DA(x)+H_x(x,u(x),Du(x))A(x)\\ &= A(x)\,F_1^{?}(x,u(x),Du(x),X_x)\,H_x(x,u(x),Du(x))\\ &= A(x)^{\perp}[H_P(x,u(x),Du(x))]^{?}\,F_1^{?}(x,u(x),Du(x),X_x)\\ &\quad H_x(x,u(x),Du(x))\end{aligned}$$

and hence $\underline{D}r(0^+) \leq 0$ because u is a D-solution on Ω. If $H_P(x,u(x),Du(x)) = 0$, then again $\underline{D}r(0^+) \leq 0$ because $A(x) = 0$. In either cases, by inequality (2.2.6) we obtain $r(t) \leq 0$ for all $t \geq 0$ and hence

$$E_1(u;O) \leq E_1(u+A;O), \quad \forall\, O \subset\subset \Omega;\ \forall\, A \in \mathcal{A}_O^{?,1}(u).$$

The theorem has been established. □

2.3.5 Proof of Corollary 2.1.1

If $u \in C^2(\Omega;\mathbb{R}^N)$, then by Lemma 2.2.2 any diffuse hessian of u satisfies $D^2u(x) = \{D^2u(x)\}$ for a.e. $x \in \Omega$. By Remark 2.3.3, we may assume this happens for all $x \in \Omega$. Therefore, the reduced support of $D^2u(x)$ is the singleton set $\{D^2u(x)\}$. Hence, for $A \in \mathcal{A}_O^{k,1}(u)$, we have that any possible affine map A satisfies $DA \in R\big(H_x(x,u(x),Du(x))\big)$ and $A(x) = 0$. In the case of $A \in \mathcal{A}_O^{?,1}(u)$, we have that any possible affine map A satisfies

$$A(x)^{\perp}H_P(x,u(x),Du(x)) = 0\ ; \quad DA \in \mathscr{L}(x,A(x),D^2u(x))\ ;$$

which gives

$$\begin{aligned}DA(x):H_P(x,u(x),Du(x)) &= A(x)\,H_{PP}(x,u(x),Du(x)):D^2u(x)+\\ &\quad + H_P(x,u(x),Du(x)):Du(x) + H_{Px}(x,u(x),Du(x)):I\\ &= A(x)\,\operatorname{Div}H_P(x,u,Du)(x).\end{aligned}$$

As a consequence, the divergence $\mathrm{Div}^{>} H_P\ ; u; Du\ (x)$ vanishes because

$$DA(x) : H_P\ x; u(x); Du(x) \quad A11.9552\ Tf\ 6.652\ 0\ Td\ [())-278(:)-277(H)]TJ/F46\ 7.9701$$

Chapter 3

Rigidity and flatness of the image of certain classes of mappings having tangential Laplacian

3.1 Introduction

Suppose that n, N are integers and Ω an open subset of $\mathbb{R}^n$. In this paper we study geometric aspects of the image $u(\Omega) \subset \mathbb{R}^N$ of certain classes of C^2 vectorial solutions $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ to the following nonlinear degenerate elliptic PDE system:

$$[\![Du]\!]^{\perp} \, \Delta u = 0 \quad \text{in} \quad \Omega. \tag{3.1.1}$$

Here, for the map u with components $(u_1, \dots, u_N)^{\top}$ the notation Du symbolises the gradient matrix

$$Du(x) = \big(D_i u_\alpha(x)\big)_{\substack{\alpha=1,\dots,N \\ i=1,\dots,n}} \in \mathbb{R}^{N \times n}, \quad D_i = \partial/\partial x_i,$$

Δu stands for the Laplacian

$$\Delta u(x) = \sum_{i=1}^{n} D^2_{ii} u(x) \in \mathbb{R}^N$$

and for any $X \in \mathbb{R}^{N \times n}$, $[\![X]\!]^{\perp}$ denotes the orthogonal projection on the orthogonal complement of the range of linear map $X : \mathbb{R}^n \to \mathbb{R}^N$:

$$[\![X]\!]^{\perp} := \mathrm{Proj}_{R(X)^{\perp}}.$$

Our general notation will be either self-explanatory, or otherwise standard as e.g. in [32, 38]. Note that, since the rank is a discontinuous function, the map $[\![\cdot]\!]^{\perp}$ is discontinuous on $\mathbb{R}^{N \times n}$; therefore, the PDE system (3.1.1) has discontinuous coefficients. The geometric meaning of (3.1.1) is that the Laplacian vector field Δu is tangential to the image $u(\Omega)$ and hence (3.1.1) is equivalent to the next

statement: there exists a vector eld

$$A : R^n \quad ! \quad R^n$$

The scalar case of $N = 1$ in (3.1.4)-(3.1.5) was pioneered by G. Aronsson in the 1960s [4–8] who initiated the field of Calculus of Variations in L^∞, namely the study of supremal functionals and of their associated equations describing critical points. Since then, the field has developed tremendously and there is an extensive relevant literature (see e.g. [16–19, 21, 26, 46, 72, 73] and the lecture notes [15, 28, 54]). In particular, although vectorial supremal functionals began to be explored early enough, the ∞-Laplace system (3.1.4) which describes the necessary critical conditions in L^∞ in the vectorial case $N \geq 2$ first arose in the early 2010s in [49]. The area is now developing very rapidly due to both the mathematical significance as well as the importance for applications in several areas (see [2, 14, 31, 36, 63], [50, 52, 53, 56–58]).

In this paper we focus on the C^2 case and establish the geometric rigidity and flatness of the images of solutions $u : \mathbb{R}^n \to \mathbb{R}^N$ to the nonlinear system (3.1.1), under the assumption that either Du has rank at most 1, or that $n = 2$ and u has an additively separated form, see (3.1.6). As a consequence, we obtain corresponding flatness results for the images of solutions to (3.1.2) and (3.1.4). Both aforementioned classes of solutions furnish particular examples which provide substantial intuition for the behaviour of general extremal maps in Calculus of Variations in L^∞, see e.g. [9, 10, 28, 50, 53, 54, 63] where solutions of this form have been studied. Obtaining further information for the still largely mysterious behaviour of ∞-Harmonic maps is perhaps the greatest driving force to isolate and study the particular nonlinear system (3.1.1). For example, it is not yet know to what extend the possible discontinuities of the coefficients relates to the failure of absolute minimality.

It is also worth clarifying that, although as it is well-known the Dirichlet problem over a bounded domain may not in general be solvable for the ∞-Laplacian not even in the scalar-valued case, if one does not prescribe boundary values (and we do not in this paper) it can be demonstrated that infinitely many non-trivial classical solutions do exist, in particular of the form arising in this paper (see for instance the explicit constructions of C^2 solutions in [50]). Therefore, the results herein are non-void and numerous solutions as those exhibited herein do exist.

Let us note that the rank-one case includes the scalar and the one-dimensional case (i.e. when $\min\{n, N\} = 1$), although in the case of $N = 1$ (in which the single ∞-Laplacian reduces to $Du \otimes Du : D^2u = 0$) (3.1.1) has no bearing since it vanishes identically at any non-critical point.

The effect of (3.1.1) to the flatness of the image can be seen through the L^∞ variational principle introduced in [52], wherein it was shown that solutions to (3.1.1) of constant rank can be characterised as those having minimal area with respect to (3.1.3)-(3.1.5). More precisely, therein the following result was proved:

3.1.1 Theorem [cf. [52, Theorem 2.7, Lemma 2.2]]

Given $N \quad n \quad 1$, let $u : R^n \quad ! \quad R^N$ be a C^2 immersion de ned on the open set (more generally u can be a map with constant rank of its gradient on). Then, the following statements are equivalent:

1. The map u solves the PDE system (3.1.1) on .
2. For all $p \ 2 \ [2; 1 \ ]$, for all compactly supported domains $O \ b$ and all C^1 vector elds $: O \ ! \quad R^N$ which are normal to the image $u(O) \quad R^N$ (without requiring to vanish on $@O$), namely those for which $= [\![D u]\!]^?$ in O, we have
$$kDuk_{L^p(O)} \quad k \ Du + D \ k_{L^p(O)}:$$
3. The same statement as in item (2) holds, but only for some $p \ 2 \ [2; 1 \ ]$.

