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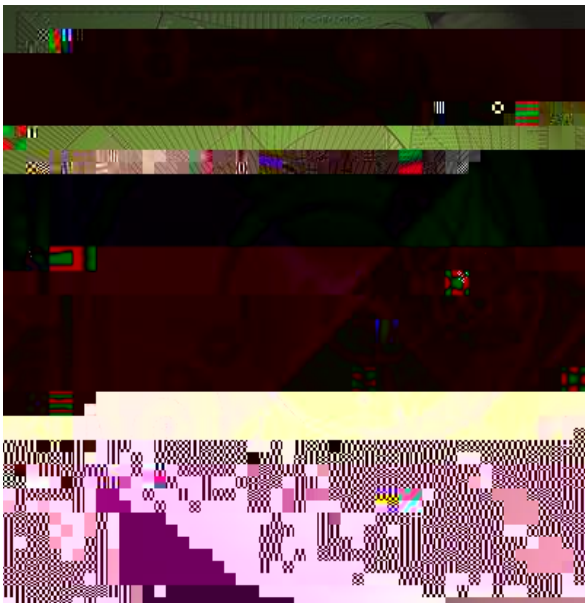

VECTORIAL VARIATIONAL PRINCIPLES IN L^∞ AND THEIR CHARACTERISATION THROUGH PDE SYSTEMS

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Abstract. We discuss two distinct minimality principles for general supremal first order functionals for maps and characterise them through solvability of associated second order PDE systems. Specifically, we consider Aronsson's standard notion of absolute minimisers and the concept of ∞-minimal maps introduced more recently by the second author. We prove that C^1 absolute minimisers characterise a divergence system with parameters probability measures and that C^2 ∞-minimal maps characterise Aronsson's PDE system. Since in the scalar case these different variational concepts coincide, it follows that the non-divergence Aronsson's equation has an equivalent divergence counterpart.

1. Introduction

Let $n, N \in \mathbb{N}$ and $\mathrm{H} \in C^2(\Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n})$ with $\Omega \subseteq \mathbb{R}^n$ an open set. In this paper we consider the supremal functional

$$\mathrm{E}_\infty(u,\mathcal{O}) := \operatorname*{ess\,sup}_{\mathcal{O}} \mathrm{H}(\cdot, u, \mathrm{D}u), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N), \quad \mathcal{O} \Subset \Omega, \tag{1.1}$$

defined on maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$. In (1.1) and subsequently, we see the gradient as a matrix map $\mathrm{D}u = (\mathrm{D}_i u_\alpha)^{\alpha=1\dots N}_{i=1\dots n} : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N\times n}$. Variational problems for (1.1) have been pioneered by Aronsson in the 1960s in the scalar case $N = 1$ ([2]-[6]). Nowadays the study of such functionals (and of their associated PDEs describing critical points) form a fairly well-developed area of vivid interest, called Calculus of Variations in L^∞. For pedagogical general introductions to the theme we refer to [10

In the scalar case of $N = 1$, Aronsson's concept of absolute minimisers turns out to be the appropriate substitute of mere minimisers. Indeed, absolute minimisers possess the desired uniqueness properties subject to boundary conditions and, most importantly, the possibility to characterise them through a necessary (and sufficient) condition of satisfaction of a certain nonlinear nondivergence second order PDE, known as the Aronsson equation ([9, 10, 12, 11, 13, 14, 17, 18, 20, 21, 27, 39, 44]). The latter can be written for functions $u \in C^2(\Omega)$ as

$$H_P(\cdot,u,Du) \cdot D\big(H(\cdot,u,Du)\big) = 0. \tag{1.3}$$

The Aronsson equation, being degenerate elliptic and non-divergence when formally expanded, is typically studied in the framework of viscosity solutions. In the above, H_P, H_η, H_x denotes the derivatives of $H(x,\eta,P)$ with respect to the respective arguments and "$\cdot$" is the Euclidean inner product.

