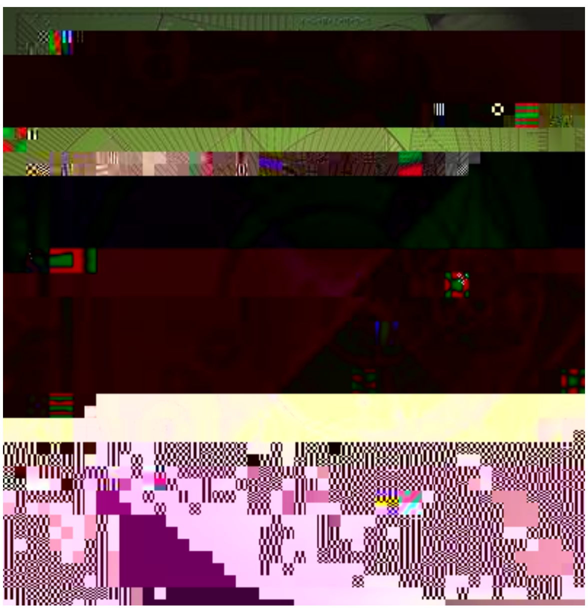

A MINIMISATION PROBLEM IN L^∞ WITH PDE AND UNILATERAL CONSTRAINTS

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Abstract. We study the minimisation of a cost functional which measures the misfit on the boundary of a domain between a component of the solution to a certain parametric elliptic PDE system and a prediction of the values of this solution. We pose this problem as a PDE-constrained minimisation problem for a supremal cost functional in L^∞, where except for the PDE constraint there is also a unilateral constraint on the parameter. We utilise approximation by PDE-constrained minimisation problems in L^p as $p \to \infty$ and the generalised Kuhn-Tucker theory to derive the relevant variational inequalities in L^p and L^∞. These results are motivated by the mathematical modelling of the novel bio-medical imaging method of Fluorescent Optical Tomography.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with C^1 boundary $\partial\Omega$ and let also $n \geq 3$. Consider the next Robin boundary value problem for a pair of coupled linear elliptic systems:

$$
\left\{
\begin{array}{llll}
(a) & \mathrm{div}(\mathrm{D}u\,\mathrm{A}) + \mathrm{K}u = S, & \text{in } \Omega, \\
(b) & \mathrm{div}(\mathrm{D}v\,\mathrm{B}) + \mathrm{L}v = -\mathrm{M}u, & \text{in } \Omega, \\
(c) & (\mathrm{D}u\,\mathrm{A})\mathrm{n} + \gamma u = s, & \text{on } \partial\Omega, \\
(d) & (\mathrm{D}v\,\mathrm{B})\mathrm{n} + \gamma v = 0, & \text{on } \partial\Omega,
\end{array}
\right. \tag{1.1}
$$

where $u, v : \Omega \to \mathbb{R}^2$ are the solutions, $\mathrm{n} : \partial\Omega \to \mathbb{R}^n$ is the outer unit normal vector field on $\partial\Omega$ and the coefficients $\mathrm{A}, \mathrm{B}, \mathrm{K}, \mathrm{L}, \mathrm{M}, s, S, \gamma$ satisfy $\gamma > 0$ and

$$
\left\{
\begin{array}{ll}
u, v, S : \Omega \to \mathbb{R}^2, & \mathrm{D}u, \mathrm{D}v : \Omega \to \mathbb{R}^{2\times n}, \\
\mathrm{K}, \mathrm{L}, \mathrm{M} : \Omega \to \mathbb{R}^{2\times 2}, & \mathrm{A}, \mathrm{B} : \Omega \to \mathbb{R}^{n\times n}_+, \\
s : \partial\Omega \to \mathbb{R}^2, & \gamma : \partial\Omega \to [0, \infty).
\end{array}
\right. \tag{1.2}
$$

Here the matrix-valued maps K, L are assumed to have the form

$$
\mathrm{K} := \begin{bmatrix} k_1 & k_2 \\ k_2 & k_1 \end{bmatrix}, \quad \mathrm{L} := \begin{bmatrix} l_1 & l_2 \\ l_2 & l_1 \end{bmatrix}. \tag{1.3}
$$

We will suppose that there exists $a_0 > 0$ such that

$$
\left\{
\begin{array}{l}
\mathrm{A}, \mathrm{B} \in VMO(\mathbb{R}^n; \mathbb{R}^{n\times n}_+), \quad \sigma(\mathrm{A}), \sigma(\mathrm{B}) \subseteq \left[a_0, \dfrac{1}{a_0} \right], \\
\mathrm{K}, \mathrm{L}, \mathrm{M} \in L^\infty(\Omega; \mathbb{R}^{2\times 2}), \quad k_1, \geq a_0, \quad l_1 \geq a_0.
\end{array}
\right. \tag{1.4}
$$

