ON A VECTOR-VALUED GENERALISATION OF VISCOSITY SOLUTIONS FOR GENERAL PDE SYSTEMS

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Abstract. We propose a theory of non-di erentiable solutions which applies to fully nonlinear PDE systems and extends the theory of viscosity solutions of Crandall-Ishii-Lions to the vectorial case. Our key ingredient is the discovery of a notion of extremum for maps which extends min-max and allows \nonlinear passage of derivatives" to test maps. This new PDE approach supports certain stability and convergence results, preserving some basic features of the scalar viscosity counterpart. In this rst part of our two-part work we introduce and study the rudiments of this theory, leaving applications for the second part.

Contents

1. Introduction 1

applicability of this approach. Without being exhaustive, some notable extensions of viscosity solution in either sense are given in [5, 6, 7, 11, 18, 21, 28, 29, 31].

It appears that the main restriction which to date has been unable to be removed is that VS apply to single equations with scalar-valued solutions, or at best to weakly coupled monotone systems which essentially can be treated componentwise as independent equations. Removing this constraint is hardly a straightforward task, as VS are essentially based on the scalar nature of the problem and on com-

de ne a tensor $S : T$ in $^{(q\ \ p)}\mathbb{R}^N$ by

$$S : T \ := \ S_{\ q:::\ \ p:::\ \ 1\, i_s:::i_1} T_{\ \ p:::\ \ 1\, i_s:::i_1}\ e_{\ q} \ \ ::: \ \ e_{\ p+1} : \tag{2.3}$$

For example, for $s = q = 2$ and $p = 1$, the tensor $S : T$ of (2.3) is a vector with components $S_{\ \ ij} T_{\ ij}$ with free index and the indices $;i;j$ are contracted. In particular, in view of (2.3), the second order linear system

$$A_{\ \ i\ \ j} \mathrm{D}^2_{ij} u \ + B_{\ \ k} \mathrm{D}_k u \ + C \ \ u \ = f\ ; \tag{2.4}$$

can be compactly written as $A : \mathrm{D}^2 u + B : \mathrm{D}u + Cu = f$, where the meaning of ":" in the respective dimensions is made clear by the context. Let now $\mathrm{P} : \mathbb{R}^n \to \mathbb{R}^N$ be linear map. We will *always* identify linear subspaces with orthogonal projections on them. Hence, we have the split $\mathbb{R}^N = [\mathrm{P}]^\top \ \ [\mathrm{P}]^\perp$ where $[\mathrm{P}]^\top$ and $[\mathrm{P}]^\perp$ denote range of P and nullspace of $\mathrm{P}^\top$ respectively. In particular, if $\in \mathbb{S}^{N-1}$, then $[\ ]^\top =$ is (the projection on) the line span$[\ ]$ and $[\ ]^\perp$ is (the projection on) the normal hyperplane I .

Symmetrised tensor products. The next notion plays a crucial role to what follows. The *symmetrised tensor product* is the operation

$$\vee \ : \ \mathbb{R}^N \ \ \mathbb{R}^N \ \to \mathbb{R}^{N\ \ N}_s \ : \ a \vee b \ := \ \frac{1}{2}\ a \ \ b + b \ \ a\ : \tag{2.5}$$

Obviously, $a \vee b = b \vee a$ and $a \vee a = a \ \ a$. Let us also record the identities

$$(a \vee b) : X \ = \ X : (a \ \ b) \ = \ a^\top X b\ ; \ \ \ X \in \mathbb{R}^{N\ \ N}_s ; \tag{2.6}$$

$$|a \ \ b|^2 \ = \ |a|^2 |b|^2\ ; \ \ \ |a \vee b|^2 \ = \ \frac{1}{2}\ |a|^2 |b|^2 + (a^\top b)^2 \ : \tag{2.7}$$

We will also need to consider tensor products "$\vee$" of higher order between $\mathbb{R}^N$ and the spaces $\mathbb{R}^N \ \ \mathbb{R}^n$ and $\mathbb{R}^N \ \ \mathbb{R}^{n\ \ n}_s$. If $\in \mathbb{R}^N$, $\mathrm{P} \in \mathbb{R}^N \ \ \mathbb{R}^n$, we view the tensor products $\ \mathrm{P}$ and $\mathrm{P}\ $ as maps $\mathbb{R}^n \to \mathbb{R}^N \ \ \mathbb{R}^N$. This allows to de ne

$$\vee \mathrm{P} \ : \ \mathbb{R}^n \ \to \mathbb{R}^{N\ \ N}_s ; \ \ \ \vee \mathrm{P} := \frac{1}{2}\ \ \ \mathrm{P} + \mathrm{P} \ \ \ : \tag{2.8}$$

c ts e P a

Obviously, $(\ \vee \mathrm{P})w = \ \vee (\mathrm{P}w)$ and $\vee \mathrm{P} = \mathrm{P} \vee$. Similarly, if $\mathrm{X} = \mathrm{X}_{\ \ ij} e \ \ e_{ij}$ J/Fe51-3.615 -8.07

Remark 4. It is evident that $\mathrm{R}_s^{Nn \quad Nn}$ is a proper subspace of $\mathrm{R}_s^{Nn \quad Nn}$ and that it can be equipped with both partial orderings \ " and \ ". It also evident that \ " is a stronger notion than \ ", in the sense that 0 implies 0. The known examples of rank-one convex quadratic form which are not convex imply that rank-one positivity is genuinely weaker that positivity.