In this paper we are interested in characterising appropriately defined minimisers of (1.1) in the general vectorial case of $N \geq 2$ through solvability of associated PDE systems which generalise the Aronsson equation (1.3). As the wording suggests and we explain below, when $N \geq 2$ Aronsson's notion of Definition 1 is no longer the unique possible L^∞ variational concept. In any case, the extension of Aronsson's equation to the vectorial case reads

$$\begin{aligned} & H_P(\cdot,u,Du)\, D\big(H(\cdot,u,Du)\big) \\ & + H(\cdot,u,Du)\,[H_P(\cdot,u,Du)]^\perp \Big(\mathrm{Div}\big(H_P(\cdot,u,Du)\big) - H_\eta(\cdot,u,Du)\Big) = 0. \end{aligned} \tag{1.4}$$

In the above, for any linear map $A : \mathbb{R}^n \to \mathbb{R}^N$, $[A]^\perp$ symbolises the orthogonal projection $\mathrm{Proj}_{R(A)^\perp}$ on the orthogonal complement of its range $R(A) \subseteq \mathbb{R}^N$. We will refer to the PDE system (1.4) as the "Aronsson system", in spite of the fact it was actually derived by the second author in [28], wherein the connections between general vectorial variational problems and their associated PDEs were first studied, namely those playing the role of Euler-Lagrange equations in L^∞. The Aronsson system was derived through the well-known method of L^p-approximations and is being studied quite systematically since its discovery, see e.g. [28]-[31], [34, 37]. The additional normal term which is not present in the scalar case imposes an extra layer of complexity, as it might be discontinuous even for smooth solutions (see [29, 31]).

For simplicity and in order to illustrate the main ideas in a manner which minimises technical complications,

(i) u is a *rank-one absolute minimiser*, namely it minimises with respect to essentially scalar variations vanishing on the boundary along xed unit directions:

$$
(1.5) \qquad \begin{array}{l} 8\, O \;\mathrm{b}\;\; ;\, 8 \;\; 2\, \mathrm{R}^N \\ 8 \;\; 2\, C^1_0(\overline{O}; \mathrm{span}[\;\;]) \end{array} \quad =) \quad \mathrm{E}_1(u; O) \quad \mathrm{E}_1(u + \;\; ; O):
$$

(ii) u has *1-minimal area*

(III) *For any* O b *and any* 2 $C^1_0(\overline{O}$

2. Proofs and a maximum-minimum principle for $H(\cdot,u,Du)$

In this section we prove our main results Theorems 4-5. Before delving into that, we establish a result of independent interest, which generalises a corresponding result from [30].

Proposition 7 (Maximum-Minimum Principles). *Suppose Let $u \in C^2(\Omega;\mathbb{R}^N)$ be a solution to (1.8), such that H satisfies*
(a) *$H_P(\cdot,u,Du)$ has full rank on Ω,*
(b) *there exists $c>0$ such that*

$$\xi^\top H_P(x,\eta,P) : \xi^\top P \geq c\,\big|\xi^\top H_P(x,\eta,P)\big|^2,$$

for all $\xi \in \mathbb{R}^N$ and all $(x,\eta,P) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

Then, for any $O \Subset \Omega$ we have:

$$\sup_O H(\cdot,u,Du) = \max_{\partial O} H(\cdot,u,Du), \tag{2.1}$$

$$\inf_O H(\cdot,u,Du) = \min_{\partial O} H(\cdot,u,Du). \tag{2.2}$$

The proof is based on the usage of the following flow with parameters:

Lemma 8. *Let $u \in C^2(\Omega;\mathbb{R}^N)$. Consider the parametric ODE system*

$$\begin{cases} \dot\gamma(t) = \xi^\top H_P(\cdot,u,Du)\big(\gamma(t)\big), & t \neq 0, \\ \gamma(0) = x, \end{cases} \tag{2.3}$$

for given $x \in \Omega$ and $\xi \in \mathbb{R}^N$. Then, we have

$$\frac{d}{dt}\Big(H(\cdot,u,Du)\big(\gamma(t)\big)\Big) = \Big(\xi^\top H_P(\cdot,u,Du)\, D\big(H(\cdot,u,Du)\big)\Big)\big(\gamma(t)\big), \tag{2.4}$$

$$\frac{d}{dt}\big(\xi^\top u \circ \gamma\big)(t) \geq c\,\Big|\xi^\top H_P(\cdot,u,Du)\big(\gamma(t)\big)\Big|^2. \tag{2.5}$$

Proof of Lemma 8. The identity (2.4) follows by a direct computation and (2.3). For the inequality (2.5), we have

$$\begin{aligned} \frac{d}{dt}\big(\xi^\top u \circ \gamma\big)(t) &= \big(\xi^\top Du \circ \gamma\big)(t) : \dot\gamma(t) \\ &= \big(\xi^\top Du \circ \gamma\big)(t) : \xi^\top H_P(\cdot,u,Du)\big(\gamma(t)\big) \\ &\geq c\,\Big|\xi^\top H_P(\cdot,u,Du)\big(\gamma(t)\big)\Big|^2. \end{aligned}$$

The lemma ensues.