If in addition $p < 1$ in (2)-(3), then we may further restrict the class of normal vector elds to those satisfying $j_{@O} = 0$ (see Figure 1).

In the paper [52], it was also shown that in the conformal class, (3.1.1) expresses the vanishing of the mean curvature vector of $u( )$.

The e ect of (3.1.1) to the atness of the image can be easily seen in the case of $n = 1 \quad N$ as follows: since
$$[\![u^0]\!]^? \ u^{00} = 0 \quad \text{in} \quad R$$
and in one dimension we have
$$[\![u^0]\!]^? = \begin{cases} I \ \dfrac{u^0 \quad u^0}{ju^0j^2}; & \text{on } f u^0 \ 6= 0 g; \\ I; & \text{on } f u^0 = 0 g; \end{cases}$$
we therefore infer that $u^{00} = f u^0$ on the open set $f u^0 \ 6= 0 g \quad R$ for some function f, readily yielding after an integration that $u( )$ is necessarily contained in a piecewise polygonal line of R^N. As a generalisation of this fact, our rst main result herein is the following:

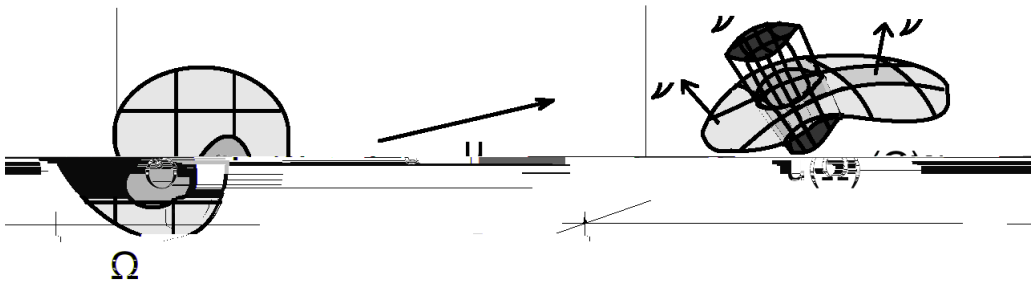


Figure 1. Illustration of the variational principle characterising (3.1.1).

3.1.2 Theorem [Rigidity and flatness of rank-one maps with tangential Laplacian]

Let $\Omega \subset \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega; \mathbb{R}^N)$ be a solution to the nonlinear system (3.1.1) in Ω, satisfying that the rank of its gradient matrix is at most one:

$$\operatorname{rk}(Du) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in $\mathbb{R}^N$, consisting of an at most countable union of affine straight line segments (possibly with self-intersections).

Let us note that the rank-one assumption for Du is equivalent to the existence of two vector fields $a : \mathbb{R}^n \to \mathbb{R}^N$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ such that $Du = a \otimes b$ in Ω.

Example 3.1.3

trivial a ne ones.

As a consequence of Theorem 3.1.2, we obtain the next result regarding the rigidity of p-Harmonic maps for $p \in [2, 1)$ which complements one of the results in the paper [53] wherein the case $p = 1$ was considered.

3.1.4 Corollary [Rigidity of p-Harmonic maps, cf. [53]]

Let $\subset R^n$ be an open set and $n; N \geq 1$. Let $u \in C^2(; R^N)$ be a p-Harmonic map in for some $p \in [2, 1)$, that is u solves (3.1.2). Suppose that the rank of its gradient matrix is at most one:

$$\mathrm{rk}(Du) \leq 1 \quad \text{in} \quad :$$

Then, the same result as in Theorem 3.1.2 is true.

In addition, there exists a partition of to at most countably many Borel sets, where each set of the partition is a non-empty open set with a (perhaps empty) boundary portion, such that, on each of these u can be represented as

$$u = \quad f:$$

Here, f is a scalar C^2 p-Harmonic function (for the respective $p \in [2, 1)$), de ned on an open neighbourhood of the Borel set, whilst $: R \to R^N$ is a Lipschitz curve which is twice di erentiable and with unit speed on the image of f.

Now we move on to discuss our second main result which concerns the rigidity of solutions $u : R^2 \to R^N$ to (3.1.1) for $N \geq 2$, having the additively separated form

$$u(x; y) = f(x) + f(y) \tag{3.1.6}$$

for some curve $f : R \to R^N$. Solutions of this form are very important in relation to the 1 -Laplacian. If $N = 1$, all 1 -Harmonic functions of this form after a normalisation reduce to the so-called Aronsson solution on R^2

$$u(x; y) = |x|^{4=3} - |y|^{4=3}$$

which is the standard explicit example of a non-C^2 1 -Harmonic function with conjectured optimal regularity. In the vectorial case, the family of separated solutions is quite large. For $N = 2$, a large class of such vectorial solutions was constructed in [50] and is given by

$$u(x; y) = \int_x^y \cos(K(t)); \sin(K(t)) \, dt$$

with K a function in $C^1(R)$ satisfying certain general conditions. The simplest non-trivial example of an 1 -Harmonic map with this form (de ned on the strip $\{|x - y| < = 4\} \subset R^2$) is given by the choice $K(t) = t$. Our second main result

asserts that solutions of separated form to (3.1.1) have images which are piecewise a ne, contained in a union of intersecting planes of $\mathbb{R}^N$. More precisely, we have:

3.1.5 Theorem [Rigidity and atness of maps with tangential Laplacian in separated form]

Let $\Omega \subseteq \mathbb{R}^2$ be an open set and let also $n, N \geq 2$. Let $u : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^N$ be a classical solution to the nonlinear system (3.1.1) in Ω, having the separated form $u(x, y) = f(x) \otimes g(y)$, for some curves $f, g \in (W^{3,p} \setminus C^2)(\mathbb{R}; \mathbb{R}^N)$ and some $p > 1$.

Then, the image $u(\Omega)$ of the solution is contained in an at most countable union of a ne planes in $\mathbb{R}^N$.

In addition, the proof of Theorem 3.1.5 shows that every connected component of the set $\{\mathrm{rk}(Du) = 2\}$ is contained entirely in an a ne plane and every connected component of the set $\{\mathrm{rk}(Du) \leq 1\}$ is contained entirely in an a ne line.

Note that our result is trivial in the case that $N = n = 2$ since the codimension $N - n$ vanishes. Additionally, due to the regularity of the solutions, if a C^2 mapping has piecewise a ne image, then second derivatives must vanish when rst derivatives vanish at the \breaking points". Further, one might also restrict their attention to domains of rectangular shape, since any map with separated form can be automatically extended to the smallest rectangle containing the domain.

Also, herein we consider only the illustrative case of $n = 2 < N$ and do not discuss more general situations, since numerical evidence obtained in [63] suggests that Theorem 3.1.5 does not hold in general for solutions in non-separated form.

In this paper we try to keep the exposition as simple as possible and therefore we refrain from discussing generalised solutions to (3.1.1) and (3.1.4) (or (3.1.2)). We con ne ourselves to merely mentioning that in the scalar case ($N = 1$), ∞-Harmonic 3o domains

to these projections,

$$Du \quad Du : D^2u + \frac{|Du|^2}{p-2}[[Du]]^{\perp} \quad u = \quad \frac{|Du|^2}{p-2}[[Du]]^{\perp} \quad u:$$

The mutual perpendicularity of the vector fields of the left and right hand side leads via a renormalisation argument (see e.g. [49, 52, 53]) to the equivalence of the p-Laplacian with the pair of systems

$$Du \quad Du : D^2u + \frac{|Du|^2}{p-2}[[Du]]^{\perp} \quad u = 0 \; ; \quad |Du|^2[[Du]]^{\perp} \quad u = 0: \tag{3.1.9}$$

The 1 -Laplacian corresponds to the limiting case of (3.1.9) as $p \to 1$, which takes the form

$$Du \quad Du : D^2u = 0 \; ; \quad |Du|^2[[Du]]^{\perp} \quad u = 0: \tag{3.1.10}$$

Hence, the1 -Laplacian (3.1.4) actually consists of the two independent systems in (3.1.10) above. The system$|Du|^2[[Du]]^{\perp} \quad u = 0$ is, at least on $\{Du \neq 0\}$, equivalent to (3.1.1). Note that in the scalar case of$N = 1$ as well as in the case of submersion solutions (for$N \quad n$), the second system trivialises.