3. Contact solutions for fully nonlinear PDE systems

In this section we introduce the basics of a theory of non-di erentiable solutions which applies to fully nonlinear systems of partial di erential equations of the form

$$\mathrm{F}(\ ; u; \mathrm{D}u; \mathrm{D}^2 u) = 0; \tag{3.1}$$

where $u : \mathrm{R}^n \quad ! \ \mathrm{R}^N$ and

$$\mathrm{F} : \tag{3.2}$$

In the following we will also need to consider closures of contact jets:

The new objects $J^{1,}$, $J^{2,}$ will be studied thoroughly later. Before presenting some explicit calculations of contact jets for a typical map to illustrate the working philosophy (which is analogous to the scalar case), we present a reformulation of Definitions 9-10.

Lemma 12 (Alternative definitions). *In the setting of Definitions 9-10, the implications* (3.11), (3.8) *can be respectively replaced by*

$$P \in J^{1,} u(x) \quad \Longrightarrow \quad F\big(x, u(x), P\big) \quad 0, \tag{3.12}$$

$$(P, X) \in J^{2,} u(x) \quad \Longrightarrow \quad F\big(x, u(x), P, X\big) \quad 0. \tag{3.13}$$

If moreover the nonlinearity F *is continuous, we can replace the* F *by* $^{>}$F.

Proof of Lemma 12. For brevity we exhibit only the second order case. Obviously, $J^{2,} u(x) \quad \overline{J}^{2,} u(x)$. Conversely, assume (3.13) and x $(P, X) \in \overline{J}^{2,} u(x)$. Then, t973846er-39437s

We now employ (4.3) to check directly that

$$(4.5)\qquad (\_ R) \quad \frac{R}{|R|} \quad = \quad \frac{R}{|R|}$$

with as in (4.1). The lemma follows.

We now show that symmetric products $\_ ( )$ coupled by the inequality induce "directed" orderings.

Proposition 14 (Induced partial orderings). *Let be in* $S^{N\ 1}$ *and* $^{?} = I$.

(i) If $v \in \mathbb{R}^N$*, then*

$$(4.6)\qquad \_ v \quad 0 \quad , \quad \begin{matrix} v = ({}^{>}v) \\ {}^{>}v \quad 0 \end{matrix} \quad , \quad \begin{matrix} {}^{?}v = 0 \\ {}^{>}v \quad 0 \end{matrix} \quad , \quad v = \quad |v| \ :$$

(ii) If $X \in \mathbb{R}^N \quad \mathbb{R}^{n\ \ n}_s$*, then*

$$\_ X \quad 0 \quad , \quad \begin{matrix} X = \quad ({}^{>}X) \\ {}^{>}X \quad 0 \end{matrix} \quad , \quad \begin{matrix} {}^{?}X = 0 \\ {}^{>}X \quad 0 \end{matrix} \quad , \quad \_ X \quad 0:$$

In particular, it follows that the orderings and coincide on the cone

$$(4.7)\qquad \Big\{ \quad \_ Y \quad \in \mathbb{R}^N; \ Y \in \mathbb{R}^N \quad \mathbb{R}^{n\ \ n}_s \Big\}$$

which is a subspace of the space (2.16) of separately symmetric tensors.

Proof of Proposition 14. (i) By Lemma 13, $\_ v \quad 0$ if and only if max $(\_ v)$ 0, hence if and only if $\frac{1}{2}(|v| + \ {}^{>}v) = 0$ and this says $v = \quad |v|$. The latter is equivalent to $v = ({}^{>}v)$ with ${}^{>}v \quad 0$ and to ${}^{?}v = 0$ with ${}^{>}v \quad 0$.
(ii) Suppose that $\_ X \quad 0$ and x $\in \mathbb{R}^N$ and $w \in \mathbb{R}^n$. Then, we have

$$(4.8)\qquad \begin{aligned} 0 &\quad (\_ X) : (\quad w) \quad (\quad w) \\ &= \frac{1}{2} \quad X_{\ ij} + \quad X_{\ ij} \quad w_i \quad w_j \\ &= \frac{1}{2} \quad X_{\ ij} w_i w_j \quad + \quad X_{\ ij} w_i w_j \\ &= \quad \_ (X : w \quad w) \ : \qquad : \end{aligned}$$

By (4.8), we obtain for any w xed that $\_ (X : w \quad w) \quad 0$. By employing (i) to the vector $v := X : w \quad w$, we see that

$${}^{>}(X : w \quad w) \quad 0 \ ; \qquad X \qquad ({}^{>}X) \ : w \quad w = 0;$$

for any w xed. Since w

and the last inequality follows by $^{>} X \quad 0$. Hence, $\_ \, X \quad 0$ as desired. Finally, the implication $\_ \, X \quad 0 \Rightarrow \_ \, X \quad 0$ is trivial.

Now we relate generalised and classical pointwise derivatives.

