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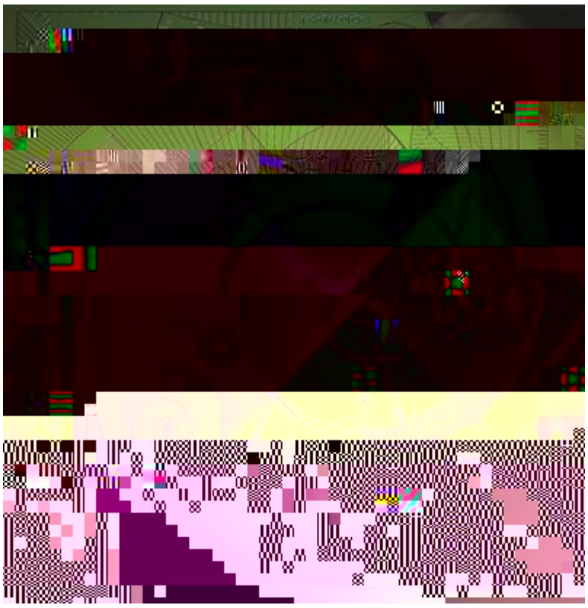

INVERSE OPTICAL TOMOGRAPHY THROUGH PDE CONSTRAINED OPTIMISATION IN L^1

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structure variations created by tumours. However, since some features imaged are not specific to the presence of actual tumour cells, the unavoidable imaging of secondary effects might lead to false diagnoses. Additionally, imaging methods using X-rays and Gamma-rays actually use ionising radiation, which is harmful for humans and animals as it is potentially cancer-inducing itself. On the other hand, FOT is an imaging method which does not use harmful radiation and can be made specific to the presence of designated cell types. Therefore, FOT is more precise and with no side effects for humans.

Technically, the aim of FOT is to reconstruct the fluorophore distribution in a solid body from measurements of light intensity through detectors placed on the boundary. The highly diffusive nature of light propagation implies that in fact FOT forms a highly nonlinear and severely ill-posed inverse problem, hence mathematically it is a very challenging problem. FOT can be modelled by a coupled system of PDEs (partial differential equations) with C-valued solutions and coefficients. The goal is to reconstruct a space-varying parameter in the system of PDEs in the interior of a body (e.g. living tissue).

Mathematically, FOT can be modelled as follows. Let $\mathbb{R}^n$ be an open bounded set with C^1 boundary ∂ and n 3. In medical applications $n = 3$, but from the mathematical viewpoint we may include greater dimensions without any ramifications. A fluorescent dye is injected into the body and in order to determine the dye concentration $= (x)$, the body is illuminated by a red light source $s = s(x)$ placed on the boundary ∂. The wavelength of the light is adjusted to the excitation wavelength of the dye, in order to force it to fluoresce. The light diffuses inside the body, and wherever dye is present, fluorescent light in the infrared range is emitted that can then be detected again at the body surface using a camera and appropriate infrared filters. The goal is then to reconstruct the distribution $= (x)$ of the dye, from these obtained surface images.

Specifically, for time-periodic light sources modulated at a specific frequency, the following system of PDEs describes at any $x \in$ the C-valued photon fluences $u = u(x)$ at the excitation wavelength and $v = v(x)$ at the fluorescent wavelength:

(1.1)

imaging applications where $n = 3$, the coe cients above take the following form:

$$
\begin{cases}
A(x) = 3 \quad {}_{ai}(x) + {}^{0}_{s}(x) + (x) \quad ^{1} \mathrm{I}_n; \\
k(x) = {}_{ai}(x) + (x) + i!c \ ^{1}; \\
h(x) = 1 \quad i! \ (x) \quad ^{1};
\end{cases}
\tag{1.3}
$$

where I_n is the identity matrix in $\mathbb{R}^n$, the di usion coe cient A describes the di usion of photons, ${}_{ai}$ is the absorption coe cient due to the endogenous chromophores, is the absorption coe cient due to the exogenous uorophore, ${}^{0}_{s}$ is the reduced scattering coe cient, is the quantum e ciency of the uorophore, is the uorophore lifetime and $!$ is the modulated light frequency and c the speed of light. Finally, $S; s$ are the light sources. In applications, some authors model the problem with either boundary sources or interior light sources (see e.g. [27] versus [12, 13]). Mathematically, we may include both types of sources without di culty.

To the best of our knowledge, although the FOT problem has been extensively
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[redacted]merically, it has not been been considered from the [redacted] this paper we utilise novel methods of Calculus of [redacted] the rigorous mathematical foundations of the FOT prob[redacted]ivated by developments underpinning the papers [38, 39, 40], we pose FOT[redacted]imisation problem in L^1 with PDE constraints as well as unilateral const[redacted]tudying the direct as well as the inverse FOT problem, both in L^p fo[redacted]d in L^1. Further, we derive the relevant variational inequalities in
;1J/F8 9.813.153 -3.: minimisers1111
L^p f[redacted]p and in L^1 that the constrained minimisers satisfy, which involve (gene[redacted]Lagrange multipliers. Additionally, we prove convergence of the corresp[redacted]^p structures to the limiting L^1 structures as $p \ ! \ 1$, in a certain fashi[redacted]ill become clear later.