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We note that our general notation will be either standard or self-explanatory, as e.g. in the textbooks [24, oks [

and let also

$$\mathrm{v}_1, \dots, \mathrm{v}_N \in L^\infty(\partial\Omega;\mathbb{R}^2) \tag{1.7}$$

be predicted (noisy) values of the solution v of (1.1)(b)-(1.1)(d) on the boundary $\partial\Omega$. Suppose that for any $i \in \{1,\dots,N\}$, the pair (u_i, v_i) solves (1.1) with coefficiens (S_i, s_i, σ). For the N-tuple of solutions $(u_1,\dots,u_N, v_1,\dots,v_N)$, we will symbolise

$$\mathbf{u},\mathbf{v} \in W^{1,\frac{m}{2}}(\Omega;\mathbb{R}^{2\times N}) \times W^{1,p}(\Omega;\mathbb{R}^{2\times N})$$

and understand $(u_i)_{i=1\dots N}$ and $(v_i)_{i=1\dots N}$ as matrix valued. Similarly, we will see the corresponding vectors of test functions as

$$\tilde{\phi}, \tilde{\psi} \in W^{1,\frac{m}{m-2}}(\Omega;\mathbb{R}^{2\times N}) \times W^{1,\frac{p}{p-1}}(\Omega;\mathbb{R}^{2\times N}).$$

Our aim is to determine some $\sigma \in L^p(\Omega;[0,\infty))$ such that all the misfits

$$(v_i - \mathrm{v}_i)\big|_{\mathrm{B}_i}$$

between the predicted approximate solution and the actual solution are minimal. We will minimise the error in L^∞ by means of approximations in L^p for large p and then take the limit $p \to \infty$. By minimising in L^∞ one can achieve uniformly small cost, rather than on average. Since no reasonable cost functional is coercive in our admissible class, we will therefore follow two different approaches to rectify this problem, but in a unified fashion. The first and more popular idea is to add a Tykhonov-type regularisation term $\varepsilon\|\sigma\|$ for small $\varepsilon > 0$ and some appropriate norm. The alternative approach is to consider that an a priori L^∞ bound is given on σ. The latter approach appears to be more natural for applications, as it does not alter the error functional. For finite $p < \infty$, we can relax this to an L^p bound, but as we are mostly interested in the limit case $p = \infty$, we will only discuss the case of L^∞ bound. In view of the above observations, we define the integral functional

$$\mathrm{I}_p(\mathbf{u},\mathbf{v},\sigma) := \sum_{i=1}^{N} \big\| v_i - \mathrm{v}_i \big\|_{\dot{L}^p(\mathrm{B}_i)} + \varepsilon\|\sigma\|_{\dot{L}^p(\Omega)}, \qquad (\mathbf{u},\mathbf{v},\sigma) \in \mathfrak{X}^p(\Omega) \tag{1.8}$$

and its supremal counterpart

$$\mathrm{I}_\infty(\mathbf{u},\mathbf{v},\sigma) := \sum_{i=1}^{N} \big\| v_i - \mathrm{v}_i \big\|_{L^\infty(\mathrm{B}_i)} + \varepsilon\|\sigma\|_{L^\infty(\Omega)} \qquad (\mathbf{u},\mathbf{v},\sigma) \in \mathfrak{X}^\infty(\Omega), \tag{1.9}$$

where the dotted L^p quantities are regularisations of the respective norms:

$$
\mathfrak{X}^p(\quad) := \left\{
\begin{array}{l}
(u, v, \quad) \in X^p(\quad) : \text{ for all } i \in \{1, \dots, N\},\ (u_i, v_i, \quad) \text{ satisfies} \\
\qquad\qquad 0 \qquad M \text{ a.e. on} \\
\text{and} \\
\left\{
\begin{array}{lll}
(a)_i & \mathrm{div}(\mathrm{D}u_i \mathrm{A}) + \mathrm{K}u_i = S_i, & \text{in } \ ; \\
(b)_i & \mathrm{div}(\mathrm{D}v_i \mathrm{B}) + \mathrm{L}v_i = \mathrm{M}u_i, & \text{in } \ ; \\
(c)_i & (\mathrm{D}u_i \mathrm{A})\mathrm{n} + \ u_i = s_i, & \text{on } \partial \ ; \\
(d)_i & (\mathrm{D}v_i \mathrm{B})\mathrm{n} + \ v_i = 0, & \text{on } \partial \ ;
\end{array}
\right. \\
\text{for } \mathrm{A}, \mathrm{B}, \mathrm{K}, \mathrm{L}, \mathrm{M}, S_i, s_i, \ , \ , p \text{ satisfying hypotheses (1.2)-(1.7)}
\end{array}
\right\}, \tag{1.13}
$$

and

$$
\mathfrak{X}^{\infty}(\quad) := \bigcap_{n < p < \infty} \mathfrak{X}^p(\quad). \tag{1.14}
$$

Note that $\mathfrak{X}^{\infty}$

We conclude this lengthy introduction with some comments about the general variational context we use herein. Calculus of Variations in L^{∞} is a modern subarea of analysis pioneered by Aronsson in the 1960s (see [6]-[9]) who considered variational problems of supremal functionals, rather than integral functional. For a pedagogical introduction we refer e.g. to [20, 36]. Except for their endogenous mathematical appeal, L

In the proofs that follow we will employ the standard practice of denoting by C a generic constant whose value might change from step to step in an estimate.