Theorem 15 (Contact jets and derivatives). *Let $u : \mathbb{R}^n \to \mathbb{R}^N$ be a map which is continuous at $x \in$.*
(a) If there exists one direction $\in \mathbb{S}^{N-1}$ *such that both $J^{1,} u(x)$ are nonempty, then u is differentiable at x and both $J^{1,} u(x)$ are singletons with element the gradient:*

$$(4.10) \qquad J^{1,} u(x) \neq \emptyset \implies J^{1,} u(x) = J^{1,} u(x) = Du(x) \, .$$

(b) If u is differentiable at x, then for all $\in \mathbb{S}^{N-1}$ the sets $J^{1,} u(x)$ are singletons with element the gradient:

$$(4.11) \qquad J^{1,} u(x) = Du(x) \, .$$

Moreover, whenever $(Du(x), X) \in J^{2,} u(x) \neq \emptyset$, we have the inequality

$$(4.12) \qquad \_ \, X \quad X^{+} \quad 0$$

which is equivalent to

$$(4.13) \qquad {}^{?} X \quad X^{+} = 0 \, , \quad {}^{>} X \quad X^{+} \quad 0.$$

(c) If u is twice differentiable at x, then for all $\in \mathbb{S}^{N-1}$ the sets $J^{2,} u(x)$ are nonempty, they contain $(Du(x), D^2u(x))$ and also

$$(4.14) \qquad J^{2,} u(x) = \Big\{ Du(x), D^2u(x) + \quad A \; : \; A \quad 0 \Big\} .$$

Moreover, we have the characterisations

$$(4.15) \qquad \begin{aligned} J^{2,} u(x) &= \Big\{ (Du(x), X) \; : \; \_ \; D^2u(x) \quad X \quad 0 \Big\} \\ &= \Big\{ (Du(x), X) \; : \; \_ \; D^2u(x) \quad X \quad 0 \Big\} \\ &= \left\{ (Du(x), X) \; : \; \begin{array}{l} {}^{>} D^2u(x) \quad X \quad 0, \\ {}^{?} D^2u(x) \quad X = 0 \end{array} \right\} . \end{aligned}$$

(d) If $v : \mathbb{R}^n \to \mathbb{R}^N$ is twice differentiable at x and $,\quad 0$, then

$$(4.16) \qquad J^{2,} (u + v)(x) = J^{2,} u(x) + Dv(x), D^2v(x) \, .$$

Proof of Theorem 15. (a) Let P $\in J^{1,} u(x) \neq \emptyset$. Then, by (3.3), we have

$$(4.17) \qquad \_ \; u(z + x) \quad u(x) \quad P \; z \; : \qquad o(|z|),$$

if $\in \mathbb{R}^N$, where "$o(1)$" is realised by $T(z)$: . We set $z := \varepsilon w$ for $\varepsilon > 0$ and for any $w \in S^{n-1}$

(c) We rst observe that by applying Proposition 14, all four sets appearing in the right hand sides of of (4.14), (

(e) We have as ¯ *3 z !* 0

Example 19 (Calculation of contact jets, cf. [24]). *Let $u : \mathbb{R} \to \mathbb{R}^N$ be given by*

$$u(z) := Az\,\chi_{(-\infty,0]}(z) + \Big(Bz + \frac{C}{2}z^2\Big)\chi_{(0,+\infty)}(z), \tag{4.38}$$

where $A, B, C \in \mathbb{R}^N$, $A + B \neq 0$. The contact jets of u at zero are

$$J^{1,\xi}u(0) = \begin{cases} \emptyset, & \xi \neq \pm\frac{A+B}{|A+B|},\\ \left\{\frac{B-A}{2} + t\frac{B+A}{2} \,:\, t \in [-1,+1]\right\}, & \xi = \pm\frac{A+B}{|A+B|}, \end{cases} \tag{4.39}$$

$$J^{2,\xi}u(0) = \begin{cases} \emptyset, & \xi \neq \pm\frac{A+B}{|A+B|},\\ \left\{\left(\frac{B-A}{2} + t\frac{B+A}{2}, X\right) \,:\, (t, X) \in S\right\}, & \xi = \pm\frac{A+B}{|A+B|}, \end{cases} \tag{4.40}$$

where

$$\begin{aligned} S := \ & \big((-1,+1) \times \mathbb{R}^N\big) \bigcup \Big(\{-1\} \times \{C - s(A+B) : s \ge 0\}\Big) \\ & \bigcup \Big(\{+1\} \times \{-s(A+B) : s \ge 0\}\Big). \end{aligned} \tag{4.41}$$

The proof of the above facts follows by a simple but lengthy computation by using directly the definition of contact jets.

5. Ellipticity and consistency with classical notions

Now we introduce the appropriate notion of ellipticity for fully nonlinear second order PDE systems and establish compatibility between classical and CS.