Proof of Proposition 7. Fix $O \Subset \Omega$. Without loss of generality, we may suppose O is connected.

Consider first the case where $\mathrm{rk}\, H_P(\cdot,u,Du) \equiv n \leq N$. Then, the matrix-valued map $H_P(\cdot,u,Du)$ is pointwise left invertible. Therefore, by (1.8

Consider now the case where $\mathrm{rk}\,\mathrm{H}_P(\cdot; u; \mathrm{D}u) \quad N \quad n$. Fix $x \in \Omega$ and a unit vector $\ \in \mathbb{R}^n$ and consider the parametric ODE system (2.3) of Lemma 8. By the fullness of the rank of $\mathrm{H}_P(\cdot; u; \mathrm{D}u)$, we have that

Since g 2 $C^1(\mathbb{R}^n$

on $\overline{O}$. Since $O(u)$ $\overline{O}$, the identity (2.6) yields u is a critical point. Since by assumption $\mathrm{H}(x; \ ; \ )$ is convex on $\mathbb{R}^N$ $\mathbb{R}^{N \ n}$, it follows that $\mathrm{E}_{1}(\ ; O)$ is convex on $W^{1; 1}(O; \mathbb{R}^N)$. Hence, the critical point u is in fact a minimum point for this class of variations. This establishes our claim.

(III) This is an immediate corollary of items (I) and (II).

Now we conclude by establishing Theorem 5.

Proof of Theorem 5. (I) $=)$ (II): If u is an absolute minimiser, then by (2.6)-(2.7) we have

$$\max_{O(u)} \ \mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad = 0;$$

for any $\ 2 \ C^1_0(\overline{O}; \mathbb{R}^N)$. By replacing with , the above yields the pointwise equality

$$\mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad = 0$$

on $O(u)$. Then, for any Radon probability measure $\ 2 \ P(\overline{O})$ with supp( ) $O(u)$, we have

$$\int_{\overline{O}} \mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad \mathrm{d} \ = 0$$

for all $\ 2 \ C^1_0(\overline{O}; \mathbb{R}^N)$. Hence, we have shown that

$$\mathrm{div} \ \mathrm{H}_P(\ ; u; \mathrm{D}u) \qquad + \mathrm{H} \ (\ ; u; \mathrm{D}u) \ = 0;$$

in the dual space $(C^1_0(\overline{O}; \mathbb{R}^N))$.

(II) $=)$ (III): By assumption we have

$$\int_{\overline{O}} \mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad \mathrm{d} \ = 0$$

for all $\ 2 \ C^1_0(\overline{O}; \mathbb{R}^N)$ and all Radon probability measures $\ 2 \ P(\overline{O})$ with supp( ) $O(u)$. Fix any $x \ 2 \ O(u)$. By choosing the Dirac measure $\ 2 \ P(\overline{O})$ given by

$$:= \ _x$$

which evidently satisfies supp( ) $= fxg$ $O(u)$, we obtain

$$\begin{aligned} &\mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad _x \\ &\qquad = \int_{\overline{O}} \mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad \mathrm{d} \\ &\qquad = 0; \end{aligned}$$

for any $x \ 2 \ O(u)$. The conclusion ensues.

(III) $=)$ (I): By assumption we have

$$\mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad 0$$

on $O(u)$. By (2.6)-(2.7), this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{E}_{1}(u + t \ ; O) = \max_{O(u)} \ \mathrm{H}_P(\ ; u; \mathrm{D}u) : \mathrm{D} \ + \mathrm{H} \ (\ ; u; \mathrm{D}u) \qquad = 0:$$

Hence, since by assumption $\mathrm{H}(x; \ ; \ )$ is convex on $\mathbb{R}^N$ $\mathbb{R}^{N \ n}$, it follows that $\mathrm{E}_{1}(\ ; O)$ is convex on $W^{1; 1}(O; \mathbb{R}^N)$. Hence, the critical point u is in fact a minimum point and this completes the proof.

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