Proof. The aim is to apply of the Lax Milgram theorem. (Note that the matrix K is not symmetric, thus this is not a direct consequence of the Riesz theorem.) We define the bilinear functional

$$\mathrm{B} : W^{1,2}(\Omega;\mathbb{R}^2) \times W^{1,2}(\Omega;\mathbb{R}^2) \to \mathbb{R},$$
$$\mathrm{B}[u,\psi] := \int_\Omega \Big[\mathrm{A} : (\mathrm{D}u^\top \mathrm{D}\psi) + (\mathrm{K}u)\cdot\psi \Big]\, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} u\cdot\psi \, \mathrm{d}\mathcal{H}^{n-1}.$$

Since A, K are L^∞, by Hölder inequality we immediately have

$$\big|\mathrm{B}[u,\psi]\big| \leq C\|u\|_{W^{1,2}(\Omega)}\|\psi\|_{W^{1,2}(\Omega)}$$

for some $C>0$ and all $u,\psi \in W^{1,2}(\Omega;\mathbb{R}^2)$. Further, since

$$(\mathrm{K}u)\cdot u = [u_1, u_2]\begin{bmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = k_1|u|^2 \geq a_0|u|^2,$$

we estimate

$$\mathrm{B}[u,u] \geq a_0\Big(\|\mathrm{D}u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\Big) + \|u\|^2_{L^2(\partial\Omega)},$$

for any $u \in W^{1,2}(\Omega;\mathbb{R}^2)$. Hence, the bilinear form B is continuous and coercive, thus the hypotheses of the Lax-Milgram theorem are satisfied (see e.g. [24]). Hence, for any $\Phi \in (W^{1,2}(\Omega;\mathbb{R}^2))^*$, exists a unique $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ such that

$$\mathrm{B}[u,\psi] = \langle \Phi, \psi \rangle, \quad \text{for all } \psi \in W^{1,2}(\Omega;\mathbb{R}^2).$$

Next, we show that the functional Φ given by

$$\langle \Phi, \psi \rangle := \int_{\partial\Omega} g\cdot\psi \, \mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \Big[f\cdot\psi + F : \mathrm{D}\psi \Big]\, \mathrm{d}\mathcal{L}^n$$

lies in $(W^{1,2}(\Omega;\mathbb{R}^2))^*$ and we will also establish the L^2 and the L^p estimates. Indeed, by the trace theorem in $W^{1,2}(\Omega;\mathbb{R}^2)$, there is a $C>0$ which allows to estimate

$$\begin{aligned}\big|\langle \Phi, \psi \rangle\big| &\leq \|g\|_{L^2(\partial\Omega)}\|\psi\|_{L^2(\partial\Omega)} + \Big(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}\Big)\|\psi\|_{W^{1,2}(\Omega)} \\ &\leq C\Big(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\Big)\|\psi\|_{W^{1,2}(\Omega)}.\end{aligned}$$

The particular choice of11g.9738 Tf 3.114 0Qar co7.749 1 for $i = 1;2$. By applying the estimate to the each of the components separately, we have

$$\|u_i\|_{W^{1;p}(\Omega)} \le C\Big(\|K\|_{L^\infty(\Omega)}\|u\|_{L^{\frac{np}{n+p}}(\Omega)} + \|f_i\|_{L^{\frac{np}{n+p}}(\Omega)} + \|F_i\|_{L^p(\Omega)} + \|g_i\|_{L^p(\partial\Omega)}\Big); \tag{2.7}$$

for $i = 1;2$. Note now that since we have assumed $p > 2n/(n-2)$, we have $2 < np/(n+p) < p$. Hence, by the L^p interpolation inequalities, we can estimate

$$\|u\|_{L^{\frac{np}{n+p}}(\Omega)} \le \|u\|^{\theta}_{L^2(\Omega)}\|u\|^{1-\theta}_{L^p(\Omega)}; \quad \text{for } \theta = \frac{2p}{n(p-2)}.$$