C[redacted]f Variations in L^1 is a modern area initiated by Aronsson in the 1960[redacted]-[9]) who was the rst to consider variational problems of functionals
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Then, (2.1) *has a unique weak solution in* $W^{1,2}(\Omega;\mathbb{R}^2)$ *satisfying*

$$\begin{cases} \displaystyle\int_\Omega \Big[\mathrm{B} : (\mathrm{D}u^\top \mathrm{D}\phi) + (\mathrm{L}u)\cdot\phi\Big]\, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} u\cdot\phi \,\mathrm{d}\mathcal{H}^{n-1} \\ \displaystyle\qquad = \int_\Omega \Big[f\cdot\phi + F : \mathrm{D}\phi\Big]\, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} g\cdot\phi\, \mathrm{d}\mathcal{H}^{n-1}, \end{cases} \tag{2.4}$$

for all $\phi \in W^{1,2}(\Omega;\mathbb{R}^2)$*. In addition, there exists* $C > 0$ *depending only on the coefficients and the domain such that*

$$\|u\|_{W^{1,2}(\Omega)} \leq C\Big(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\Big). \tag{2.5}$$

In (2.4), the notation "$:$" symbolises the Euclidean (Frobenius) inner product in the matrix space $\mathbb{R}^{n\times n}$ and "$\cdot$" the Euclidean inner product in $\mathbb{R}^2$.

Proof. As we have already mentioned, the aim is to apply of the Lax Milgram theorem. (Note that the matrix L involved in the zeroth order term is not symmetric, therefore this is not a direct consequence of the Riesz representation theorem.) To this end, we define the bilinear form

$$\mathbb{B} : W^{1,2}(\Omega;\mathbb{R}^2) \times W^{1,2}(\Omega;\mathbb{R}^2) \longrightarrow \mathbb{R}$$

by setting

$$\mathbb{B}[u,\phi] := \int_\Omega \Big[\mathrm{B} : (\mathrm{D}u^\top \mathrm{D}\phi) + (\mathrm{L}u)\cdot\phi\Big]\, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} u\cdot\phi\, \mathrm{d}\mathcal{H}^{n-1}.$$

Since B,L are L^∞, we immediately have by Hölder inequality that

$$\mathbb{B}[u,\phi] \leq C\|u\|_{W^{1,2}(\Omega)}\|\phi\|_{W^{1,2}(\Omega)}$$

for some $C > 0$ and all $u,\phi \in W^{1,2}(\Omega;\mathbb{R}^2)$. Further, since

$$\begin{aligned} (\mathrm{L}u)\cdot u &= [u_R, u_I]\begin{bmatrix} l_R & -l_I \\ l_I & l_R \end{bmatrix}\begin{bmatrix} u_R \\ u_I \end{bmatrix} \\ &= l_R(u_R)^2 + l_R(u_I)^2 \\ &= l_R|u|^2 \\ &\geq \lambda_0|u|^2, \end{aligned}$$

we have that

$$\mathbb{B}[u,u] \geq \lambda_0 \|\mathrm{D}u\|\|$$

and also

$$\begin{aligned}\|u_l\|_{W^{1,p}(\Omega)} \leq C\Big(&\|f_l\|_{L^{\frac{np}{n+p}}(\Omega)} + \|F_l\|_{L^p(\Omega)} \\ &+ \|g_l\|_{L^p(\partial\Omega)} + \|\mathrm{L}\|_{L^\infty(\Omega)}\|u\|_{L^{\frac{np}{n+p}}(\Omega)}\Big).\end{aligned} \tag{2.8}$$

Note now that since by assumption $p > \frac{2n}{n-2}$, we have

$$2 < \frac{np}{n+p} < p.$$

Hence, by the L^p interpolation inequalities, we can estimate

$$\|u\|_{L^{\frac{np}{n+p}}(\Omega)} \leq \|u\|^{\theta}_{L^2(\Omega)}\|u\|^{1-\theta}_{L^p(\Omega)}, \quad \text{where } \theta = \frac{2p}{n(p-2)}.$$

By the Young inequality (for $a, b \geq 0$, $\varepsilon > 0$, $r > 1$ and $r/(r-1) = r'$)

$$ab \leq \frac{r-1}{r}(\varepsilon r)^{\frac{1}{1-r}}\, b^{\frac{r}{r-1}} + \varepsilon a^r, \tag{2.9}$$

for the choice $r := (1-\theta)^{-1}$, we have

$$r = \frac{n(p-2)}{p(n-2)-2n}, \quad \frac{r}{r-1} = \frac{n(p-2)}{2p}, \quad 1-\theta = \frac{n(p-2)}{p(n-2)-2n},$$