Definition 20 (Degenerate elliptic second order systems). Let $u : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^N$ be a C^2 map. The PDE system (3.1) is called *degenerate elliptic* when for all $(x, \eta, P) \in \Omega \times \mathbb{R}^N \times (\mathbb{R}^N \otimes \mathbb{R}^n)$ the map $\mathcal{F}(x, \eta, P, \cdot) : \mathbb{R}^N \otimes \mathbb{R}^{n \times n}_s \to \mathbb{R}^N$

Proof of Lemma 21. By Proposition 14, (5.3) is equivalent to

$$
\begin{array}{l} {}^{?}\ X \quad Y \ = 0; \\ {}^{>}\ X \quad Y \quad\ 0 \end{array} \quad =) \quad {}^{>}\ G(X) \quad G(Y) \quad 0: \tag{5.5}
$$

Assuming (5.5), we have ${}^{>}(X \quad Y) \quad 0$ and ${}^{>}(X \quad Y) = X \quad Y$ and also ${}^{>}\ G(X) \quad G(Y) \quad 0$. These relations yield

$$
\begin{aligned} 0 \quad & {}^{>}\ G(X) \quad G(Y) \quad {}^{>}(X \quad Y) \\ & = \ G(X) \quad G(Y) \end{aligned}
$$

(I ... $\in C^1(0,\infty)$ with $(0^+) = 0$, such that as

$$\frac{\big| u(z+x) - u(x) - Pz \big|^2}{(|z|)|z|} + (|z|)|z| \ldots$$

... nd contact jets). *Let* $u : \mathbb{R}^n \ldots \to \mathbb{R}^N$ *be* ...

... $N-1$ *and* $(P, X) \in \mathbb{R}^N \otimes (\mathbb{R}^n \otimes \mathbb{R}^n_s \ldots n)$. *Then, the* ...

). ...

case only 1=2 of the derivatives can be interpreted weakly, the rest 1=2 must exist classically".

In order to prove Theorems 26-27, we need a technical tool. Let R 2 R^N and 2 S$^{N\ \ 1}$. By Lemma 13, the tensor product _ R $_1$ R_

(i) $0 \in J^{p,-}R(0)$, *that is,* $\max\big(\ \_\, R(z)\big) \ \ o(|z|^p)$ *as* $\Omega \ni z \to 0$.
(ii) We have

$$(6.9) \qquad {}^{\top}R(z) \quad o(|z|^p) \quad \text{and} \quad \frac{\big({}^{\perp}R(z)\big)^2}{|R(z)|} = o(|z|^p) \quad \text{as } \Omega \ni z \to 0.$$

(iii) There exist maps $: \mathbb{R}^n \to {}^{\perp}\_\,\mathbb{R}^N$ *and* $: \mathbb{R}^n \to [0,\infty)$ *satisfying that* $(z) = o(|z|^{p/2})$ *and* $(z) = o(|z|^p)$ *as* $\Omega \ni z \to 0$ *and also*

$$(6.10) \qquad {}^{\top}R \qquad \text{and} \qquad {}^{\perp}R = \ \ \frac{1}{2}|\ \ |^2 + \Big(\frac{1}{2}|\ \ |^2\Big)^2 + \big({}^{\top}R\big)^2\ ^{1/2} \qquad \text{on } \Omega.$$

(iv) If we set $T := \big\{ |{}^{\perp}R| \ \ |{}^{\top}R| \big\}\ \Omega$, *then*

$$(6.11) \qquad \left.\begin{array}{l} {}^{\top}R(z) \quad o(|z|^p); \\ \dfrac{\big({}^{\perp}R(z)\big)^2}{{}^{\top}R(z)} = o(|z|^p); \end{array}\right\} \text{as } \Omega \cap T \ni z \to 0; \qquad |R(z)| = o(|z|^p); \quad \text{as } \Omega \setminus T \ni z \to 0.$$

We observe that when $N = 1$, then ${}^{\perp} \ \ 0$ and we recover a single inequality along $= \mathbb{R}$ $|^p\ z)\ |$

as $\Omega \ni z \to 0$. Since on $\Omega \cap \{ \,^{>}R > 0\}$ we have $1 \; |R| + \,^{>}R = |R|$ and on $\Omega \cap \{ \,^{>}R \; 0\} \cap \{R \neq 0\}$ we have $1 \; |R| \; \,^{>}R = |R|$, we infer that

$$\text{(6.16)} \qquad \frac{|R|}{4}\, \mathrm{sgn}(R) \quad s(R) \quad ^2 \qquad \frac{|R|^2 \quad | \,^{>}R|^2}{2|R|}(z) \;=\; o(|z|^p);$$

as $\Omega \ni z \to 0$. Thus, (6.9) and (6.16) imply (6.12), which is equivalent to (i). Let us now prove the equivalence between (ii) and (iii). If we assume (ii), we de ne

$$\text{(6.17)} \qquad := \quad \,^{>}R \;\; ^{+}; \qquad := \quad \,^{?}R|R| \;\; ^{\frac{1}{2}} \quad _{\{R\neq 0\} \cap \Omega}.$$

It follows that $\;,\;$ have the desired properties and by (6.17), we have $|R|| \;|^2 = | \,^{?}R|^2 = |R|^2 \quad | \,^{>}R|^2$. It follows that $|R|$ is the positive solution of the quadratic equation $t^2 \quad | \;|^2 t \quad | \,^{>}R|^2 = 0$. Hence,

$$\text{(6.18)} \qquad |R| \;=\; \frac{1}{2}| \;|^2 \;+\; \frac{1}{2}| \;|^2 \;\; ^2 \;+\; | \,^{>}R|^2 \;\; ^{1=2}.$$