By Young's inequality

$$ab \le \frac{r-1}{r}(\varepsilon r)^{\frac{1}{1-r}}\, b^{\frac{r}{r-1}} + \varepsilon a^r; \tag{2.8}$$

which holds for $a;b;\varepsilon > 0$, $r > 1$ and $r/(r-1) = r'$, the choice $r := 1/(1-\theta)$ yields

$$1-\theta = \frac{n(p-2)}{p(n-2)-2n}; \quad r = \frac{n(p-2)}{p(n-2)-2n}; \quad \frac{r}{r-1} = \frac{n(p-2)}{2p};$$

and hence we can estimate

$$\begin{aligned}\|u\|_{L^{\frac{np}{n+p}}(\Omega)} &\le \|u\|_{L^p(\Omega)}^{\frac{p(n-2)-2n}{n(p-2)}}\,\|u\|_{L^2(\Omega)}^{\frac{2p}{n(p-2)}} \\ &\le \varepsilon\Big(\|u\|_{L^p(\Omega)}^{\frac{p(n-2)-2n}{n(p-2)}}\Big)^{r} + \frac{r-1}{r}(\varepsilon r)^{\frac{1}{1-r}}\Big(\|u\|_{L^2(\Omega)}^{\frac{2p}{n(p-2)}}\Big)^{\frac{r}{r-1}} \\ &= \varepsilon\|u\|_{L^p(\Omega)} + 4\,\frac{2p}{n(}\end{aligned} \tag{2.9}$$

it satis es
(2.11)8

A PROBLEM IN L^1

is weakly closed. To this aim, let ${}^{j} \rightharpoonup {}_{p}$ in $L^p(\ )$ as $j_k \to \infty$. Then, for any measurable set E with positive measure $\mathcal{L}^n(E) > 0$, by integrating the last inequality over E, the averages satisfy

$$0 \quad \int_E {}^{j} \, d\mathcal{L}^n \quad M$$

and therefore

$$\int_E {}_{p} \, d\mathcal{L}^n = \lim_{j_k \to \infty} \int_E {}^{j} \, d\mathcal{L}^n \in [0, M].$$

By selecting $E := \mathbb{B}(x)$ for $x \in$ and $\in (0, \mathrm{dist}(x, \partial\ ))$, the Lebesgue di erentiation theorem allows us to infer

$$_{p}(x) = \lim_{\to 0} \int_{\mathbb{B}(x)} {}_{p} \, d\mathcal{L}^n \in [0, M], \quad \text{for a.e. } x \in .$$

To conclude that $(u_p, v_p, {}_{p}) \in \mathfrak{X}^p(\ )$, we must pass to the weak limit in the equations $(a)_i$ $(d)_i$ in (1.13). The only convergence that needs to be justi ed that of the nonlinear source term $\mathrm{M}u_i$ in $(b)_i$. To this end, note that by our assumption $p > \frac{2n}{n\ 2}$, we have the inequality

$$\frac{p}{p\ 1} < \frac{n}{2} \quad \frac{m}{2}.$$

Thus, since $u_i^j \to u_{pi}$ in $L^{\frac{m}{2}}(\ ;\mathbb{R}^2)$ as $j_k \to \infty$, we have that

$$u_i^j \to u_{pi} \ \text{ in } L^{\frac{p}{p\ 1}}(\ ;\mathbb{R}^2)$$

as $j_k \to \infty$. Hence, since ${}^{j} \rightharpoonup {}_{p}$ in $L^p(\ )$, it follows that

$$\int {}^{j} \mathrm{M}u_i^j \quad d\mathcal{L}^n \to \int {}_{p} \mathrm{M}u_{pi} \quad d\mathcal{L}^n$$

for any $\in C_c^{\infty}(\ ;\mathbb{R}^2)$ as $j_k \to \infty$, as a consequence of the weak-strong continuity of the duality pairing between $L^p(\ )$ and $L^{\frac{p}{p\ 1}}$

for any $M > 0$. If on the other hand (1.15) is satisfied, then by the weak lower-semicontinuity of the functional $\|\cdot\|_{L^q(\Omega)}$ on $L^q(\Omega)$, we have

$$
\begin{aligned}
\|\xi_\infty\|_{L^\infty(\Omega)} &= \lim_{q\to\infty} \|\xi_\infty\|_{L^q(\Omega)} \\
&\leq \liminf_{q\to\infty} \ \liminf_{p_j\to\infty} \|\xi_p\|_{L^q(\Omega)} \\
&\leq \liminf_{q\to\infty} \ \liminf_{p_j\to\infty} \left(\frac{1}{N} \sum_{i=1}^{N} \big\| v_i \big\|_{L^\infty(B_i)} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \big\| v_i \big\|_{L^\infty(B_i)}.
\end{aligned}
$$