and hence we can further estimate

$$\begin{aligned}\|u\|_{L^{\frac{np}{n+p}}(\Omega)} &\leq \|u\|^{\theta}_{L^2(\Omega)}\|u\|^{1-\theta}_{L^p(\Omega)} \\ &= \Big(\|u\|_{L^2(\Omega)}\Big)^{\frac{2p}{n(p-2)}}\Big(\|u\|_{L^p(\Omega)}\Big)^{\frac{p(n-2)-2n}{n(p-2)}} \\ &\leq \frac{r-1}{r}(\varepsilon r)^{\frac{1}{1-r}}\Big(\|u\|_{L^2(\Omega)}^{\frac{2p}{n(p-2)}}\Big)^{\frac{r}{r-1}} + \Big(\|u\|_{L^p(\Omega)}^{\frac{p(n-2)-2n}{n(p-2)}}\Big)^{r} \\ &= \left\{\frac{2p}{n(p-2)}\left(\frac{\varepsilon n(p-2)}{p(n-2)-2n}\right)^{-\frac{p(n-2)-2n}{2p}}\right\}\|u\|_{L^2(\Omega)} + \varepsilon\|u\|_{L^p(\Omega)} \\ &=: C(\varepsilon; p; n)\|u\|_{L^2(\Omega)} + \varepsilon\|u\|_{L^p(\Omega)}.\end{aligned} \tag{2.10}$$

By (2.7), (2.8) and (2.10), by choosing $\varepsilon > 0$ small enough, we infer that

$$\|u\|_{W^{1,p}(\Omega)} \leq C\Big(\|f\|_{L^{\frac{np}{n+p}}(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{L^p(\partial\Omega)} + \|u\|_{L^2(\Omega)}\Big).$$

The desired estimate (2.6) ensues by combining the above estimate with our earlier $W^{1,2}$ estimate (2.5) from Theorem 1, together with Hölder inequality and the fact that $\min\{p, \frac{np}{n+p}\} > 2$. The theorem has been established.

3. Well-posedness of the direct Optical Tomography problem

In this section we utilise the well-posedness results of Section 2 to show that the direct problem of Fluorescent Optical Tomograph5 Tf -2n1-303(lem)tauorTf 11.623 9eir(oe66ec [(),)c [(),)c [(),)F

is bounded and uniformly continuous, valued in the positive matrices and its eigenvalues are uniformly bounded on $\overline{\Omega}$ away from zero. Additionally, it is evident that $\mathrm{K} = \mathrm{K} + \gamma\mathrm{I}_2 \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$ and that it satisfies the structural assumptions in (2.2). Hence, by Theorems 1-2 applied to the Robin boundary value problem (1.4)(a)-(1.4)(c) for $p = m/2$ and noting that

$$\frac{m}{2} > \frac{n}{2} > \frac{2n}{n-2},$$

for any $S \in L^{\frac{nm}{2n+m}}(\Omega;\mathbb{R}^2)$ and any $s \in L^{\frac{m}{2}}(\partial\Omega;\mathbb{R}^2)$ there exists a unique solution $u \in W^{1,\frac{m}{2}}(\Omega;\mathbb{R}^2)$ satisfying (3.3)(a) for all $\phi \in W^{1,\frac{m}{m-2}}(\Omega;\mathbb{R}^2)$, as well as the estimate (3.4)(a). The only thing which is not already stated in the estimate (2.6) is the estimate on $\|u\|_{L^m(\Omega)}$, which follows by the Sobolev inequalities.

Again by Theorems 1-2 applied to the Robin boundary value problem (1.4)(b)-(1.4)(d), there exists a unique solution $v \in W^{1,p}(\Omega;\mathbb{R}^2)$ satisfying (3.3)(b) for all $\psi \in W^{1,\frac{p}{p-1}}(\Omega;\mathbb{R}^2)$. Further, by applying (2.6), by Hölder inequality we estimate

$$\begin{aligned}\|v\|_{W^{1,p}(\Omega)} &\leq C\|\mathrm{H}u\|_{L^{\frac{np}{n+p}}(\Omega)}\\ &\leq C\|\kappa\|_{L^\infty(\Omega)}\|u\|_{L^{\frac{np}{n+p}}(\Omega)}\\ &\leq C\|\kappa\|_{L^\infty(\Omega)}\|u\|_{L^m(\Omega)},\end{aligned}$$

since $m > n$. The estimate (3.4)(b) therefore follows by the above estimate together with the Sobolev inequalities. The proof is complete.

4. The inverse problem through PDE-constrained minimisation

Now that the forward fluorescent optical tomography problem is understood, we proceed with the solvability of the inverse problem associated with (1.4). Throughout this and subsequent sections we assume that the hypotheses of Theorem 3 are satisfied for a domain $\Omega \Subset \mathbb{R}^n$ with $n \geq 3$ and which *from now is assumed to have $C^{1,1}$ regular boundary*.