Thus, (6.17) and (6.18) imply (6.10). Conversely, if we assume (iii) and let S be de ned by the formula giving R above, then S solves the equation $t^2 \quad | \;|^2 t$ $| \,^{>}R|^2 = 0$. Hence, by the above and perpendicularity, we have

$$\text{(6.19)} \qquad S^2 \;=\; | \,^{>}R|^2 \;+\; | \;|^2 S \;=\; |R|^2 \quad | \,^{?}R|^2 \;+\; | \;|^2 S \;=\; |R|^2.$$

This, $S = |R|$ and hence $\,^{?}R = \; |R|^{\frac{1}{2}}$. As a result, (6.10) implies (6.9) as claimed. We conclude by proving that (iv) is equivalent to (ii). For, let us split $\mathbb{R}^N$ as 2-15.571 Td [(and)-33313 Td [(+)]TJ/F14 9.9626 Tf 1d [3 and equip it with the norm $\|R\| := \max \; | \,^{>}R|, | \,^{?}R|$

Let $f\ _1; :::; \ _{N}\ _1 g$ be an orthonormal base of the hyperplane $^?$ $\mathbb{R}^N$. Then, R and the identity I can be written as:

$$(6.24)\qquad R = (\ ^> R) \ + \sum_{=1}^{N\ 1} (\ ^> R) \ ; \quad I = \ + \sum_{=1}^{N\ 1} \ :$$

By plugging (6.24) into (6.23), we obtain

$$(6.25)\qquad \_ \left[(\ ^> R) \ + \sum_{=1}^{N\ 1} (\ ^> R) \right] \ + \sum_{=1}^{N\ 1} \ :$$

By applying \: " to (6.25) and employing orthonormality of the base, we infer that $^> R +$ 0:. Let now $t\ 2\ \mathbb{R} n f0g$ and $2\ f1; :::; N\ \ 1g$ be xed and apply again \: $(t \ + \ )$ $(t \ + \ )$" to (6.25) to obtain

$$j\ j^4\ ^> R + \sum_{=1}^{N\ 1} \ ^> R \ ^>(t \ + \ ) \ ^>(t \ + \ ) \qquad j\ j^4 + \sum_{=1}^{N\ 1} \ ^>(t \ + \ )^2 :$$

By orthogonality of the base, we deduce that $t\ ^> R$ $t^2 +$ $+$ $^> R$. Since this holds for both t, we infer that

$$(6.26)\qquad ^> R \quad jtj + \frac{^> R}{jtj}$$

and the choice $t := (\ ^> R)= \ ^{1=2}$ in (6.26) implies $^> R^{\ 2}$ 4 $+$ $^> R$. By summing with respect to , we obtain

$$^? R^{\ 2} = \sum_{=1}^{N\ 1} \ ^> R^{\ 2} \quad 4(N \ 1) \quad ^> R + 4(N \ 1) \quad :$$

The above estimate implies the direction \=) " of Claim 32 for the choice $(jwj) :=$ $jwj\ ^1(4(N\ 1)\ (jwj))^{1=p}$. Conversely, assume the validity of the inequality in (6.22) for such a and set $(w) := (\ (jwj)jwj)^p$. Then, we have

$$(6.27)\qquad ^? R^{\ 2} \qquad ^> R + \ ;$$

locally near $0\ 2\ \mathbb{R}^n$. Since > 0 near zero, (6.27) readily gives $^> R$ $(w) =$ $o(jwj^p)$ as $w\ !\ 0$. By setting $T :=$ $^? R$ $^> R$ and $:=$ $^> R >$, the inequality (6.27) implies on $T\ \backslash$ that

$$\frac{^? R^{\ 2}}{^> R}(w) \qquad \frac{(w) \quad ^> R(w) + \ (w)}{^> R(w)} \qquad (w) + \frac{^2(w)}{^> R(w)} \qquad 2\ (w);$$

as $T\ \backslash\ \ 3\ w\ !\ 0$. Hence, by the implication (iv) =) (i) of Theorem 31, we obtain

$$(6.28)\qquad \max \ \_ R(w) \qquad o(jwj^p);$$

as $T\ \backslash\ \ 3\ w\ !\ 0$. On the other hand, on $T\ n$ we have $^> R$ and also $^? R$ $^> R$, hence by Lemma 13 we estimate

$$\begin{aligned} \max \ \_ R(w) &= \frac{1}{2}\ jR(w)j + \ ^> R(w) \quad jR(w)j = \\ &= \ ^> R^{\ 2} + \ ^? R^{\ 2\ \frac{1}{2}} \quad ^2(w) + \ ^2(w)^{\ \frac{1}{2}} = \ ^p\overline{2}\ (w); \end{aligned}$$

as $z \to 0$. By increasing the $o(1)$ functions appearing in the summands appropriately, we incorporate the $o(|z|^4)$ term in the rst summand and therefore obtain that $(\mathrm{P}, \mathrm{X} \quad A) \in J^{2,} u(x)$, as claimed.