We conclude the introduction with a geometric interpretation of the nonlinear system (3.1.1), which can be expressed in a more geometric language as follows:[1] Suppose that$u( )$ is a C^2 manifold and let $A(u)$ denote its second fundamental form. Then

$$[[Du]]^{\perp} \quad u = \quad \operatorname{tr} A(u)(Du; Du):$$

The tangential part $[[Du]]^{k} \quad u$ of the Laplacian is commonly called the tension field in the theory of Harmonic maps and is symbolised by (u) (see e.g. [69]). Hence, we have the orthogonal decomposition

$$u = \quad (u) \quad \operatorname{tr} A(u)(Du; Du):$$

Therefore, in the case of higher regularity of the image of, we obtain that the nonlinear system

$$u = \quad (u) \quad \text{in} \quad ; \tag{3.1.11}$$

is a further geometric reformulation of our PDE system (3.1.1).

3.2 Proofs

In this section we prove the results of this paper. Before delving into that, we present a result of independent interest in which we represent explicitly the vector field A arising in the parametric system $u = Du A$, in the illustrative case of $n = 2$.

We will be using the symbolisations \cof",\det" and \rk" to denote the cofactor

[1] This fact has been brought to our attention by Roger Moser.

matrix, the determinant function and the rank of a matrix, respectively.

3.2.1 Lemma [Representation of A]

Let $u \in C^2(\ ; R^N)$ be given, $\ R^2$ open, $N \ 2$. The following are equivalent:

1. The map u is a solution to the PDE system (3.1.1).
2. There exists a vector field $A : R^2 \ ! \ R^N$ such that

$$u = Du\,A \quad \text{in} \quad :$$

In (2), as A one might choose

$$A := \begin{cases} \dfrac{\operatorname{cof}\,Du^{>}Du\ ^{>}}{\det\,Du^{>}Du}(Du)^{>}\ u; & \text{on } \{rk(Du) = 2\}; \\ (\ u)^{>}\dfrac{Du\,Du^{>}}{|Du\,Du^{>}|^2}Du; & \text{on } \{rk(Du) = 1\}; \\ 0; & \text{on } \{rk(Du) = 0\}. \end{cases}$$

A is uniquely determined on $\{rk(Du) = 2\}$ but not on $\{rk(Du) < 2\}$ and any other A has the form $A + V$, where $V(x)$ lies in the nullspace of $Du(x)$, $x \in \ $.

3.2.2 Proof of Lemma 3.2.1

The equivalence between (1)-(2) is immediate, therefore it suffices to show that A satisfies $\ u = Du\,A$ and is unique on $\{rk(Du) = 2\}$. Let A be as in (2). On $\{rk(Du) = 2\}$, the $2 \ 2$ matrix-valued map $Du^{>}Du$ is invertible and

$$Du^{>}Du\ ^{1} = \frac{\operatorname{cof}\,Du^{>}Du\ ^{>}}{\det\,Du^{>}Du}.$$

Since $Du^{>}\ u = Du^{>}Du\,A$, we obtain that $A = A$.

The claim being obvious for $\{rk(Du) = 0\} = \{Du = 0\}$, it suffices to consider only the set $\{rk(Du) = 1\}$ in order to conclude. Thereon we have 0 Td cu45dja(20sy 0 Td d

and hence $u = \ j\ j^2$ and also $^{>}Du = aj\ j^2$. On the other hand, since

$$Du\,Du^{>} = (\quad a)(a\quad) = \quad ; \qquad Du\,Du^{>} = j\ j^2$$

we infer that

$$A = \ a\ = \ \frac{u}{j\ ^2j}\ \ \frac{^{>}Du}{j\ ^2j} \ = \ \frac{u^{>}(\quad)Du}{j\quad j^2} \ = (\quad u)^{>}\frac{Du\,Du^{>}}{jDu\,Du^{>}j^2}Du;$$

as claimed. □

We now continue with the proof of the main results.

The main analytical tool needed in the proof of Theorem 3.1.2 is the next rigidity theorem for maps whose gradient has rank at most one. It was established in [53] and we recall it below for the convenience of the reader and only in the case needed in this paper.

3.2.3 Theorem [Rigidity of Rank-One maps, cf. [53]]

Suppose R^n is an open set and u is in $C^2(\ ;R^N)$. Then, the following are equivalent:

(i) The map u satis es that $rk(Du)\quad 1$ on . Equivalently, there exist vector elds $:\quad !\quad R^N$ and $a:\quad !\quad R^n$ with $a\ 2\ C^1(\ ;R^n)$ and $\ 2\ C^1(\quad n\,f a = 0g;R^N)$ such that

$$Du = \quad a; \quad on \quad :$$

(ii) There exists Borel subset $fB_i g_{i\,2\,N}$ of such that

$$= \ \bigcup_{i=1}^{1} B_i$$

and each B_i equals a non-empty connected open set with a (possibly empty) bound-
) ansati552 Tf -232.047 -35.157 Td [(and)-224(eac)28(h)]TJ/F20 11.9552 Tf 46.645 0 11.955

the same properties as above if is contractible (namely, homotopically equivalent to a point).

We may now prove our rst main result.

3.2.4 Proof of Theorem 3.1.2

Suppose that $u : R^n \quad ! \quad R^N$ is a solution to the nonlinear system (3.1.1) in $C^2(\ ; R^N)$ which in addition satis es that $\mathrm{rk}(D\,u) \quad 1$ in . Since $f\,Du = 0\,g$ is closed, necessarily its complement in which is $f\,\mathrm{rk}(D\,u) = 1\,g$ is open.

By invoking Theorem 3.2.3, we have that there exists a partition of the open subset $f\,\mathrm{rk}(D\,u) = 1\,g$ to countably many Borel sets $(B_i)_1^1$ with respective functions $(f_i)_1^1$ and curves $(\ _i)_1^1$ as in the statement such that (3.2.1)-(3.2.2) hold true and in addition

$$Df_i \ 6 \ 0 \quad \text{on } B_i;\ i\ 2\ N:$$

Consequently, on each B_i we have

$$[\![Du]\!]^? = [\![(\ _i^0\ f_i) \quad Df_i]\!]^? = I \quad \frac{(\ _i^0\ f_i) \quad (\ _i^0\ f_i)}{j\ _i^0\ f_i j^2};$$

$$u = (\ _i^{00}\ f_i) j Df_i j^2 + (\ _i^0\ f_i) \quad f_i:$$

Hence, (3.1.1) becomes

) (]TJ/F46 7.9701 Tf -0.754 -7.294 Td; i.

$$I \quad (\ _i^0\ f_i) \quad (\ _i^0\ f_i) u \quad {}_i^0\ f_i j_2 \quad {}_{i2} \quad ;$$

Suppose u is as in the statement of the corollary. By Theorem 3.2.3, there exists, a partition of $\quad$ to Borel sets $\{B_i\}_{i \in \mathbb{N}}$

respectively. Let f_α denote the α-component of f, $\alpha = 1, \ldots, N$. By subtracting (3.2.5) from (3.2.6) we get for $t \neq 0$ that

$$2\frac{f'(z+t) - f'(z)}{t} = a(z+t;z) + b(z;z+t)\,\frac{f'(z+t) - f'(z)}{t} + f'(z)\,\frac{a(z+t;z) - a(z;z+t)}{t} + \frac{b(z;z+t) - b(z+t;z)}{t} \tag{3.2.7}$$

for $\alpha = 1, \ldots, N$. On the set $\{f' = 0\}$, equation (3.2.7) becomes

$$2f'''(z) = a(z;z) + b(z;z)\,f''(z) \tag{3.2.8}$$

as $t \to 0$. Note also that $\{f' = 0\}$ is closed and its complement $\{f' \neq 0\}$ is open. Now let us set

$$C(z;t) := \frac{a(z+t;z) - a(z;z+t)}{t} + \frac{b(z;z+t) - b(z+t;z)}{t}.$$

On $\{f' \neq 0\}$, (3.2.7) yields that

$$C(z;t) = \frac{1}{f'(z)}\left[2\frac{f''(z+t) - f''(z)}{t} - a(z+t;z) + b(z;z+t)\,\frac{f'(z+t) - f'(z)}{t}\right].$$

Fix an index $\alpha \in \{1, \ldots, N\}$, $\varepsilon > 0$, an infinitesimal sequence $(t_m)_1^1$ and consider the inner ε-neighbourhood O of the set $\{f'$

as $m \to 1$ along a subsequence of indices $(m_k)_1^1$. As a result, (3.2.7) becomes

$$2f'''(z) = a(z;z) + b(z;z)\, f''(z) + f'(z)C(z) \quad \text{on } \{f' \neq 0\}; \tag{3.2.10}$$

for any $= 1;:::;N$: Combining equations (3.2.8) and (3.2.10), we infer that there exist measurable functions $A; B : R \to R$ such that

$$f''' = Af' + Bf'' \quad \text{a.e. on } R: \tag{3.2.11}$$

The goal in now to show that (3.2.11) implies that the torsion of the curve f vanishes, at least on a union of subintervals of R. The idea to project on three-dimensional subspaces of R^N in order to utilise standard ideas of elementary differential geometry of curves.