Fix an integer $N \in \mathbb{N}$, $m > n$, $M, \lambda, \varepsilon > 0$ and $p > \max\left\{n, \frac{2n}{n-2}\right\}$. Consider Borel sets

$$\Gamma_1, \dots, \Gamma_N \subseteq \partial\Omega \tag{4.1}$$

and light sources

$$S_1, \dots, S_N \in L^{\frac{nm}{2n+m}}(\Omega;\mathbb{R}^2), \quad s_1, \dots, s_N \in L^{\frac{m}{2}}(\partial\Omega;\mathbb{R}^2) \tag{4.2}$$

in the interior and on the boundary respectively. Let also

$$\mathrm{v}_1, \dots, \mathrm{v}_N \in L^\infty(\partial\Omega;\mathbb{R}^2) \tag{4.3}$$

be predicted approximate values of the solution v of (1.4)(b)-(1.4)(d) on the boundary $\partial\Omega$, at noise (error) level ε. Suppose that for any $i \in \{1, \dots, N\}$, the pair (u_i, v_i) solves (1.4) with data $(S_i, s_i; \kappa)$. For the N-tuple of solutions $(u_1, \dots, u_N; v_1, \dots, v_N)$, we will be using the notation

$$(\mathbf{u}, \mathbf{v}) \in W^{1,\frac{m}{2}}(\Omega;\mathbb{R}^{2\times N}) \times W^{1,p}(\Omega;\mathbb{R}^{2\times N})$$

The goal of the inverse problem associated with (1.4) is to *determine a non-negative $\gamma \in L^p(\Omega;[0,\infty))$ such that the errors $(v_i - \bar v_i)_{|\Gamma_i}$ which describe the misfit between the predicted approximate solution and the actual solution become as small as possible*. We will minimise the error in L^∞ by means of approximations in L^p for large p and then take the limit $p\to\infty$. The benefit of minimisation in L^∞ is that one can achieve uniformly small error rather than on average. Since no reasonable error functional is coercive in the admissible class of N-tuples of PDE solutions without additional constraints, we add an extra Tykhonov-type regularisation term $\varepsilon\|\gamma\|$ for a small parameter $\varepsilon > 0$ and some appropriate norm.

In view of the above observations, we define for $p > \max\left\{n, \frac{2n}{n-2}\right\}$ the functional

$$
\mathrm{E}_p(\mathbf{u},\mathbf{v},\gamma) := \sum_{i=1}^N \dot{\|} v_i - \bar v_i \dot{\|}_{L^p(\Gamma_i)} + \varepsilon\, \dot{\|} D^2\gamma \dot{\|}_{L^m(\Omega)}, \quad (\mathbf{u},\mathbf{v},\gamma) \in \mathfrak{X}^p(\Omega) \tag{4.4}
$$

and its limiting counterpart

$$
\mathrm{E}_\infty(\mathbf{u},\mathbf{v},\gamma) := \sum_{i=1}^N \| v_i - \bar v_i \|_{L^\infty(\Gamma_i)} + \varepsilon\, \dot{\|} D^2\gamma \dot{\|}_{L^m(\Omega)}, \quad (\mathbf{u},\mathbf{v},\gamma) \in \mathfrak{X}^\infty(\Omega), \tag{4.5}
$$

where the dotted L^p-functionals are the next regularisations of the respective norms

$$
\dot{\|} g \dot{\|}_{L^p(\Gamma_i)} := \left(⨍_{\Gamma_i} (|g|_{(p)})^p \, \mathrm{d}\mathcal{H}^{n-1} \right)^{1/p}, \quad \dot{\|} f \dot{\|}_{L^m(\Omega)} := \left(⨍_{\Omega} (|f|_{(m)})^m \, \mathrm{d}\mathcal{L}^{n} \right)^{1/m} \tag{4.6}
$$

and $|\cdot|_{(p)}$ is a regularisation of the Euclidean norm away from zero in the corresponding space, given by

$$
|\cdot|_{(p)} := \sqrt{|\cdot|^2 + p^{-2}}. \tag{4.7}
$$

The slashed integral symbolises the average with respect to either the Lebesgue measure $\mathcal{L}^n$ or the Hausdorff measure $\mathcal{H}^{n-1}$. The respective admissible classes $\mathfrak{X}^p(\Omega)$ and $\mathfrak{X}^\infty(\Omega)$ are defined by

$$
\mathfrak{X}^p(\Omega) := \left\{
\begin{array}{l}
(\mathbf{u},\mathbf{v},\gamma) \in \mathcal{W}^p(\Omega) : \text{ for any } i \in \{1,\dots,N\},\ (u_i,v_i,\gamma) \text{ satisfies} \\[4pt]
\qquad\qquad 0 \le \gamma \le M \text{ a.e. on } \Omega \\[4pt]
\text{and} \\
\left\{
\begin{array}{llll}
(a)_i & -\mathrm{div}(A\,Du_i) + \gamma K u_i = S_i, & \text{in } \Omega, \\
(b)_i & -\mathrm{div}(A\,Dv_i) + \gamma K v_i = \gamma H u_i, & \text{in } \Omega, \\
(c)_i & (A\,Du_i)\cdot n + \beta u_i = s_i, & \text{on } \partial\Omega, \\
(d)_i & (A\,Dv_i)\cdot n + \beta v_i = 0, & \text{on } \partial\Omega,
\end{array}
\right. \\[4pt]
\text{for } A, K, H, S_i, s_i, \beta, \Gamma_i, \varepsilon, M, p \text{ satisfying the hypotheses (1.5)-} \\
\text{(1.6), (1.7), (3.1) and (4.1)-(4.3)}
\end{array}
\right\}, \tag{4.8}
$$

and

$$
\mathfrak{X}^\infty(\Omega) := \bigcap_{n<p<\infty} \mathfrak{X}^p(\Omega). \tag{4.9}
$$

whilst the Banach space $W^p(\Omega)$ involved in the definition of the admissible class $\mathfrak{X}^p(\Omega)$ is

$$W^p(\Omega) := W^{1,\frac{m}{2}}(\Omega;\mathbb{R}^{2\otimes N}) \ W^{1,p}(\Omega;\mathbb{R}^{2\otimes N}) \ W^{2,m}(\Omega). \tag{4.10}$$

