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by

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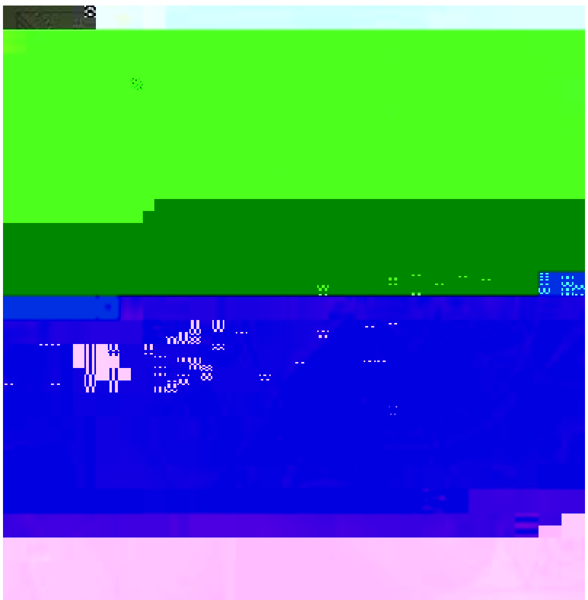

Surface permeability, capillary transport and the Laplace-Beltrami problem.

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We have established previously, in a lead-in study, that the spreading of liquids in particulate porous media at low saturation levels, characteristically less than 10% of the void space, has very distinctive features in comparison to that at higher saturation levels. In particular, we have found that the dispersion process can be accurately described by a special class of partial di erential equations, the super-fast non-linear di usion equation. The results of mathematical modelling have demonstrated very good agreement with experimental observations. However, any enhancement of the accuracy and predictive power of the model, keeping in mind practical applications, requires the knowledge of the e ective surface permeability of the constituent particles, which de nes the global, macroscopic permeability of the particulate media. In the paper, we demonstrate how this quantity can be determined through the solution of the Laplace-Beltrami Dirichlet problem, we study this using the well-developed surface nite element method.

I. INTRODUCTION

Liquid distributions and transport in particulate porous media, such as sand, at low saturation level**s**, de ned in our study as the ratio of the liquid volume V_L to the volume of available voids V_E in a sample volume element V, $s = \frac{V_L}{V_E}$, have many distinctive features. Theoretically, as we have shown previously, the liquid dispersion can be described by a special class of mathematical models, the superfast non-linear di usion equation [1]. Unlike in the standard porous medium equation, which is a paradigm of research in porous media [2], in this special case, the non-linear coe cient of di usion $D(s)$ demonstrates divergent behaviour as a function of saturation s, $D(s) / \ (s \quad s_0)^{\ 3=2}$, where s_0 is some minimal saturation level.

In practical applications, the analysis of this regime of wetting is crucial for studies of biological processes, such as microbial activity, and spreading of persistent (non-volatile) liquids in soil compositions and dry porous media commonly found in arid natural environments and industrial installations [1, 3].

If we consider liquid distributions on the grain size length scale, one would observe that when the saturation level s is reduced to (or below) the critical level $s_c \quad 10\%$, the liquid domain predominantly consists of isolated liquid bridges formed at the point of particle contacts [1, 4{8], see Fig. 1 for illustration. The formation of liquid bridges is characteristic for the so-called pendular regime of wetting. In this regime, the liquid bridges are only connected via thin lms formed on the rough particle surfaces and serve as variable volume reservoirs, where the capillary pressure p depends directly on the amount of the liquid in the bridge V_b

$$p \qquad p_0 \left(\frac{R^3}{V_b}\right)^{1=2}: \qquad (1)$$

Here, $p_0 = \frac{2}{R}$, is the coe cient of the surface tension of the liquid and R is an average radius of the porous medium particles [1, 4]. The spreading process in such conditions only occurs over the rough surface of the elements of the particulate porous media, Fig. 1.

Microscopically, the liquid creeping ow through the surface roughness of each particle can be described by a local Darcy-like relationship between the surface ux density q and averaged (over some area containing many surface irregularities) pressure in the grooves

$$\frac{m}{\ } r \quad = q: \qquad (2)$$

Here, is liquid viscosity and k_m is the local coe cient of permeability of the rough surface, which proportional to the average amplitude of the surface roughness $_R$, that is the width of the surface layer conducting the liquid ux, $k_m / \ {}^2_R$ [9].