To this end, let $P_3 : R^N \to R^N$ be the orthogonal projection on a 3D subspace $V_3 \quad P_3(R^N)$ of R^N: The choice of 3-dimensional subspaces owes to the fact that we would like to use the classical formulas of di erential geometry of curves in the Euclidean space. Then P_3

Chapter 4

Explicit 1 -harmonic functions in high dimensions

4.1 Introduction

Let R^n be an open set and$u$ 2 C^2( ) a continuous twice di erentiable function. In this paper we study the existence of solutions to the PDE

$$_1 u := \sum_{i,j=1} D_i u D_j u D^2_{ij} u = 0 \qquad (4.1.1)$$

of the form

$$u(x) = \prod_{i=1} f_i(x_i);$$

where f_i are possibly non-linear for 1 i n

utilised (see e.g. [54]).

In this paper all the separated 1 -harmonic functions are found for $n = 2$ in polar coordinates and for all $n \geq 2$ in cartesian coordinates. Some of these new solutions derived herein coincide with previously known classes of solutions. For instance, the well-known G. Aronsson's solution $u(x, y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$ which has a $C^{1,1=3}$ regularity, described in Remark 4.2.2. Also M.-F. Bidaut-Veron, M. Garcia Huidobro and L. Veron have found solutions in [20] which coincide with first two solutions of the theorem 4.1.1

following

$$\ln|t| + c =$$

and

$$|f_j(x_j)| = |f_j(x_j^0)|\, e^{\int_{x_j^0}^{x_j} F_j(t)\,dt};$$

where F_j satisfies

$$t + c = \frac{1}{}$$

Case (A) Let A and G B, then (4.2.3) gives A B 0 or A^2 $A + B^2 = 0$ which can be rewritten as A $\frac{1}{2})^2 + B^2 = \frac{1}{4}$ and as the consequent of substitutions $f(r) = r^A$ and $g(\ ) = e^{B\ }$ up to a constants.

Case (D) Let α and G be non-constant functions, then there exist $r_1 \neq r_2$ and $\alpha_1 \neq \alpha_2$ such that $\alpha(r_1) = \alpha(r_2)$ and $G(\alpha_1) = G(\alpha_2)$ satisfying (4.2.3). Thus

$$r_1\alpha(r_1)^2\alpha_r(r_1)\alpha^3(r_1) + \alpha(r_1)^4 + 2\alpha(r_1)^2G(\alpha)^2 + G(\alpha)^2G(\alpha) + G(\alpha)^4 - \alpha(r_1)G(\alpha)^2 = 0 \tag{4.2.8}$$

$$r_2\alpha(r_2)^2\alpha_r(r_2)\alpha^3(r_2) + \alpha(r_2)^4 + 2\alpha(r_2)^2G(\alpha)^2 + G(\alpha)^2G(\alpha) + G(\alpha)^4 - \alpha(r_2)G(\alpha)^2 = 0: \tag{4.2.9}$$

Subtracting (4.2.8) and (4.2.9) we get for any α

$$G^2(\alpha)(\alpha(r_1) - \alpha(r_2))(2(\alpha(r_1) + \alpha(r_2)) - 1) = r_2\alpha^2(r_2)\alpha_r(r_2) - \alpha(r_2)^3 + \alpha(r_2)^4 - r_1\alpha^2(r_1)\alpha_r(r_1) + \alpha(r_1)^3 - \alpha(r_1)^4: \tag{4.2.10}$$

Let's consider two cases.

Case (I) If there exists $r_1 \neq r_2$ such that $2(\alpha(r_1) + \alpha(r_2)) - 1 \neq 0$, then (4.2.10) gives that $G^2(\alpha)$ is a constant, hence $G(\alpha)$ is a step function.

Case (II) For any $r_1 \neq r_2$ we have $2(\alpha(r_1) + \alpha(r_2)) - 1 = 0$, hence $\alpha(r)$ is a step function.

For both cases we have a contradiction to $C^{1,\alpha}$ regularity for 1-Harmonic functions in two dimensions (see [39], [75]), since $\alpha(r) = \frac{1}{r}\frac{u_r}{u}$ and $G(\alpha) = \frac{u}{u}$ have to have at least $C^{0,\alpha}$ regularity.

Finally integrating $\int \frac{f'}{f} = \int \frac{1}{r}$, $\int \frac{g'}{g} = \int G$ and substituting we get $|f(r)| = |f(r_0)|e^{\int_{r_0}^{r} \frac{\alpha(t)}{t}dt}$ and $|g(\alpha)| = |g(\alpha_0)|e^{\int_{\alpha_0}^{\alpha} G(t)dt}$, which completes the proof. □

4.2.2 Remark [The Arronson solution]

Let $A = \frac{4}{3}$ in the Theorem 4.1.1 ii, then $A^2 - A > 0$ and function G satisfies

$$t + c = -\arctan\frac{3}{4}G(t) + \frac{1}{2}\arctan\frac{3}{2}G(t);$$

which can be rewritten as

$$27G^3(t) + 54G^2(t)\tan 2(t + c) + 32\tan 2(t + c) = 0:$$

Solving a third degree equation with respect to $G(t)$, we get

$$G(t) = \frac{4}{3}\frac{\tan^{\frac{1}{3}}(t + c) + \tan^{\frac{5}{3}}(t + c) + \tan(t + c)}{1 - \tan^2(t + c)}:$$

Therefore

$$\int G(t)\,dt = \ln\frac{1 - \tan^{\frac{2}{3}}(t + c)}{\tan^{\frac{4}{3}}(t + c)}\left(\frac{1 + \tan^{\frac{2}{3}}(t + c)}{\tan^{\frac{2}{3}}(t + c) + 1}\right)^{\frac{1}{3}} + c:$$

Hence

$$e^{\int_{\theta_0} G(t)\,dt} = \frac{|1 - \tan^{\frac{2}{3}}(\theta + c)|\,|1 + \tan^{\frac{2}{3}}(\theta + c)|^{\frac{1}{3}}}{|\tan^{\frac{4}{3}}(\theta + c) - \tan^{\frac{2}{3}}(\theta + c) + 1|^{\frac{2}{3}}}\, c(\theta_0)$$
$$= \frac{|1 - \tan^{\frac{4}{3}}(\theta + c)|}{|1 + \tan^{2}(\theta + c)|^{\frac{2}{3}}}\, c(\theta_0)$$
$$= |\cos^{\frac{4}{3}}(\theta + c) - \sin^{\frac{4}{3}}(\theta + c)|\, c(\theta_0).$$

We can ignore$c(\theta_0)$ since if$c_1 u + c_2$ is a solution then$u$ is also a solution.