Note that $\mathfrak{X}^\infty(\Omega)$ is a subset of a Frechet space, rather than of a Banach space, but this will not cause any added difficulties.

Remark 4. It might be quite surprising that in the Tikhonov term we include the L^m norm of the Hessian of α, rather than as one would expect the L^m norm of either the gradient or α itself. It turns out that one cannot regularise enough by adding "$+\ \|\alpha\|_{L^m(\Omega)}$" to obtain minimisers (this would be redundant anyway because of the unilateral constraint). On the other hand, by adding "$+\ \|D\alpha\|_{L^m(\Omega)}$", one can indeed recover all the results up to and including Section 5, but not the results of Section 6, as we cannot obtain the variational inequalities in L^∞ without higher regularity in the coefficients of the PDE systems in (4.8) due to the emergence of quadratic gradient terms.

The main result in this section concerns the existence of E_p-minimisers in the admissible class $\mathfrak{X}^p(\Omega)$, the existence of E_∞-minimisers in the admissible class $\mathfrak{X}^\infty(\Omega)$ and the approximability of the latter by the former as $p\to\infty$.

Theorem 5 (E_∞-error minimisers, E_p-error minimisers and convergence as $p\to\infty$). (A) *The functional* E_p *given by* (4.4) *has a constrained minimiser* $(\mathbf{u}_p,\mathbf{v}_p,\alpha_p)$ *in the admissible class* $\mathfrak{X}^p(\Omega)$*:*

$$\mathrm{E}_p\big(\mathbf{u}_p,\mathbf{v}_p,\alpha_p\big) = \inf\Big\{\mathrm{E}_p\big(\mathbf{u},\mathbf{v},\alpha\big) \ : \ (\mathbf{u},\mathbf{v},\alpha) \in \mathfrak{X}^p(\Omega)\Big\}. \tag{4.11}$$

(B) *The functional* E_∞ *given by* (4.5) *has a constrained minimiser* $(\mathbf{u}_\infty,\mathbf{v}_\infty,\alpha_\infty)$ *in the admissible class* $\mathfrak{X}^\infty(\Omega)$*:*

$$\mathrm{E}_\infty\big(\mathbf{u}_\infty,\mathbf{v}_\infty,\alpha_\infty\big) = \inf\Big\{\mathrm{E}_\infty\big(\mathbf{u},\mathbf{v},\alpha\big) \ : \ (\mathbf{u},\mathbf{v},\alpha) \in \mathfrak{X}^\infty(\Omega)\Big\}. \tag{4.12}$$

Additionally, there exists a subsequence of indices $(p_j)_1^\infty$ *such that the sequence of respective* E_{p_j}*-minimisers* $\big(\mathbf{u}_{p_j},\mathbf{v}_{p_j},\alpha_{p_j}\big)$ *satisfies*

$$\begin{cases} \mathbf{u}_p \rightharpoonup \mathbf{u}_\infty, & \text{in } W^{1,\frac{m}{2}}(\Omega;\mathbb{R}^{2\otimes N}),\\ \mathbf{u}_p \to \mathbf{u}_\infty, & \text{in } L^{\frac{m}{2}}(\Omega;\mathbb{R}^{2\otimes N}),\\ \mathbf{v}_p \rightharpoonup \mathbf{v}_\infty, & \text{in } W^{1,q}(\Omega;\mathbb{R}^{2\otimes N}), \text{ for all } q\in(1,\infty),\\ \mathbf{v}_p \to \mathbf{v}_\infty, & \text{in } C(\overline{\Omega};\mathbb{R}^{2\otimes N}),\\ \alpha_p \rightharpoonup \alpha_\infty, & \text{in } W \ldots \end{cases} \tag{4.13}$$

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Proof. Let us begin by noting that $\mathfrak{X}^p(\ ) \neq ;$, and, as we will show right next, in fact it is a weakly closed subset of the re exive Banach space $W^p(\ )$ with cT4 Td [n64p and a subsequence $(j_k)_1^\infty$ such that along which we have

$$
\left\{
\begin{array}{lll}
\mathbf{u}^j & \rightharpoonup \mathbf{u}_p, & \text{in } W^{1,\frac{m}{2}}(\Omega;\mathbb{R}^{2\times N}); \\
\mathbf{u}^j & \longrightarrow \mathbf{u}_p, & \text{in } L^{\frac{m}{2}}(\Omega;\mathbb{R}^{2\times N}); \\
\mathbf{v}^j & \rightharpoonup \mathbf{v}_p, & \text{in } W^{1,p}(\Omega;\mathbb{R}^{2\times N}); \\
\mathbf{v}^j & \longrightarrow \mathbf{v}_p, & \text{in } C(\overline{\Omega};\mathbb{R}^{2\times N}); \\
{}^j & \rightharpoonup {}_p, & \text{in } W^{2,m}(\Omega); \\
{}^j & \longrightarrow {}_p, & \text{in } C^1(\overline{\Omega});
\end{array}
\right.
$$