Further, by passing to the limit as $p_j \to \infty$ in $(a)_i$–$(d)_i$ of (1.13) as in the proof of Proposition 6, we see that the limit $(u_\infty, v_\infty, \xi_\infty)$ lies in $\mathfrak{X}^\infty(\Omega)$. It remains to prove that $(u_\infty, v_\infty, \xi_\infty)$ is a minimiser of I_∞ and that the energies converge. Fix an arbitrary $(u, v, \xi) \in \mathfrak{X}^\infty(\Omega)$. Since $p_j \geq q$ for large $j \in \mathbb{N}$, by minimality we have

$$
\begin{aligned}
I_\infty(u_\infty, v_\infty, \xi_\infty) &= \lim_{q\to\infty} I_q(u_\infty, v_\infty, \xi_\infty) \\
&\leq \liminf_{q\to\infty}
\end{aligned}
$$

Proof. To see (2.17), note that if $M = \infty$, then by testing in (1.20) against $\psi := \sigma_p + \phi$ where $\phi \in \mathrm{L}^p(\Omega;[0,\infty))$, we obtain

$$\int_\Omega \left(\frac{\mathrm{d}[\sigma_p(\nu_p)]}{\mathrm{d}\mathcal{L}^n} + \sum_{i=1}^N \mathrm{M}u_{pi}\, \psi_{pi} \right) \phi \, \mathrm{d}\mathcal{L}^n \geq 0,$$

for any $\phi \in \mathrm{L}^p(\Omega;[0,+\infty))$, which yields

$$\frac{|\sigma_p|^{p-2}_{(p)}\sigma_p}{\mathcal{L}^n(\Omega)\|\sigma_p\|^{p-1}_{\mathrm{L}^p(\Omega)}} + \sum_{i=1}^N \mathrm{M}u_{pi}\, \psi_{pi} \geq 0, \quad \text{a.e. on } \Omega. \tag{2.19}$$

From the above inequality we readily deduce (2.17). To see (2.18), we fix a point $x \in \{\sigma_p > 0\}$, $t > 0$ small and $\rho \in (0, \mathrm{dist}(x, \partial\Omega))$ and test against the function

$$\psi := \sigma_p - t\chi_{\{\sigma_p > t\} \cap \mathbb{B}_\rho(x)} \in \mathrm{L}^p(\Omega;[0,\infty)).$$

Then, by (1.20) we get

$$t \int_{\mathbb{B}_\rho(x)} \chi_{\{\sigma_p > t\}} \left(\frac{\mathrm{d}[\sigma_p(\nu_p)]}{\mathrm{d}\mathcal{L}^n} + \sum_{i=1}^N \mathrm{M}u_{pi}\, \psi_{pi} \right) \mathrm{d}\mathcal{L}^n \leq 0,$$

which by diving by $t\mathcal{L}^n(\mathbb{B}_\rho(x))$, letting $t \to 0$, using the Dominated Convergence theorem and letting $\rho \to 0$ yields

$$\lim_{\rho \to 0} \fint_{\mathbb{B}_\rho(x)} \chi_{\{\sigma_p > 0\}} \left(\frac{\mathrm{d}[\sigma_p(\nu_p)]}{\mathrm{d}\mathcal{L}^n} + \sum_{i=1}^N \mathrm{M}u_{pi}\, \psi_{pi} \right) \mathrm{d}\mathcal{L}^n \leq 0.$$

Now, (2.18) follows as a consequence of the Lebesgue differentiation theorem and (2.19). The proof is complete. $\square$

The proof of Theorem 2 consists of a few sub-results. We begin by computing the derivative of I_p.

Lemma 9. *The functional* $\mathrm{I}_p : \mathcal{X}^p(\Omega) \longrightarrow \mathbb{R}$ *is Frechet differentiable and its derivative*

$$\mathrm{dI}_p : \mathcal{X}^p(\Omega) \longrightarrow \mathcal{X}^p(\Omega)^*$$

which maps

$$(\mathbf{u}, \mathbf{v}, \sigma) \mapsto \mathrm{dI}_p\big|_{(\mathbf{u},\mathbf{v},\sigma)}$$

is given for all $(\mathbf{u}, \mathbf{v}, \sigma), (\mathbf{z}, \mathbf{w}, \tau) \in \mathcal{X}^p(\Omega)$ *by the formula*

$$\mathrm{dI}_p\big|_{(\mathbf{u},\mathbf{v},\sigma)}(\mathbf{z}, \mathbf{w}, \tau) = p\int_{\partial\Omega} \mathbf{w} : \mathrm{d}[\tilde{\mu}_p(\mathbf{v})] + p\int_\Omega \tau\, \mathrm{d}[\sigma_p(\sigma)]. \tag{2.20}$$

Proof. The Frechet differentiability of I_p follows from well-known results on the differentiability of norms on Banach spaces and our p-regularisations in (1.10)-(1.11). To compute the Frechet derivative, we use directional differentiation. For

Let us also define for any $M \in [0,\infty]$ the following weakly closed convex subset of the Banach space $X^p(\Omega)$:

$$X^p_M(\Omega) := W^{1,\frac{m}{2}}(\Omega;\mathbb{R}^{2\times N}) \times W^{1,p}(\Omega;\mathbb{R}^{2\times N}) \times L^p(\Omega;[0,M]). \tag{2.25}$$

Then, in view of (2.21)-(2.25), we may reformulate the admissible class $\mathfrak{X}^p(\Omega)$ of the minimisation problem (1.17) as

$$\mathfrak{X}^p(\Omega) = \Big\{ (\mathbf{u},\mathbf{v},\gamma) \in X^p_M(\Omega) : \mathrm{J}\big(\mathbf{u},\mathbf{v},\gamma\big) = 0 \Big\}. \tag{2.26}$$

We now compute the derivative of J above and prove that it is a C^1 submersion.