Finally

$$|g(\theta)| = |g(\theta_0)|\, e^{\int_{\theta_0} G(t)\,dt}$$
$$= |g(\theta_0)|\, |\cos^{\frac{4}{3}}(\theta + c) - \sin^{\frac{4}{3}}(\theta + c)|;$$
$$|f(r)| = r^{\frac{4}{3}}.$$

Thus, one of the possible solutions is

$$u(r, \theta) = f(r)g(\theta)$$
$$= r^{\frac{4}{3}}$$

4.2.5 Proof of Theorem 4.1.3

We can assume that $u \neq 0$ since if u is a solution then $u + c$ is also a solution then equation (4.1.4) can be written as

$$\sum_{i,j=1}^{n} \frac{D_i u}{u}\frac{D_j u}{u}\frac{D_{ij}^2 u}{u} = 0. \tag{4.2.11}$$

Let $F_i = \frac{D_i u}{u}$ then $D_i F_i + F_i^2 = \frac{D_{ii} u}{u}$ and $F_i F_j = \frac{D_{ij} u}{u}$. Thus (4.2.11) becomes

$$\sum_{i=1}^{n} F_i^2(x_i) \Big)^2 + \sum_{i=1}^{n} F_i^2(x_i) D_i F_i(x_i) = 0. \tag{4.2.12}$$

Since $u(x) = \prod_{i=1}^{n} f_i(x_i)$, then F_i depends only on x_i, consequently

$$D_i F_i(x_i) = F_i'(x_i).$$

Set $x^1, x^2 \in$ such that $x^1 = (x_1; x_2; \ldots; x_j^1; \ldots; x_n)$ and $x^2 = (x_1; x_2; \ldots; x_j^2; \ldots; x_n)$, where $x_j^1 \neq x_j^2$ in (4.2.12) and subtract these two equations. We find

$$F_j^2(x_j^1) - F_j^2(x_j^2) - 2\sum_{i\neq j} F_i^2(x_i) + 2F_j^2(x_j^1) + 2F_j^2(x_j^1) + F_j^2(x_j^1)F_j'(x_j^1) - F_j^2(x_j^2)F_j'(x_j^2) = 0;$$

assuming $F_j^2(x_j^1) \neq F_j^2(x_j^2)$, we have

$$-2\sum_{i\neq j} F_i^2(x_i) = \frac{F_j^2(x_j^1)F_j'(x_j^1) - F_j^2(x_j^2)F_j'(x_j^2)}{F_j^2(x_j^1) - F_j^2(x_j^2)} - 2F_j^2(x_j^1) - 2F_j^2(x_j^1). \tag{4.2.13}$$

LHS of (4.2.13) does not depend on x_j^1 and x_j^2 so

$$\sum_{i\neq j} F_i^2(x_i) = c$$

for all x_i. Then $F_i(x_i) = A_i$, where A_i is a constant for all $1 \leq i \neq j \leq n$ and hence $|f_i(x_i)| = |f_i(x_i^0)|\, e^{A_i(x_i - x_i^0)}$. Thus (4.2.12) gives

$$\sum_{i\neq j} A_i^2 + F_j^2(x_j) \Big)^2 + F_j(x_j)^2 F_j'(x_j) = 0;$$

consequently

$$dx_j = \left(\frac{F_j^2}{\sum_{i \neq j} A_i^2 + F_j^2} \right)^{1/2} dF_j ;$$

$$\int_{x_j^0}^{x_j} F_j(t)\, dt$$

hence $|f_j(x_j)| = |f_j(x_j^0)|\, e$, where $F_j(t)$ satisfies

$$t + c = \frac{1}{2} \sqrt{\sum_{i \neq j} A_i^2} \arctan \frac{F_j(t)}{\sqrt{\sum_{i \neq j} A_i^2}} + \frac{F_j(t)}{2 \sum_{i \neq j} A_i^2 + F_j^2(t)}; \text{ if } \sum_{i \neq j} A_i^2 \neq 0 :$$

Otherwise (i.e. if $\sum_{i \neq j} A_i^2 = 0$)

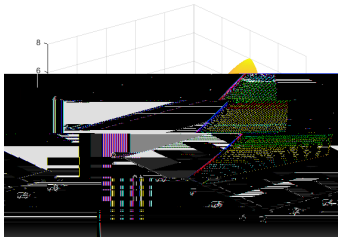
$$F_j^4(x_j) + F_j^2(x_j) F_j'(x_j) = 0 ;$$

so

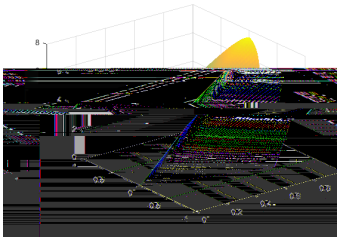
$$F_j^2(x_j) + F_j'(x_j) = 0 ; \text{ since we assume } F_j^2(x_j^1) \neq F_j^2(x_j^2): \qquad (4.2.14)$$

Solving (4.2.14) we get $F_j(x_j) = \frac{1}{x_j + c}$. Hence $|f_i(x_i)| = c_i$ for all $i \neq j$ and

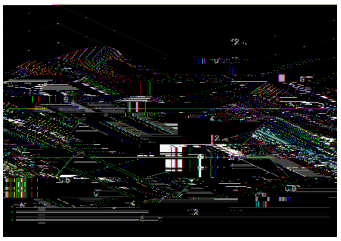
Figure 4.2: The approximation tou of the Theorem 4.1.1 ii, depending on the parameterA.



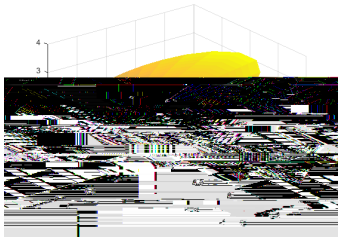
(a) A = 4=3,
min = 0, max = 6



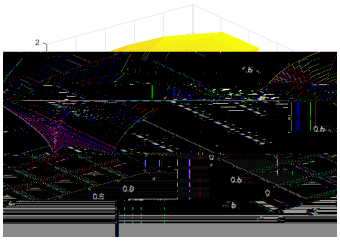
(b) A = 1:15,
min = 0, max = 7



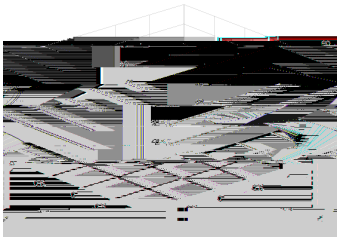
(c) A = 1,
min = 0, max = 10



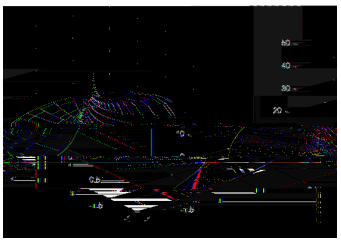
(d) A = 1=3,
min = 0, max = 3



(e) A = 0:15,
min = 0, max = 1.8

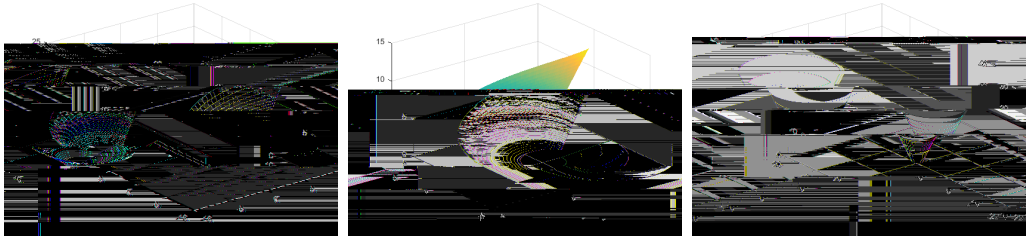


(f) A = 0,
min = 1, max = 39

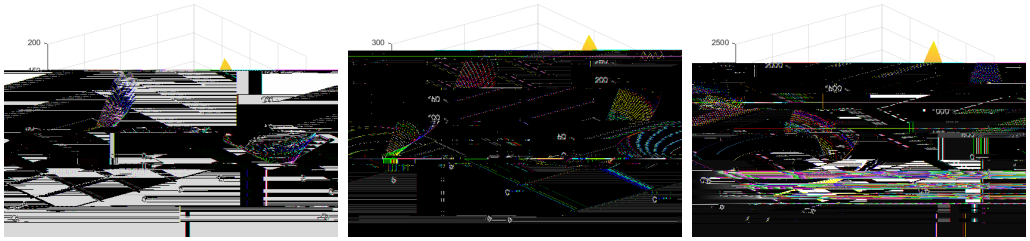


(g) A = 0:15,
min = 0, max = 50

Figure 4.3: The approximation tou of the Theorem 4.1.1 iii, depending on the parameterB.