as $j_k \to \infty$. We note that in this paper we will utilise the common practice of

for any $(\mathbf{u}, \mathbf{v}, \ ) \in \mathfrak{X}^\infty(\ )$. Hence $(\mathbf{u}_\infty, \mathbf{v}_\infty, \ _\infty)$ is a minimiser of E_∞ over $\mathfrak{X}^\infty(\ )$. The particular choice of $(\mathbf{u}, \mathbf{v}, \ ) := (\mathbf{u}_\infty, \mathbf{v}_\infty, \ _\infty)$ in the above inequality yields

$$\lim_{p_j \to \infty} E_p(\mathbf{u}_p, \mathbf{v}_p, \ _p) = E_\infty(\mathbf{u}_\infty, \mathbf{v}_\infty, \ _\infty).$$

The proof of Proposition 7 is now complete.

5. Kuhn-Tucker theory and Lagrange multipliers for the p-error

In this section we return the L^p-minimisation problem (4.11) solved in Theorem 5 for finite $p < \infty$ (Section 4). Given the presence of both PDE and unilateral constraints, in general one cannot have an Euler-Lagrange equation, but an one-sided variational inequality with Lagrange multipliers. The goal here is to derive the relevant variational inequality associated with (4.11). The main result is therefore the following.

Theorem 8 (The variational inequalities in L^p). *In the setting of Section 4 and under the same assumptions, for any $p > \max\{n, 2n/(n-2)\}$, there exist Lagrange multipliers*

$$(\tilde{\ }_p, \tilde{\ }_p) \in W^{1,\frac{m}{m-2}}(\ ; \mathbb{R}^{2\times N}) \times W^{1,\frac{p}{p-1}}(\ ; \mathbb{R}^{2\times N})$$

associated with the constrained minimisation problem (4.11) for E_p in the admissible class $\mathfrak{X}^p(\ )$, such that the constrained minimiser $(\mathbf{u}_p, \mathbf{v}_p, \ _p) \in \mathfrak{X}^p(\ )$ satisfies the next three relations:

$$(5.1)\qquad \frac{m}{p}\int (D^2 \ - D^2 \ _p) : (D^2 \ _p)\, d\mathcal{L}^n \ \ \sum_{i=1}^{N} \int (\ - \ _p)\Big[\mathrm{H}u_{pi}\ \ _{pi} + \underline{r}(\cdot, \ _p)\, Du_{pi} : D\ _{pi} + Dv_{pi} : D\ _{pi} + u_{pi}\ \ _{pi} + v_{pi}\ \ _{pi}\Big]\, d\mathcal{L}^n$$

for any $\ \in W^{2,m}(\ ; [0, M])$; further,

$$(5.2)\qquad \int_{\partial} \mathbf{w} : d[\tilde{\ }_p(\mathbf{v}_p)] = \sum_{i=1}^{N} \int \Big[\mathrm{A}_{\ _p} : (Dw_i^\top D\ _{pi}) + \mathrm{K}_{\ _p} w_i \ \ _{pi}\Big]\, d\mathcal{L}^n + \int_{\partial} (\ w_i)\ \ _{pi}\, d\mathcal{H}^{n-1};$$

for any $\mathbf{w} \in W^{1,p}(\ ; \mathbb{R}^{2\times N})$, and finally

$$(5.3)\qquad \sum_{i=1}^{N} \int \Big[\mathrm{A}_{\ _p} : (Dz_i^\top D\ _{pi}) + (\mathrm{K}_{\ _p} z_i)\ \ _{pi}\Big]\, d\mathcal{L}^n + \int_{\partial} (\ z_i)\ \ _{pi}\, d\mathcal{H}^{n-1} = \sum_{i=1}^{N} \int \ _p\ \mathrm{H}z_i\ \ _{pi}\, d\mathcal{L}^n;$$

for any $\mathbf{z} \in W^{1,\frac{m}{2}}(\ ; \mathbb{R}^{2\times N})$.

In the relations (5.1)-(5.2), $\ (V)$ is defined for any $V \in L^m(\ ; \mathbb{R}^{n\times n})$ as

$$(5.4)\qquad \ (V) := \ (|V|_{(m)}$$ ETd [(2)]TJ/F1,]TJ7E any

which by the chain rule yields

$$
\begin{aligned}
\mathrm{dE}_{p\,(u,v,\,)}(z,w,\,) = p & \left(\sum_{i=1}^{N} \int_{i} \left| v_i - v_i \right|_{(p)}^{p} \mathrm{d}H^{n-1} \right)^{\frac{1}{p}-1} \\
& \int_{i} |v_i - v_i|_{(p)}^{p-2} (v_i - v_i) \, w_i \, \mathrm{d}H^{n-1} \\
+ \; m & \left(\int |\mathrm{D}^2 \,|_{(m)}^{m} \, \mathrm{d}L^n \right)^{\frac{1}{m}-1} \int |\mathrm{D}^2 \,|_{(m)}^{m-2} \mathrm{D}^2 \, : \mathrm{D}^2 \, \mathrm{d}L^n .
\end{aligned}
$$

Hence, (5.6) follows in view of the definitions (5.4)-(5.5). The lemma ensues.