Lemma 10. *The map* J *defined by* (2.21)-(2.25) *is a continuously differentiable submersion and its Frechet derivative*

$$\mathrm{dJ} : X^p(\Omega) \longrightarrow \mathcal{L}\Big(X^p(\Omega); \Big[W^{1,\frac{m}{m-2}}(\Omega;\mathbb{R}^2) \times W^{1,\frac{p}{p-1}}(\Omega;\mathbb{R}^2) \Big]^N \Big), \tag{2.27}$$

which maps

$$(\mathbf{u},\mathbf{v},\gamma) \mapsto \mathrm{dJ}_{(\mathbf{u},\mathbf{v},\gamma)}$$

is given by

$$\Big\langle \mathrm{dJ}_{(\mathbf{u},\mathbf{v},\gamma)}(\mathbf{z},\mathbf{w},\theta), (\boldsymbol{\phi},\boldsymbol{\psi}) \Big\rangle = \begin{bmatrix} \Big\langle \mathrm{dJ}^1_1{}_{(\mathbf{u},\mathbf{v},\gamma)}(\mathbf{z},\mathbf{w},\theta), \phi_1 \Big\rangle \\ \Big\langle \mathrm{dJ}^2_1{}_{(\mathbf{u},\mathbf{v},\gamma)}(\mathbf{z},\mathbf{w},\theta), \psi_1 \Big\rangle \\ \vdots \\ \Big\langle \mathrm{dJ}^1_N{}_{(\mathbf{u},\mathbf{v},\gamma)}(\mathbf{z},\mathbf{w},\theta), \phi_N \Big\rangle \\ \Big\langle \mathrm{dJ}^2_N{}_{(\mathbf{u},\mathbf{v},\gamma)}(\mathbf{z},\mathbf{w},\theta), \psi_N \Big\rangle \end{bmatrix}. \tag{2.28}$$

In (2.28), *for each* $i \in \{1,\dots,N\}$ *and* $j \in \{1,2\}$, *the component* $\mathrm{dJ}^1_N{}_{(\mathbf{u},\mathbf{v},\gamma)}$ *of the derivative is given for any test functions*

$$(\boldsymbol{\phi},\boldsymbol{\psi}) \in \Big(W^{1,\frac{m}{m-2}} \dots \Big)^N$$

associated with the minimisation problem (1.17)*, such that the constrained minimiser* $(u_p, v_p, \ _p) \in \mathfrak{X}^p(\ )$ *satisfies for any* $(z, w, \ )$ *in the convex set* $X^p_M(\ )$ *that*

$$\frac{1}{p}\, \mathrm{d}I_p\,_{(u_p, v_p, \ _p)}\, z, w, \quad _p \quad \sum_{i=1}^{N} \mathrm{d}J^1_i\,_{(u_p, v_p, \ _p)}\, z, w, \quad _p \, ; \quad _{pi} \qquad (2.32)$$
$$+ \sum_{i=1}^{N} \mathrm{d}J^2_i\,_{(u_p, v_p, \ _p)}\, z, w, \quad _p \, ; \quad _{pi} \; :$$

Proof. By Lemmas 9-10, I_p is Frechet differentiable and J is a continuously Frechet differentiable submersion on $X^p(\ )$. Also, the set X^p_M -58 (4)]TJ/F10293Jo0 9.1 (2]TJ 7.4/F3610 6.9

for any $(\bar{z}, \bar{w}, \ ) \in X^p_M(\ )$.

We conclude this section by obtaining the further desired information on the variational inequality (2.34).

Lemma 13. *In the setting of Corollary 12, the variational inequality* (2.34) *for the constrained minimiser* $\bar{u}_p, \bar{v}_p, \ _p$ *is equivalent to the triplet of relations* (1.20)-(1.22).

Proof. The inequality (1.20) follows by setting $\bar{z} = \bar{w} = 0$ in (2.34), and recalling the definition of Radon-Nikodym derivative of the absolutely continuous measure $\ _p(\ _p)$. The identity (1.21) follows by setting $\ = \ _p$ and $\bar{z} = 0$ in (2.34) and by recalling that $W^{1,p}(\ ; \mathbb{R}^{2\ N})$ is a vector space, so the inequality we obtain in fact holds for both $\ w$. Finally, the identity (1.22) follows by setting $\ = \ _p$ and $\bar{w} = 0$ in (2.34) and by recalling again that $W^{1,\frac{m}{2}}(\ ; \mathbb{R}^{2\ N})$ is a vector space, so the inequality holds for both $\ z$.