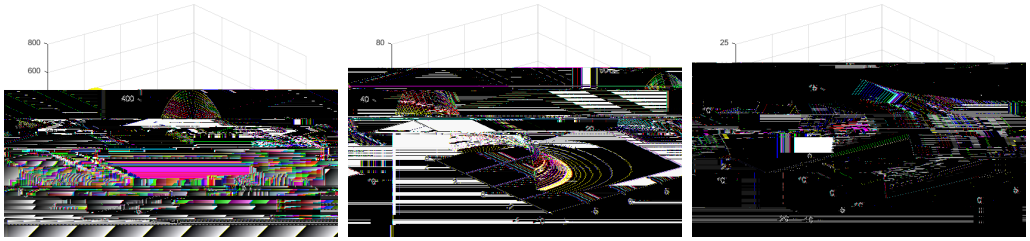


(a) $B = 1{=}3$, min = 0.12, max = 20 (b) $B = 1{=}2$, min = 0.04, max = 12 (c) $B = 0$, min = 1, max = 40

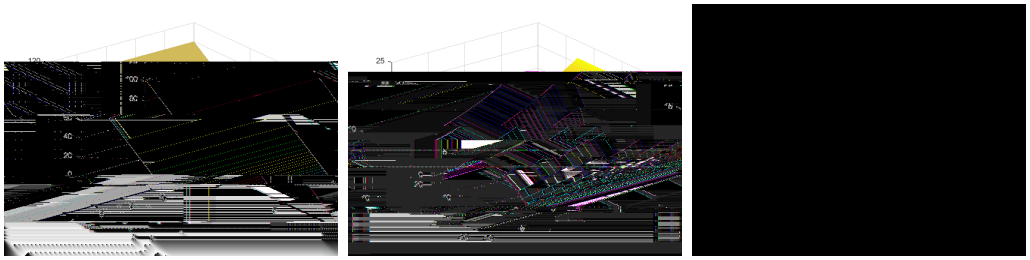


(d) $B = 1{=}3$, min = 1, max = 163.58 (e) $B = 1{=}2$, min = 1, max = 293.26 (f) $B = 1$, min = 1, max = 2304

Figure 4.4: The approximation tou of the Theorem 4.1.2 i, depending on the parameterA.



(a) $A = 0{:}5$, min = 0.0067, max = 663 (b) $A = 0{:}25$, min = 0.0821, max = 76 (c) $A = 0{:}05$, min = 0.0665, max = 21



(d) $A = 0$, min = 1, max = 100 (e) $A = 0{:}05$, min = 0.6065, max = 21 (f) $A = 0{:}25$, min = 0.0821, max = 76

5.1.1 De nition [Absolute Minimiser]

Let $u \in W^{1,\infty}_{loc}(\Omega;\mathbb{R}^N)$. We say that u is an absolute minimiser of (5.1.1) on Ω if

$$\forall\, O \Subset \Omega, \quad \forall\, \varphi \in W^{1,\infty}_0(O;\mathbb{R}^N) \quad \Longrightarrow \quad E_\infty(u;O) \leq E_\infty(u+\varphi;O). \tag{5.1.2}$$

In the scalar case of $N = 1$, Aronsson's concept of absolute minimisers turns out to be the appropriate substitute of mere minimisers. Indeed, absolute minimisers possess the desired uniqueness properties subject to boundary conditions and, most importantly, the possibility to characterise them through a necessary (and sufficient) condition of satisfaction of a certain nonlinear nondivergence second order PDE, known as the Aronsson equation ([3, 13, 15–18, 25–27, 29, 48, 68, 78]). The latter can be written for functions $u \in C^2(\Omega)$ as

$$H_P(x;u;Du)\cdot D_x H(x;u;Du) = 0. \tag{5.1.3}$$

The Aronsson equation, being degenerate elliptic and non-divergence when formally expanded, is typically studied in the framework of viscosity solutions. In the above, H_P, H_x, H_u denotes the derivatives of $H(x;u;P)$ with respect to the respective arguments and "$\cdot$" is the Euclidean inner product.

In this paper we are interested in characterising appropriately defined minimisers of (5.1.1) in the general vectorial case $N \geq 2$ through solvability of associated PDE systems which generalise the Aronsson equation (5.1.3). As the wording suggests and we explain below, when $N \geq 2$ Aronsson's notion of Definition 5.1.1 is no longer the unique possible L^∞ variational concept. In any case, the extension of Aronsson's equation to the vectorial case reads

$$\begin{aligned} &H_P(x;u;Du)\cdot D_x H(x;u;Du) \\ &+ H(x;u;Du)\,[H_P(x;u;Du)]^\perp \cdot \mathrm{Div}_x H_P(x;u;Du) - H_u(x;u;Du) = 0. \end{aligned} \tag{5.1.4}$$

In the above, for any linear map $A : \mathbb{R}^n \to \mathbb{R}^N$, $[A]^\perp$ symbolises the orthogonal projection $\mathrm{Proj}_{R(A)^\perp}$ on the orthogonal complement of its range $R(A) \subset \mathbb{R}^N$. We will refer to the PDE system (5.1.4) as the "Aronsson system", in spite of the fact it was actually derived by N.Katzourakis in [49], wherein the connections between general vectorial variational problems and their associated PDEs were first studied, namely those playing the role of Euler-Lagrange equations in L^∞. The Aronsson system was derived through the well-known method of L^p-approximations and is being studied quite systematically since its discovery, see e.g. [49]-[50], [57, 66]. The additional normal term which is not present in the scalar case imposes an extra layer of complexity, as it might be discontinuous even for smooth solutions (see [50, 53]).

For simplicity and in order to illustrate the main ideas in a manner which minimises technical complications, in this paper we restrict our attention exclusively to

regular minimisers and solutions In general, solutions to (5.1.4) are nonsmooth and the lack of divergence structure combined with its vectorial nature renders its study beyond the reach of viscosity solutions. To this end, the theory of $\mathcal{D}$-solutions introduced in [57] and subsequently utilised in several works (see e.g. [14, 31, 56, 57]) o ers a viable alternative for the study of general locally Lipschitz solutions to (5.1.4), and in fact it works far beyond the realm of Calculus of Variations in L^{∞}. We therefore leave the generalisation of the results herein to a lower regularity setting for future work.

Additionally to absolute minimisers, for reasons to be explained later, in the paper [52] a special case of the next 1st variational concept was introduced (therein for $H(x; ;P) = |P|^2$):

5.1.2 De nition [1 -Minimal Map]

Let $u \in C^1(; R^N)$. We say that u is an 1 -minimal map for (5.1.1) on if (i) and (ii) below hold true:

(i) u is a rank-one absolute minimiser, namely it minimises with respect to essentially scalar variations vanishing on the boundary along xed unit directions:

$$\begin{array}{l} 8 \ O \ b \ ; \ 8 \ \in R^N \\ 8 \ \in C_0^1(\overline{O}; \text{span}[ ]) \end{array} \quad =) \quad E_1(u; O) \quad E_1(u + \ ; O): \qquad (5.1.5)$$

(ii) u has 1 -minimal area, namely it minimises with respect to variations which are normal to the range of the matrix eld $H_P(; u; Du)$ and free on the boundary:

$$\begin{array}{l} 8 \ O \ b \ ; \ 8 \ \in C^1(R^n; R^N) \\ \text{with } {}^{>}H_P(; u; Du) = 0 \text{ on } O \end{array} \quad =) \quad E_1(u; O) \quad E_1(u + \ ; O): \qquad (5.1.6)$$

In the above,

$$C_0^1(\overline{O}; R^N) := \quad \in C^1(R^n; R^N) : \quad = 0 \text{ on } @O :$$

=;1

(II) If u has $\mathbb{1}$-minimal area for (5.1.1) on Ω (Definition 5.1.2(ii)), then it solves

$$H(\cdot,u,Du)\,[\![H_P(\cdot,u,Du)]\!]^{*}\ \mathrm{Div}\ H_P(\cdot,u,Du) - H_{\cdot}(\cdot,u,Du) = 0 \ \text{ on } \Omega. \tag{5.1.9}$$

The converse statement is true if in addition for any $x \in \Omega$, $H(x,\cdot,\cdot)$ is convex on $\mathbb{R}^n \times \mathbb{R}^{N\times n}$.