With the aim of deriving the variational inequality which is the necessary condition of the minimisation problem (4.11), we compute the Frechet derivative of the mapping G above and prove that it is a submersion.

Lemma 11.

The exact form of the Gateaux derivative of G is a simple consequence of the de nitions of A ; K and the next computations:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} H(\ \ + \varepsilon\ \ )(u_i + \varepsilon z_i) = H(\ \ u_i + \ \ z_i);$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} A_{\ \ +\varepsilon\ \ } : (Du_i + \varepsilon D$$

and

$$\begin{cases} \quad \operatorname{div}(\mathrm{A} \quad \mathrm{D}w_i) + \mathrm{K}\, w_i = \quad g_i + \quad \mathrm{H}z_i \quad \operatorname{div} G_i; & \text{in} \quad ; \\ (\mathrm{A} \quad \mathrm{D}w_i \quad G_i)\mathrm{n} + \quad w_i = 0; & \text{on } @ \quad ; \end{cases}$$

for all $i\ 2\ f1; :::; Ng$ and with $\mathrm{A}; \mathrm{K}; \mathrm{H}; u_i; \ ; \ ; f_i; F_i; g_i; G_i$ being xed coe cients and parameters. The solvability of the above systems follows from Theorems 1-3. The result is therefore complete.

Now we derive the variational inequality through the generalised Kuhn-Tucker theory of Lagrange multipliers.

Proposition 12 (The variational inequality). *For any $p > 2n=(n \quad 2)$, there exist*

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for any $(\bar{z}; \bar{w}; \ )$ in the convex set $W^p_M(\ )$. Recall now that (5.11) implies that the convex subset $W^p_M(\ )$ of the Banach space $W^p(\ )$ can be written as the Cartesian product of the vector spaces

$$W^{1;\frac{m}{2}}(\ ;\mathbb{R}^{2\ N}) \quad W^{1;p}(\ ;\mathbb{R}^{2\ N})$$

with the convex set $W^{2;m}(\ ;[0;M])$, we may replace $\bar{z}$ by $\bar{z} + \bar{u}_p$ and we may also replace $\bar{w}$ by $\bar{w} + \bar{v}_p$ in (5.19) to arrive at (5.18). The proof of Proposition 12 is now complete.

We now use Proposition 12 to deduce that the variational inequality takes the form (5.20) below, as a direct consequence of Lemmas 10-11, (5.6), (5.4), (5.5), (5.13)-(5.16).

Corollary 13. *In the setting of Proposition 12, in view of the form of the Frechet derivatives of* E_p *and* G*, the variational inequality* (5.18) *takes the form*

(5.20)

$$\begin{aligned}
&\bar{w} : \mathrm{d}[\tilde{\ }_p(\bar{v}_p)] + \frac{m}{p} \quad \mathrm{D}^2 \quad \mathrm{D}^2\ _p\ :\ (\mathrm{D}^2\ _p)\,\mathrm{d}\mathcal{L}^n \\
&\sum_{i=1}^{N} \Big[\mathrm{A} : (\mathrm{D}z_i^{>}\mathrm{D}\ _{pi}) + (\mathrm{K}z_i) \quad _{pi} \Big] \mathrm{d}\mathcal{L}^n + \int_{\partial} (\ z_i) \quad _{pi}\,\mathrm{d}\mathcal{H}^{n\ 1} \\
&+ \sum_{i=1}^{N} \Big[\mathrm{B} : (\mathrm{D}w_i^{>}\mathrm{D}\ _{pi}) + \ \mathrm{L}w_i \quad \mathrm{H}\ (\quad _p)u_{pi} + \ _p z_i \quad\quad _{pi} \Big] \mathrm{d}\mathcal{L}^n \\
&+ \int_{\partial} (\ w_i) \quad _{pi}\,\mathrm{d}\mathcal{H}^{n\ 1} \quad + \sum_{i=1}^{N} (\quad\ _p) \ \underline{r}(\ ;\ _p) \Big[\mathrm{D}u_{pi} : \mathrm{D}\ _{pi} \\
&+ \mathrm{D}v_{pi} : \mathrm{D}\ _{pi} \ + u_{pi} \quad _{pi} + v_{pi} \quad _{pi} \Big] \mathrm{d}\mathcal{L}^n
\end{aligned}$$

for any $(\bar{z}; \bar{w}; \ ) \ 2\ W^p_M(\ )$.

We conclude this section by obtaining the further desired information on the variational inequality (5.20).