We conclude by establishing our last main result.

Proof of Theorem 3. We first show that for any $p > n$ and any

$$(\bar{v}, \ ) \in W^{1,p}(\ ; \mathbb{R}^{2\ N}) \quad L^p(\ ),$$

we have the next total variations bounds for the measures (1.23)-(1.24):

$$\tilde{\ }_p(\bar{v}) \ (\partial \ ) \quad N, \qquad\qquad \partial$$

To see (2.37), we argue as follows. First, note that if $\ {}_{\infty} = 0$ a.e. on , then by the positivity of $\ {}_{p}$ and $\ {}_{p}$ we trivially have

$$\liminf_{p_j \to \infty} \ {}_p \, d[\ {}_p(\ {}_p)] \quad 0 = \| \ {}_\infty \|_{L^1(\ )}$$

and hence (2.37) ensues. Therefore, we may assume $\| \ {}_\infty \|_{L^1(\ )} > 0$. Next, note that by (1.11) we have

$$\begin{aligned} {}_p \, d[\ {}_p(\ {}_p)] &= \frac{| \ {}_p|_{(p)}^{\ p-2} | \ {}_p|^2}{\| \ {}_p\|_{L^p(\ )}^{\ p-1}} \, dL^n \\ &= \frac{| \ {}_p|_{(p)}^{p}}{\| \ {}_p\|_{L^p(\ )}^{\ p-1}} \, dL^n \quad \frac{1}{p^2} \quad \frac{| \ {}_p|_{(p)}^{\ p-2}}{\| \ {}_p\|_{L^p(\ )}^{\ p-1}} \, dL^n; \end{aligned}$$

which by Hölder inequality gives

$$\begin{aligned} {}_p \, d[\ {}_p(\ {}_p)] &= \| \ {}_p\|_{L^p(\ )} \quad \frac{1}{p^2} \| \ {}_p\|_{L^p(\ )}^{\ 1-p} \quad | \ {}_p|_{(p)}^{\ p-2} \, dL^n \\ & \| \ {}_p\|_{L^p(\ )} \quad \frac{1}{p^2 \| \ {}_p\|_{L^p(\ )}}: \end{aligned}$$

Hence, for any $k \quad 1$ xed and $p \quad k$, we have

$$ {}_p \, d[\ {}_p(\ {}_p)] \quad \| \ {}_p\|_{L^k(\ )} \quad \frac{1}{p^2 \| \ {}_p\|_{L^k(\ )}}:$$

Since by Theorem 1 we have $\ {}_p \ \overset{*}{\ } \ {}_\infty$ in $L^k(\ )$ for any $k \in (1, \infty)$, by the weak lower semi-continuity of the convex functional $\| \ \|_{L^k(\ )}$ on $L^k(\ )$, it follows that

$$\begin{aligned} \liminf_{p_j \to \infty} \ {}_p \, d[\ {}_p(\ {}_p)] & \liminf_{p_j \to \infty} \| \ {}_p\|_{L^k(\ )} \quad \limsup_{p_j \to \infty} \frac{1}{p^2} \ \frac{1}{\liminf_{p_j \to \infty} \| \ {}_p\|_{L^k(\ )}} \\ & \| \ {}_\infty\|_{L^k(\ )}: \end{aligned}$$

We therefore discover (2.37) by letting $k \to \infty$.

Now we proceed with establishing (I) and (II) of the theorem.

(I) Suppose that $C_\infty = 0$. Then, we have pi

$$\tilde{\ }_p, \tilde{\ }_p \ \to \ \tilde{0}, \tilde{0} \quad \text{in } W^{1,\frac{m}{m-2}}(\ ;\mathbb{R}^{2\ N}) \quad BV(\ ;\mathbb{R}^{2\ N}) \tag{2.38}$$

as $p_j \to \infty$, where $\tilde{\ }_p, \tilde{\ }_p$ are the Lagrange multipliers associated with the constrained minimisation problem (1.18). In view of (2.37) and (1.19), the inequality (1.20) implies