(III) If u is $\mathbb{1}$-minimal map for (5.1.1) on Ω, then it solves the (reduced) Aronsson system

$$\begin{aligned} A_{\mathbb{1}}u := {} & H_P(\cdot,u,Du)\,D\,H(\cdot,u,Du) \\ & + H(\cdot,u,Du)\,[\![H_P(\cdot,u,Du)]\!]^{*}\ \mathrm{Div}\ H_P(\cdot,u,Du) - H_{\cdot}(\cdot,u,Du) = 0. \end{aligned}$$

The converse statement is true if in addition H does not depend on $\in \mathbb{R}^N$, $H_P(\cdot,Du)$ has full rank on Ω and for any $x \in \Omega$ $H(x,\cdot)$ is convex in $\mathbb{R}^{N\times n}$.

The emergence of two distinct sets of variations and a pair of separate PDE systems comprising (5.1.4) might seem at first glance mysterious. However, it is a manifestation of the fact that the (reduced) Aronsson system in fact consists of two linearly independent differential operators because of the perpendicularity between $[\![H_P]\!]^{*}$ and H_P; in fact, one may split $A_{\mathbb{1}}u = 0$ to

$$\begin{cases} H_P(\cdot,u,Du)\,D\,H(\cdot,u,Du) = 0; \\ H(\cdot,u,Du)\,[\![H_P(\cdot,u,Du)]\!]^{*}\ \mathrm{Div}\ H_P(\cdot,u,Du) - H_{\cdot}(\cdot,u,Du) = 0. \end{cases}$$

Theorem 5.1.4 makes clear that Aronsson's absolute minimisers do not characterise the Aronsson system when $N \geq 2$, at least when the additional natural assumptions hold true. This owes to the fact that, unlike the scalar case, the Aronsson system admits arbitrarily smooth non-minimising solutions, even in the model case of the ∞-Laplacian. For details we refer to [66].

Since Aronsson's absolute minimisers do not characterise the Aronsson system, the natural question arises as to what is their PDE counterpart. The next theorem which is our second main result answers this question:

5.1.5 Theorem [Divergence PDE characterisation of Absolute minimisers]

Let $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ be a map in $C^1(\Omega;\mathbb{R}^N)$. Fix also $O \subset\subset \Omega$ and consider the following statements:

(I) u is a vectorial minimiser of $E_{\mathbb{1}}(\cdot,O)$ in $C^1_u(\overline{O};\mathbb{R}^N)$[1].

[1] We remind the reader that $u \in C^1_g(\overline{O};\mathbb{R}^N)$ means $u - g \in C^1_0(\overline{O};\mathbb{R}^N)$.

(II) We have

$$\max_{\text{Argmax}\{H(\cdot;u;Du):\overline{O}\}} \left[H_P(\cdot;u;Du) : D \quad + H \quad (\cdot;u;Du) \right] \qquad 0;$$

for any $\quad 2\ C_0^1(\overline{O};R^N)$.

(III) For any $\quad 2\ C_0^1(\overline{O};R^N)$, there exists a non-empty compact set

$$K \qquad K \qquad \text{Argmax} \quad H(\cdot;u;Du) \ : \ \overline{O} \qquad (5.1.10)$$

such that,

$$H_P(\cdot;u;Du) : D \quad) \ )]TJ/F24\ 11.9552\ Tf\ 4.552\ 0\ Td\ [(\ )]TJ/F20\ 11.9552\ Tf\ 3$$

$u(x;y) = jxj^{4=3} \quad j \; yj^{4=3}$, as well as for any other 1 -Harmonic function which is nowhere Eikonal (i.e. $jDuj$ is non-constant on all open subsets).

We conclude this introduction by noting that the two vectorial variational concepts we are considering herein (De nitions 5.1.1-5.1.2) do not exhaust the plethora variational concepts in L^1. In particular, in the paper [76] the concept of tight maps was introduced in the case of $H(x; ;P) = kPk$ where $k \quad k$ is the operator norm on $R^{N \quad n}$. Additionally, in the papers [14, 56] a concept of special a ne variations was considered which also characterises the Aronsson system, in fact in the generality of merely locally Lipschitz D-solutions. Finally, in the paper [12] new concepts of absolute minimisers for constrained minimisation problems have been proposed, whilst results relevant to variational principles in L^1 and applications appear in [21, 22, 26, 45, 73, 74].

5.2 Proofs and a maximum-minimum principle for $H(\ ;u;Du)$

In this section we prove our main results Theorems 5.1.4-5.1.5. Before delving into that, we establish a result of independent interest, which generalises a corresponding result from [52].

5.2.1 Proposition [Maximum-Minimum Principles]

Suppose Let $u \; 2 \; C^2(\ ; \; R^N)$ be a solution to (5.1.8), such that H satis es

(a) $H_P(\ ;u;Du)$ has full rank on ,

(b) there exists $c > 0$ such that

$$^{>}H_P(x; \; ;P) \qquad ^{>}P) \quad c \quad ^{>}H_P(x; \; ;P)^{\;2};$$

for all $\quad 2 \; R^N$ and all $(x; \; ;P) \; 2 \qquad R^N \quad R^{N \quad n}$.

Then, for any O [redacted] $c > 0$ such that

5.2.2 Lemma

Let $u \in C^2(\ ;\ R^N)$. Consider the parametric ODE system

$$(\quad \_(t) = \quad {}^{>}H_P($$

We will now show that the trajectory (t) reaches @O in nite time. To this end, we estimate

$$kDuk_{L^1(O)}diam(O) \quad k\ Duk_{L^1(O)} \quad (t) \qquad (0) \qquad \frac{d}{dt}\ {}_{\hat{t}}\ {}^{>}u(\ (t))\ t;$$

for some $\hat{t}$ 2 (0; t), by the mean value theorem. Hence,

$$\begin{aligned} kDuk_{L^1(O)}diam(O) \quad & \frac{d}{dt}\ {}_{\hat{t}}\ {}^{>}u(\ (t))\ t \\ = \quad & {}^{>}Du(\ (\hat{t}))\ \_(\hat{t})\ t \\ = \quad & {}^{>}Du(\ (\hat{t})) \quad {}^{>}H_P(\ ;u;Du)\ {}_{(\hat{t})}\ t \\ & c_0\ {}^{>}H_P(\ ;u;Du)\ {}_{(\hat{t})}\ {}^{2}\ t \\ & (c_0c_1^2)\ t: \end{aligned}$$

This proves the desired claim. Further, since u solves (5.1.8), by (5.2.4) of Lemma 5.2.2 it follows that H(; u; Du) is constant along the trajectory. Thus, if x 2 O is chosen as a point realising either the maximum or the minimum in $\overline{O}$, then by moving along the trajectory, we reach a point y 2 @O such that H(; u; Du) $_x$ = H(; u; Du) $_y$. This establishes both the maximum and minimum principle. The proposition ensues. □

5.2.5 Remark [Danskin's theorem]

The central ingredient in the proofs of Theorems 5.1.4-5.1.5 is the next consequence of Danskin's theorem: for any O b and any u; 2 C^1(; R^N), we have the identities

$$\begin{aligned} & \frac{d}{dt}\ {}_{t=0^+}\ E_1(u+t\ ;\ O) = \max_{O(u)}\ H_P(\ ;u;Du) : D \quad + H\ (\ ;u;Du) \quad ; \\ & \frac{d}{dt}\ {}_{t=0}\ E_1(u+t\ ;\ O) = \min_{O(u)}\ H_P(\ ;u;Du) : D \quad + H\ (\ ;u;Du) \quad ; \end{aligned} \qquad (5.2.6)$$

where

$$O(u) := \text{Argmax}\ \ H(\ ;u;Du) : \overline{O}\ :$$

Indeed, by [34

This establishes the rst identity of (5.2.6). The second one follows through the substitutions ; , t ; t.

satis ed at x . If on the other hand

$$D_{\xi} H(x, u, Du)_{x} \neq 0$$

then $\partial\{H(x) \text{ is a } C^1$ manifold near x and the gradient above is the normal vector at the point x. Due to the interior sphere condition, this implies that this is also the normal vector to the sphere $\partial B_{\cdot}(x)$ at x. Thus, there exists $\neq 0$ such that

$$x - x = D_{\xi} H(x, u, Du)_{x}. \tag{5.2.10}$$

By inserting (5.2.10) into (5.2.9) and noting that $|x - x| = $, we infer that

$$2 \; ^{>} H_P(x, u, Du) D_{\xi} H(x, u, Du)_{x} = 0.$$

By dividing by 2 and letting $\to 0$, we deduce that (5.1.8) is satis ed at the arbitrary $x \in$.