Lemma 14. *In the setting of Corollary 13, the variational inequality* (5.20) *for the constrained minimiser* $\bar{u}_p; \bar{v}_p;\ _p$ *is equivalent Td [1e9 0 Td [()]TJ/726 Tf TJ/F11 85(triple01)]TJ/51(l]TJ/(mi* the admissible class (4.9

for any $w \in C_0^\infty(\Omega;\mathbb{R}^{2\times N})$*, and finally*

$$\sum_{i=1}^{N} \Big[A_1 : (Dz$$

limit as $p_j \to \infty$, since the rescaled Lagrange multipliers $\tilde{\ }_p/C_p, \tilde{\ }_p/C_p$ are bounded in the product space

$$W^{1,\frac{m}{m-2}}(\ ;\mathbb{R}^{2\times N}) \times BV(\ ;\mathbb{R}^{2\times N})$$

and therefore the sequence is weakly* precompact. (Recall also that on a reflexive space the weak and the weak* topology coincide.) Note first that we have

$$\begin{aligned} \int \mathrm{D}^2(\ \ _p) : \mathrm{D}^2\ _p\, \mathrm{d}\mathcal{L}^n &= \int \mathrm{D}^2(\ \ _p : \frac{(|\mathrm{D}^2\ _p|_{(m)})^{m-2}\,\mathrm{D}^2\ _p}{\|\mathrm{D}^2\ _p\|_{L^m(\ )}^{m-1}}\, \mathrm{d}\mathcal{L}^n \\ &\le \int \mathrm{D}^2(\ \ _p)\ \frac{(|\mathrm{D}^2\ _p|_{(m)})^{m-1}}{\|\mathrm{D}^2\ _p\|_{L^m(\ )}^{m-1}}\, \mathrm{d}\mathcal{L}^n \\ &\le C\ \|\mathrm{D}^2(\ \ _p)\|_{L^m(\ )}, \end{aligned}$$

by the definition of and by Hölder inequality. In order to conclude, we need to justify the weak* convergence as $p_j \to \infty$ of the quadratic terms

$$\mathrm{D}u_{pi} : \frac{\mathrm{D}\ _{pi}}{C_p}, \quad \mathrm{D}v_{pi} : \frac{\mathrm{D}\ _{pi}}{C_p}.$$

To this end, we will show that under the higher regularity assumptions on the coefficients, we in fact have the next strong modes of convergence for the p-minimisers:

$$\mathrm{D}u_{pi} \longrightarrow \mathrm{D}u_{\infty i} \ \text{ in } L^{\frac{m}{2}}_{\mathrm{loc}}(\ ;\mathbb{R}^2), \tag{6.9}$$

$$\mathrm{D}v_{pi} \longrightarrow \mathrm{D}v_{\infty i} \ \text{ in } C(\ ;\mathbb{R}^2), \tag{6.10}$$

as $p_j \to \infty$, for all $i \in \{1,\ldots,N\}$. Before proving (6.9)-(6.10), we demonstrate how to conclude by assuming them. Since we have

$$\frac{\mathrm{D}\ _{pi}}{C_p} \overset{*}{\rightharpoonup} \mathrm{D}\ _{\infty i} \ \text{ in } L^{\frac{m}{m-2}}(\ ;\mathbb{R}^2), \tag{6.11}$$

$$\frac{\mathrm{D}\ _{pi}}{C_p}\mathcal{L}^n \overset{*}{\rightharpoonup} \mathrm{D}\ _{\infty i} \ \text{ in } \mathcal{M}(\ ;\mathbb{R}^2) \tag{6.12}$$

and also $\ _p \longrightarrow \ _\infty$ in $C^1(\overline{\ })$ as $p_j \to \infty$, by choosing any $O \Subset$ with Lipschitz boundary (for instance the union of finitely many balls), $(\ _{p_j})_1^\infty \subseteq W^{2,m}(\ ;[0,M])$ and $\ \in W^{2,m}(\ ;[0,M])$ with

$$\ _p \equiv \ \ _p \text{ on } \ \setminus O, \quad \ _p \longrightarrow \ \text{ in } W^{2,m}(O) \cap C^1(\overline{O}),$$

we have $\ \ _p \in W^{2,m}_0(O)$ and

$$\ _p\ _p \longrightarrow \ \ _\infty \ \text{ in } W^{2,m}_0(O)$$

as $p_j \to \infty$. Hence, (6.4)-(6.6) follow by (6.9)-(6.12), together with the weak-strong continuity of the duality pairing between $L^{\frac{m}{2}}(O;\mathbb{R}^2)$ and $L^{\frac{m}{m-2}}(O;\mathbb{R}^2)$ and the weak*-strong continuity of the duality pairing between $C_0(O;\mathbb{R}^2)$ and $\mathcal{M}(O;\mathbb{R}^2)$, at least for test functions $\ \in W^{2,m}_{\ \infty}(O;[0,M])$. The general case for test functions $\ \in C_\infty(\ ;[0,M])$ follows by a standard approximation argument.

Now we establish (6.9)-(6.10). Fix $i \in \{1,\ldots,N\}$, $e \in \mathbb{R}^n$ with $|e| = 1$ (with [0981 0 Td TJ/F14 9.9626 Tf 4.68.5

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