7. The extremality notion of contact maps

So far, our central objects of study have been contact jets, a certain type of generalised pointwise derivatives. Jets in fact introduce in an implicit non-trivial fashion an extremality notion for maps, which we will now exploit. This notion extends min and max of scalar functions to the vector-valued case, e ectively extending the "Maximum Principle calculus" ($\mathrm{D}u = 0$ and $\mathrm{D}^2u \quad 0$ at maxima of u) to the vectorial case. This device allows the "nonlinear passage of derivatives to test maps". This extremality notion, although simple in its form, presents peculiarities and is not obvious how it arises. Hence, we have chosen to base the PDE theory of CS to jets rather than to extrema, since jets seem more reasonable due to the formal resemblance to their scalar counterparts.

Motivation. We begin by motivating the notions that follow. Let $u : \mathbb{R} \to \mathbb{R}^N$ be a smooth curve. Every reasonable de nition of extremal point $u(x) \in \mathbb{R}^N$ at $x \in \mathbb{R}$ must imply that $|u'(x)| = 0$. However, this is impossible if $N \quad 2$ as the example of unit speed curves certi es for which $|u'| \quad 1$. In order to succeed we must radically change our point of view of "extremals". The idea is to relax the pointwise notion to a exible *functional notion of "extremal map"* which takes into account the possible "twist". Our viewpoint is the following: if $N = 1$ and $u : \mathbb{R} \to \mathbb{R}$ has a maximum $u(x) \in \mathbb{R}$ at $x \in \mathbb{R}$, then we can identify the extremum $u(x)$ with the constant function $\quad u(x) : \mathbb{R} \to \mathbb{R}$ which passes through x (Figure 1(a)).

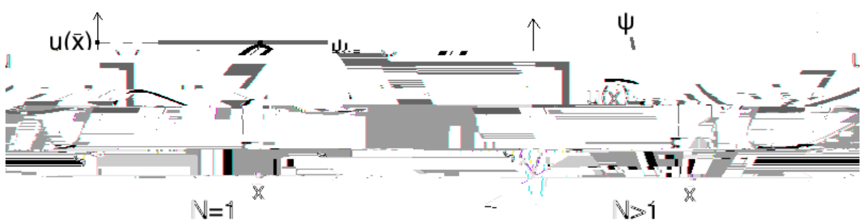

Figure 1(a). Figure 1(b).

When $N \quad 2$ we can view extrema as maps $\quad : \mathbb{R} \to \mathbb{R}^N$ passing at x through $u(x)$ which generally are *nonconstant* (Figure 1(b)).

Going back to $N = 1$, we see that maximum can be viewed as a constant function "touching u" at x in the direction $\quad = +1$ and minimum as "touching u" at x in the direction $\quad = \quad 1$. When $N \quad 2$, there still exists a sti er notion of "touching u" at x by a ma11.956 Td [(u)]TJ/F8 9.966 Tf 5.703 0 T3i3[(N1(d51(function)]T55ed)-251(as)-251(a)-251(consta

De nition 34 (Contact maps)**.** Let $u : \mathrm{R}^n \quad \bar{} \quad ! \ \mathrm{R}^N$ be continuous and x $x\ 2\ \bar{}$ and $\ 2\ \mathrm{S}^{N\ \ 1}$.

(1) *The map $\ 2\ C^1(\mathrm{R}^n)^N$ is a rst contact -map of u at x* if $(x) = u(x)$ and for every cone C_x, there is a neighbC

We will shortly see that contact maps constitute an appropriate notion of extremum for PDE theory. Let us first connect contact maps to contact jets. We will consider only the second order case and $x \in$ and we refrain from providing details for the first order case and boundary points which can be done by simple modifications. Given a continuous $u : \mathbb{R}^n \to \mathbb{R}^N$, $x \in$ and $\in \mathbb{S}^{N-1}$, set

(7.6)
$$D^{2,} u(x) := \left\{ \big(D(x), D^2(x)\big) \;\middle|\; \begin{array}{l} \in C^2(\mathbb{R}^n)^N;\ (x) = u(x)\ \&\ \forall \text{ cone } C_x; \\ ^{\perp}(u\ \ )^{2}\ \ (C_x)^2\ \ \ ^{\top}(u\ \ )\ \text{ near } x \end{array} \right\}.$$

Theorem 35 (Equivalence between extremality and jets). *If $u : \mathbb{R}^n \to \mathbb{R}^N$ is continuous, $x \in$ and*

Hence,

$$
\begin{aligned}
{}^{?}u \quad T_{2;x} \quad {}^{2} &= 2 \Big[\quad {}^{>}u \quad T_{2;x} \quad \Big] + \Big\{ 4 \, {}^{2} + 2 \quad {}^{?}R_{2;x} \quad {}^{2} \Big\} \\
& \quad 2 \quad + |\, {}^{?}R_{2;x} \,| \Big[\quad {}^{>}u \quad T_{2;x} \quad + 2 \quad + |\, {}^{?}R_{2;x} \,| \quad \Big] .
\end{aligned}
$$

The \contact principle calculus" result we establish below explains why contact maps of the solution play the role of smooth \test maps" for the PDE system.