FIG. 1. Illustration of the liquid distribution in particulate porous media (grey) with pendular rings (blue) at low saturation levels.

Macroscopically, that is after averaging over some vol-

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which, if it is found, allows to calculate the total flux through the particle element

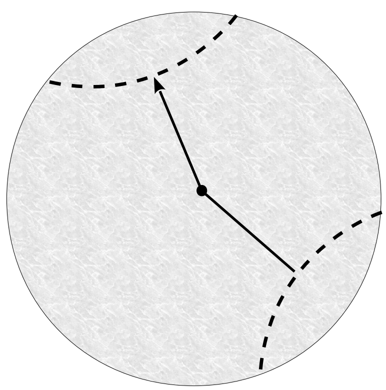
$$Q_T = {}_R \frac{m}{} \int_{\partial \ {}_1} \frac{\partial}{\partial n}\, dl = {}_R \frac{m}{} \int_{\partial \ {}_2} \frac{\partial}{\partial n}\, dl;$$

where n is the normal vector to the domain boundaries $\partial \ {}_{1;2}$ on the surface, ${}_R$ is the average amplitude of the surface roughness, that is the width of the surface layer conducting the liquid flux and the line integral is taken along a closed curve in ${}_0$, for example the boundary $\partial \ {}_1$.

If the total flux Q_T is determined, one can define the global permeability coefficient of a single particle K_1. This can be done, if we assume that the particle has a characteristic size D and so that it can be enclosed in a volume element $V = D^3$ with the characteristic side surface area D^2. Then, the effective flux density Q can be represented in terms of K_1 (and the total flux Q_T)

$$Q = \frac{Q_T}{D^2} = \frac{K_1}{} \frac{{}_2 \quad {}_1}{D};$$

if the flow is driven by the constant pressure difference ${}_2 \quad {}_1$ applied to the sides of the volume element.



How does the result a ect the super-fast di usion model (3), and basically how can it be incorporated into the main di usion equation? If we approximate the permeability coe cient K by K_1 obtained in the azimuthally symmetric case at $_1 = \ _0$, and, using an approximate relationship between the radius of curvature $R \sin \ _0$ of the boundary contour @ $_1$ and the pendular ring volume [6], one can show

$$\sin^2 \ _0 \quad \sqrt{s \quad s_0}$$

and at $_0 \quad 1$ or $(s \quad s_0) \quad 1$

$$K \quad 2\frac{R}{R}\frac{k_m}{j \ln(s \quad s_0)j}: \tag{11}$$

As one can see from (11), the distinctive particle shape results in logarithmic correction to the main non-linear superfast-di usion coe cient $D(s) = \frac{D_0}{(s \quad s_0)^{3=2}}$, such that

$$D(s) \ / \ \frac{1}{j \ln(s \quad s_0)j(s \quad s_0)^{3=2}}:$$

Apparently, the correction will mitigate to some extent the divergent nature of the dispersion at the very small saturation levels $s \quad s_0$, smoothing out the characteristic dispersion curves.

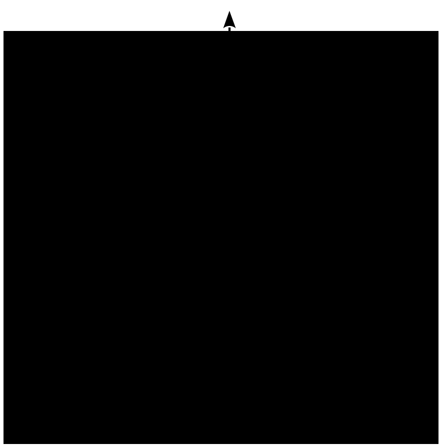


FIG. 3. Illustration of the triangular tessellation of the truncated spherical surface domain $_0$ with a normal vector n_0 at $= 150$ and $_0 = \ _{1grmd}$ j113 7582t-455(lll7h)-rr.962680(p)-on((trun7h)i8(trt-455(lll7h)-4ergy62680(n)180.962680([17]f 5567lV