(2.39)

$$ d[\ {}_p(\ {}_p)] + \sum_{i=1}^{N} (\quad {}_p) \ Mu_{pi} \quad {}_{pi} \, dL^n \quad o(1)_{p_j \to \infty} + \| \ {}_\infty\|_{L^1(\ )}:$$

for any $\ \in C_0^0(\ ;[0,M])$. Note now that Hölder's inequality gives

$$(\quad {}_p) Mu_{pi} \quad {}^{m}$$

duality pairing between $\mathrm{L}^{\frac{m}{2}}(\ )$ and $\mathrm{L}^{\frac{m}{m-2}}(\ )$, by letting $p \to \infty$ along the sequence $(p_j)_1^\infty$, (2.39) yields

$$\mathrm{d}\ _\infty \quad \| \ _\infty \|_{\mathrm{L}^\infty(\ )};$$

for any $\ \in C_0^0(\ ; [0; M])$. Hence, if $\ > 0$, we see that $\ _\infty = 0$ a.e. on $\ $. Again by (1.25) and (2.38), by passing to the limit as $p_j \to \infty$ in (1.21), we obtain

$$\int_{\partial} w : \mathrm{d}\tilde{\ }_\infty = 0 = \lim_{p_j \to \infty} \sum_{i=1}^{N} \int \Big[\mathrm{B} : (\mathrm{D}w_i^\top \mathrm{D} \ _{pi}) + \mathrm{L}w_i \ _{pi} \Big] \mathrm{d}\mathcal{L}^n$$

$$+ \int_{\partial} (\ w_i) \ _{pi} \, \mathrm{d}\mathcal{H}^{n-1} ;$$

for any $w \in C_0^1(\overline{\ }; \mathbb{R}^{2 \ N})$. Therefore, $\tilde{\ }_\infty = 0$, as claimed.

(II) Suppose now that $C_\infty > 0$. Then, the desired relations (1.27)-(1.29) would follow directly from (2.39) and (1.21)-(1.22) by rescaling $\tilde{\ }_p; \tilde{\ }_p$ and passing to the limit as $p_j \to \infty$ since the rescaled multipliers $\tilde{\ }_p = C_p; \tilde{\ }_p = C_p$ are bounded in the product space

$$\mathrm{W}^{1; \frac{m}{m-2}}(\ ; \mathbb{R}^{2 \ N}) \quad \mathrm{BV}(\ ; \mathbb{R}^{2 \ N})$$

and therefore the sequence is sequentially weakly* compact, once we justify the convergence

$$\int \ _p \ \mathrm{M}u_{pi} \ \frac{\ _{pi}}{C_p} \mathrm{d}\mathcal{L}^n \ \to \ \int \ _\infty \ \mathrm{M}u_{\infty i} \ \ _{\infty i} \, \mathrm{d}\mathcal{L}^n; \tag{2.40}$$

as $p_j \to \infty$. To this end, we estimate

$$\int \ _p \ \mathrm{M}u_{pi} \ \frac{\ _{pi}}{C_p} \mathrm{d}\mathcal{L}^n \ \int \ _\infty \ \mathrm{M}u_{\infty i} \ \ _{\infty i} \, \mathrm{d}\mathcal{L}^n$$

$$\int j \ _p j \ \mathrm{M}u_{pi} \ \frac{\ _{pi}}{C_p} \ \mathrm{M}u_{\infty i} \ \ _{\infty i} \ \mathrm{d}\mathcal{L}^n \tag{2.41}$$

$$+ \int (\ _p \ \ _\infty) \ \mathrm{M}u_{\infty i} \ \ _{\infty i} \ \mathrm{d}\mathcal{L}^n :$$

i 9 -1.49as9626 Tf 5.811 -8.07-342.06 [([(95)]TJ/F13 6.9
Note now that by Theorem 1 we have $\ _p \ \overset{*}{\ } \ \ _\infty$ in L(J/F1 [(in)-382(L)]TJ/F10 6.973r[(i)]TJ/F8 9.9626 Tf :

and the latter inequality is true by the definition of ε. In conclusion, by Hölder's inequality and the above arguments, (2.46) yields

(2.47)

$$\int_\Omega \left| \mathrm{M}u_{p_i} - \frac{\Phi_{p_i}}{C_p}\right|^r d\mathcal{L}^n \le \left(\int_\Omega \left|\mathrm{M}u_{p_i}\right|^{\frac{nm(1-\varepsilon)}{2n-m}} d\mathcal{L}^n\right)^{\frac{1}{s}} \left(\int_\Omega \left|\frac{\Phi_{p_i}}{C_p}\right|^{\frac{n(1-\varepsilon)}{n-1}} d\mathcal{L}^n\right)^{\frac{r(n-1)}{n(1-\varepsilon)}}.$$

In view of (2.44)-(2.45), (2.42) ensues from (2.47) for any $t \in (1, r)$. Finally, (2.43) also follows from (2.47) and the Vitali convergence theorem, as from (2.44)-(2.45) we already know

$$\mathrm{M}u_{p_i} - \frac{\Phi_{p_i}}{C_p} \longrightarrow \mathrm{M}u_{\infty} - \Phi_{\infty} \quad \text{a.e. on } \Omega,$$

as $p_j \to \infty$, because $\mathrm{M} \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$. The theorem ensues.

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