Theorem 38 (Nonlinear passage of derivatives to contact maps). *Suppose $u : \mathbb{R}^n$! $\mathbb{R}^N$ is a continuous map, twice di erentiable at x 2 . Let 2 $C^2(\mathbb{R}^n)^N$ and 2 $\mathbb{S}^{N\ 1}$ be given. Consider the following statements:*

(i) is a second contact -map of u at x 2 .
(ii) We have

$$\begin{cases} \mathrm{D}\ u \quad (x) = 0; \\ \_\ \mathrm{D}^2\ u \quad (x) \quad 0: \end{cases}$$

Then, (i) implies (ii). Moreover, (ii) implies (i) if moreover $^{>}\mathrm{D}^2\ u \quad (x) < 0$.

Trivial modi cations in the arguments of the proof of Theorem 38 that follows readily imply the following consequence.

adding a "viscosity term", there exists some extra information which is trivial in the scalar case and allows convergence of the approximating solutions. In order to make this statement precise, we introduce an auxiliary notion of sequential derivatives needed in the exploitation of stability and approximation.

Definition 43 (Approximate derivative)**.** let $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ be a continuous map. The set of *Approximate first jets* of u at $x \in \Omega$ is

$$A^1u(x) := \left\{ P \in \mathbb{R}^N \otimes \mathbb{R}^n \ \Big|\ \liminf_{r \to 0} \max_{|z|=r} \frac{\big|u(z+x) - u(x) - Pz\big|}{r} = 0 \right\} \tag{8.2}$$

The set of *Approximate second derivatives* of u at $x \in \Omega$ is

$$A^2u(x) := \Bigg\{ (P, X) \in \mathbb{R}^N \otimes \mathbb{R}^n \times \mathbb{R}^N \otimes \mathbb{R}^{n \times n}_s \ \Big|$$

$$\liminf_{r \to 0} \max_{|z|=r} \frac{\big|u(z+x) - u(x) - Pz - \frac{1}{2} X : z \otimes z\big|}{r^2} = 0 \Bigg\} \tag{8.3}$$

Remark 44. Obviously, if u is (twice) differentiable at x, then $A^1u(x) = \{Du(x)\}$ and $A^2u(x) = \{(Du(x), D^2u(x))\}$. In general, approximate derivatives may exist at non-differentiability points, as it happens for the Lipschitz continuous function $u : \mathbb{R} \longrightarrow \mathbb{R}$ given by $u(z) := z\cos(1/|z|)$ for $z \neq 0$ and $u(0) = 0$ for which $[-1, +1] = A^1u(0) \neq \emptyset$, while $u'(0)$ does not exist. This follows from the observation $\max_{|z|=r}(|u(z) - u(0) - pz|/r) = |\cos(1/r) - p|$.

The following is the main approximation result for contact jets. It follows that *contact jets perturb to contact jets under weak* convergence in the local Lipschitz space*, together with a *technical assumption* which appears to be satisfied in the cases of interest. This assumption requires convergence of codimension-one projections of sequential jets along a sequence of hyperplanes.

Theorem 45 (Approximation of contact jets)**.** *Let* $u : \mathbb{R}^n \supseteq \Omega \longrightarrow$ [illegible]

Corollary 47. *By* (8.6) *and Lemma* 2, *we deduce that* $\mathrm{Y} = e^{\perp}\mathrm{X}$ *on the hyperplane* $(e^{\perp})^{\perp}$ *of* $\mathbb{R}^N$ *and* $(e^{\perp})^{\top}(\mathrm{Y} \quad e^{\perp}\mathrm{X}) \quad 0$ *along* $(e^{\perp}) \quad (e^{\perp})$ *(Figures 6(a),(b)).*

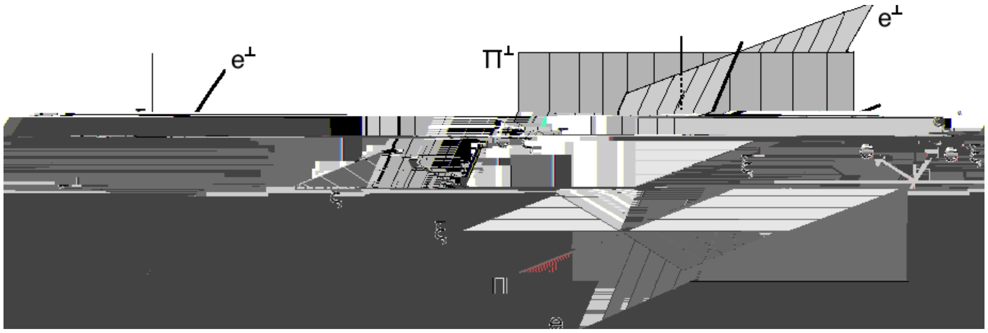

Figure 6(a): Illustration for $N = 2$ *Figure 6(b): Illustration for* $N = 3$

Remark 48. Proposition 46 is optimal, as the example $u(z) := \quad |z| \cos^2(1/z) \quad_{\mathbb{R}\setminus\{0\}}$ shows: indeed, we have $J^{1,+}u(0) = \{0\} = A^1u(0)$ but $u'(0)$ does not exist, although $J^{1,+}u(0)$ and $A^1u(0)$ coincide and are singletons. Namely, some "loss of information" occurs when $e = \quad$ (i.e. when $e^{\perp} \quad = 0$).