Conversely, suppose that u satis es (5.1.8) on , together with the additional assumptions of the statement. Fix O b and $\in C_0^1(\overline{O}; \text{span}[\ ])$. Without loss of generality, we may suppose O is connected. Since $= (\ ^{>}\ )$, for convenience we set $g := \ ^{>}$ and then we may write $= g$ with $g \in C_0^1(\overline{O})$. Then, the matrix-valued map $H_P(x, Du)$ is pointwise left invertible. Therefore, by (5.1.8)

$$H_P(x, Du)^{\ 1} H_P(x, Du) D_{\xi} H(x, Du) = 0 \text{ on } O,$$

which, by the connectivity of O, gives

$$H(x, Du) \equiv \text{const on } O.$$

Since $g \in C^1(R^n)$ with $g = 0$ on ∂O, there exists at least one interior critical point $x \in O$ such that $Dg(x) = 0$

$H_P(\ ;u;Du)^{>}$ at x. This implies that there exists a C^1 extension $\ 2\ C^1(R^n;R^N)$ such that $\ (x) =\ $ and $(\ )^{>}H_P(\ ;u;Du) = 0$ on the closed ball $B_{"}(x)$ for some $"\ 2\ (0;\ )$. By di erentiating the relation $(\ )^{>}H_P(\ ;u;Du) = 0$ and taking its trace, we obtain

$$\operatorname{div}\ H_P(\ ;u;Du)\ \ +\ D\ \ :H_P(\ ;u;Du) = 0; \tag{5.2.11}$$

on $B_{"}(x)$. Since u has 1 -minimal area and $\ $ is an admissible normal variation, by using Remark 5.2.5 and arguing as in the beginning of part (I), it follows that

$$H\ (\ ;u;Du) + D\ \ :H_P(\ ;u;Du)\ \ _{x_{"}} = 0 \tag{5.2.12}$$

for some $x_{"}\ 2\ (B_{"}(x))(u)$, where

$$(B_{"}(x))(u) = \operatorname{Argmax}\ \ H(\ ;u;Du)\ :\ B_{"}(x)\ \ :$$

By (5.2.11)-(5.2.12), we infer that

$$(x_{"})\quad \operatorname{div}\ H_P(\ ;u;Du)\ \ \ H\ (\ ;u;Du)\ \ _{x_{"}} = 0$$

and by letting $"\ !\ \ 0$, we deduce that

$$\operatorname{div}\ H_P(\ ;u;Du)\ \ \ H\ (\ ;u;Du)\ \ _{x} = 0;$$

for any $\ 2\ \aleph\ H_P(\ ;u;Du)^{>}\ _{x}$. Hence, u satis es (5.1.9) at the arbitrary $x\ 2\ $.

Conversely, suppose that [(Con)3r0-tlvwse (5.1.9)5 0 Td [(satis e6(w)27(e)9(Fi Tf 9.973 0 Td
$\operatorname{div}(\ ;u;Du)\ :\ \ ;\ 9$ 11.95527.9701 Tf 15.348 -1.794 Td [(P)]TJ/F41 11

which evidently satis es supp( ) = f xg O (u), we obtain

$$\begin{aligned} H_P(\ ;u;Du) : D \quad & + H\ (\ ;u;Du) \\ & = \int_{\overline{O}} H_P(\ ;u;Du) : D \quad + H\ (\ ;u;Du) \qquad d^{x} \\ & = 0; \end{aligned}$$

for any x 2 O (u). The conclusion ensues with K =O(u).

(III) =)

yields

$$0 \quad f'(0^+) \quad \liminf_{s \to 0^+} p_s < 1 :$$

Chapter 6

Conclusions and future work

6.1 Conclusions

We would like to mention that this thesis is a collection of published papers presented as chapters consist of original results. This work includes new results in the eld of Calculus of Variations in L^1. The new results are improved previous theorems by generalising and relaxing some of the conditions. Chapter 2 and Chapter 5 are joint papers with my supervisor Dr. N. Katzourakis. Chapter 3 is a joint paper with Dr. N. Katzourakis and Dr. H. Abugirda. While chapter 4 is single author paper.

The main result of Chapter 2 is that we characterise local minimiser of the following functional

$$E_1(u;O) := \operatorname*{ess\,sup}_{O} H(\ ;u;Du); \quad u \in W^{1;1}_{loc}(\ ;R^N); \quad O \text{ b } :$$

for appropriate classes of a ne variations of the energy as generalised solutions of associated PDE system which plays the role of Euler-Lagrange equation. Similar result was proven for $H(x;\ ;P) = |P|^2$ in [53]. That makes our result a generalisation of result in [53] since the Hamiltonian function H depends not only on gradient of the function but also on the function itself and the domain.

Chapter 3 is the joint paper with Dr. N. Katzourakis and Dr. H. Abugirda. The author of this thesis gave an idea which partly impacted on the proof of the main result \Rigidity and atness of maps with tangential Laplacian in separated form", which states let $\ \subset R^2$ be an open set and let also $N \geq 2$. Let $u : R^n \to R^N$ be a classical solution to the nonlinear system

$$[\![Du]\!]^2 \ u = 0 \quad \text{in} \quad ;$$

having the separated form $u(x;y) = f(x) \ f(y)$, for some curve $f \in (W^{3;p} \cap C^2)(R;R^N)$ and some $p > 1$. Then, the image $u(\ )$ of the solution is contained in an at most countable union of a ne planes in R^N.

Chapter 4 investigated with authors own initiative and he found a new classical 1 -harmonic functions in high dimensions, particularly when domains are two dimensional in polar coordinates and at least two dimensional in Cartesian coordinates. The challenges were the technical computations and the regularity of the solutions on uncertain domains which was assumed to be well de ned.

We have two main outcomes of Chapter 5 which is a joint paper with Dr. N. Katzourakis. First result \Variational Structure of Aronsson's system" coincides with the result in [49] when the Hamiltonian function depends only on the gradient function, namely $H(x; ; P) = |P|^2$. The result characterises C^2 1 {minimal maps as solution of the Aronsson system and vice versa. One of the di culty of proving this theorem was that we can not di erentiate

$$( )^{>} H_P(; u; Du) = 0 :$$

if $2 C^1(R^n; R^N)$. We avoided that using reduced nullspace in the de nition 5.1.3. The second result \Divergence PDE characterisation of Absolute minimisers" is a completely new original result. Lets highlight the main di erences of this theorem with previous results. Firstly it has been proved for C^1 maps and for C^1 variations vanishing on compactly contained boundaries. Secondly we did not use approximation techniques of the L^p space but rather techniques of the L^1 space, namely Danskin's theorem.

6.2 Future work

We believe that the work in this eld is interesting and there are still many open problems one can work on, for example:

1. It is common that a solutions to a PDEs might have less regularity than we require. So it is natural to work on extending the result of the theorem 5.1.5 from C^1 to Sobolev spaces or D-solutions.

2. Theorem 5.1.5 gives us characterization of Absolute minimiser only on Argmax set. One of the methods to ll the gap is to study vectorial L^1-absolute minimisers on using vectorial L^p-absolute minimisers on , i.e. for every O b if u_p is minimiser of $E_p(u; O) := |H(; u; Du)|_{L^p(O)}$, then one can study \convergence" of u_p to u_1 as $p ! 1$, where u_1 is minimiser of $E_1(u; O) := |H(; u; Du)|_{L^1(O)}$.

3. All known explicit solutions have at least $C^{1;}$ regularity. So it is challenging to nd explicit solutions of the theorem 5.1.5 and nd out how and/or why other results have to fail.

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