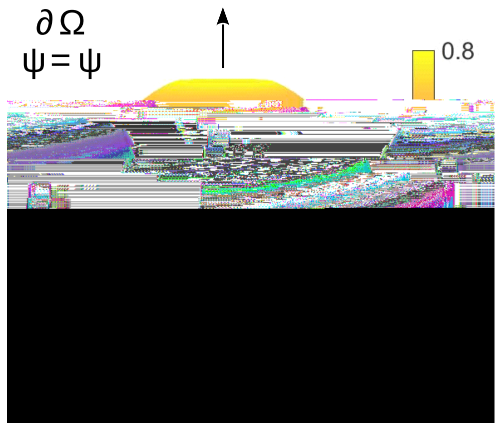


FIG. 5. Distribution of non-dimensional pressure $=$ ${}_c$ (${}_c =$ $=R$) on a unit sphere $R = 1$ at ${}_1 = 0{:}8$, ${}_2 = 0{:}2$, ${}_1 =$ ${}_0 =$ $22{:}5$ and $= 150$.

approximation of the geometry. It is, however, well understood appearing as a 'variational crime' [18]. We then discretise the Laplace-Beltrami operator over the polygon using piecewise linear nite elements. To test our numerical model we examine the azimuthally symmetric case, where the exact solution is known and given in (9). We then check convergence of the nite element approximation to (9). The results are shown in Fig. 4.

We make use of the numerical model generated to examine the dependency of the total ux, and hence the permeability of the truncated spherical element as a function of the tilt angle , that is the position of the boundaries on the sphere. As in the azimuthally symmetric case, without much loss of generality, we consider circular boundaries. The size of the boundary contour, that is its radius $R \sin {}_0$ (or $R \sin {}_1$), will be characterized by the polar angle ${}_0$ (or ${}_1$) counted from the axis of symmetry of each contour and the particle radius R.

C. Results of numerical analysis and discussion

The distribution of pressure on the spherical surface is illustrated in Fig. 5, while the typical total ux dependence on the tilt angle is presented in Fig. 6 at ${}_0 =$ ${}_1$. The distribution of pressure demonstrates relatively smooth variations in the range bounded by the prescribed boundary values, such that, as is expected in a di usion problem, ${}_2$ ${}_1$. The value of the total liquid ux Q_T through the spherical element decreases when the tilt angle increases and the boundary contours move further away from each other. At the same time, one readily observes, Fig. 6, that at relatively large tilt angles, close to the re ex angle in the azimuthal symmetrical case, the total ux value and hence permeability of the surface elements, is close to that predicted on the basis of the azimuthally symmetric solution (10). This implies that the analytical result (10) and (11) can be used in practical applications to obtain rst order corrections to the e ective non-linear coe cient of dispersion in the super-fast di usion model. One may notice that even at small tilt angles, when the two boundaries are located close to each other, one can still approximate coe cient of permeability with the accuracy of 50 %. We have veri ed numerically that in the general case the permeability coe cient of the particles demonstrates the same trends with variations of parameters ${}_0$ and ${}_1$ as in the azimuthally symmetric case.

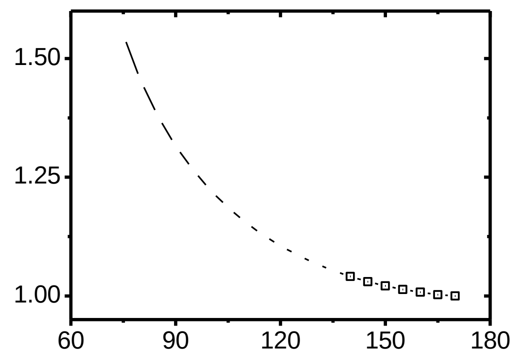


FIG. 6. Non-dimensional total ux $Q_T = Q_0$ as a function of the tilt angle at ${}_0 =$ ${}_1$. Here Q_0 is the total ux value at $= 180$.

CONCLUSIONS

We have demonstrated how the permeability coe cient of constituent particle surface elements of a porous matrix can be estimated on the basis of a solution to the Laplace-Beltrami problem using, as an example, truncated spherical particles with arbitrary oriented boundaries. In the azimuthally symmetric case, we obtained an observable analytical solution, which has been incorporated into the macroscopic super-fast dispersion model to calculate a correction to the e ective non-linear coe cient of di usion. We have shown, that in the case of arbitrary oriented boundaries, the analytical solutions provide a reasonable approximation in the general case. The analytical, (10) and (11), and numerical solutions are the main results of our paper. The methodology developed in our study can be used in practical applications involving more sophisticated shapes of constituent elements. This will be the subject of future studies.

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