Proof of Proposition 46. Since $\mathrm{Q} \in A^1(e^{\perp}u)(x)$, exists $r_j \to 0$ such that

$$e^{\perp} \quad u(wr_j + x) \quad u(x) \quad r_j \mathrm{Q}w \quad = o(r_j) \tag{8.7}$$

as $j \to \infty$, for all $w \in \mathbb{S}^{n-1}$. Fix $\quad \in e^{\perp}$. If $e \neq \quad$, then $e^{\perp} n \quad^{\perp} \neq \emptyset$, and for any $\varepsilon > 0$, exists $\quad_\varepsilon \in e^{\perp} n \quad^{\perp}$ with $| \quad_\varepsilon | \quad \varepsilon$. Since $(\quad_\varepsilon \quad _\varepsilon)e^{\perp} = \quad_\varepsilon \quad _\varepsilon$, (8.7) gives

$$o(r_j) = \quad (\quad^{\top}_\varepsilon) \quad^{\top}_\varepsilon \Big[u(wr_j + x) \quad u(x) \quad r_j \mathrm{Q}w \Big] \tag{8.8}$$

as $j \to \infty$. Since $\mathrm{P} \in J^{1,} u(x)$, we have

$$o(|z|) = (\quad^{\top}_\varepsilon) \quad^{\top}_\varepsilon \Big[u(z + x) \quad u(x) \quad (e^{\perp}\mathrm{P})z \Big] \tag{8.9}$$

as $z \to 0$. By choosing $z = wr_j$ in (8.9) and summing (8.9) and (8.8), we get $(\quad^{\top}_\varepsilon) \quad^{\top}_\varepsilon \quad \mathrm{Q} \quad e^{\perp}\mathrm{P} \quad w \quad o(1)$ as $r_j \to 0$. By interchanging w with $\quad w$, using that $j \quad^{\top}_\varepsilon \quad j > 0$ and letting $j \to \infty$, we get $\quad^{\top}_\varepsilon \quad \mathrm{Q} \quad e^{\perp}\mathrm{P} \quad = 0$. By letting $\varepsilon \to 0$, we find $\quad^{\top} \quad \mathrm{Q} \quad e^{\perp}\mathrm{P} \quad = 0$. Since $\quad \in e^{\perp}$ is arbitrary and $\mathrm{Q} = e^{\perp}\mathrm{Q}$, we conclude that $\mathrm{Q} = e^{\perp}\mathrm{P}$. If $e = \quad$ and moreover $J^{2,} u(x) \neq \emptyset$, Corollary 28 and Definition 43 imply that $\mathrm{Q} = e^{\perp}\mathrm{P} = \mathrm{D}(e^{\perp}u)(x)$. Further, if $(\mathrm{Q}, \mathrm{Y}) \in A^2(e^{\perp}u)(x)$, it trivially follows that $\mathrm{Q} \in A^1(e^{\perp}u)(x)$. Since $(\mathrm{P}, \mathrm{X}) \in J^{2,} \quad (x) \neq \emptyset$, part (a) implies that $\mathrm{Q} = e^{\perp}\mathrm{P}$. Fix $\quad \in e^{\perp}$. By arguing as eaelier, there exists $r_j \to 0$ such that

$$(8.10) \quad o(r_j^2) \qquad : \quad \_ \quad \Big[u(wr_j + x) \quad u(x) \quad r_j \mathrm{Q}w \quad \frac{r_j^2}{2}(e^{\perp}\mathrm{X}) : w \quad w \quad \Big] ;$$

$$o(r_j^2) \qquad : \quad \_ \quad \Big[$$ J/F11 9.9626 Tf 21.585 0 Td [( )]TJ/F14 9.9626 Tf 7. 7o 3051

By passing to the limit we conclude that $(e^{\perp})\vee(Y \quad e^{\perp}X) \quad 0$.

Proof of Theorem 45. Since $(P;X) \in J^{2,}\ u(x)$, we have $(\ ^{\top}P;\ ^{\top}X) \in J^{2,+}(\ ^{\top}u)(x)$. By the C^0 convergence $\ ^{\top}u_m \to\ ^{\top}u$ as $m \to \infty$, standard arguments of the scalar case (see e.g. [14, 24]) imply that there exists $x_m \to x$ and $(p_m;X_m) \in J^{2,+}(\ ^{\top}u_m)(x_m)$ such that *(tains) -4.11Tu;*

$$(p_m;X_m) \to (\ ^{\top}P;\ ^{\top}X)\,; \quad \text{as } m \to \infty. \tag{8.12}$$

Since $u_m \in C^2(\ )^N$, it follows that $p_m = \ ^{\top}Du_m(x_m)$ and $X_m \quad ^{\top}D^2u_m(x_m)$. By Theorem 15, the set $J^{2,}\ u_m(x_m)$ contains

$$(P_m;X_m) := \Big(\tag{8.13}$$

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