Speaking about the system (1.1), we would like to mention that the system (1.1) is called the\1 -Laplacian" and it arises as a sort of Euler-Lagrange PDE of vectorial variational problems in L^1 for the supremal functional

$$E_1 (u; O) := kH(Du)k_{L^1(O)}; \quad u\ 2\ W^{1;1}_{loc}(\ ; R^N); \ O\ b \quad ; \tag{1.4}$$

when the Hamiltonian (the non-negative function $H\ 2\ C^2\ R^{N \quad n}$) is chosen to be $H(Du) = \frac{1}{2} Du^{2}$, with $:$ is the Euclidean norm on the space $R^{N \quad n}$. the 1 -Laplacian is a special case of the system

$$_1 u := \quad H \tag{1.5}$$

Interestingly, even when the operator Δ_∞ is applied to C^1 maps, which may even be solutions, (1.1) may have discontinuous coefficients. This further difficulty of the vectorial case is not present in the scalar case. As an example consider

$$u(x,y) := e^{ix} - e^{iy}, \quad u : \mathbb{R}^2 \to \mathbb{R}^2. \tag{1.8}$$

Katzourakis has showed in [11] that even though (1.8) is a smooth solution of the ∞-Laplacian near the origin, we still have the coefficient $|Du|^2[\![Du]\!]^\perp$ of (1.1) is discontinuous. This is because when the dimension of the image changes, the projection $[\![Du]\!]^\perp$ "jumps". More precisely, for (1.8) the domain splits to three components according to the rk(Du), the "2D phase Ω_2", whereon u is essentially 2D, the "1D phase Ω_1", whereon u is essentially 1D (which is empty for (1.8)) and

(i) On Ω_2 we have $\mathrm{rk}(Du) = 2$, and the map $u : \Omega_2 \to \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$|Du|^2 = C^2 > 0. \tag{1.10}$$

The constant C may vary on different connected components of Ω_2.

(ii) On Ω_1 we have $\mathrm{rk}(Du) = 1$ and the map $u : \Omega_1 \to \mathbb{R}^N$ is given by an essentially scalar ∞-Harmonic function $f : \Omega_1 \to \mathbb{R}$:

$$u = a + \xi f, \quad \Delta_\infty f = 0, \quad a \in \mathbb{R}^N, \quad \xi \in \mathbb{S}^{N-1}. \tag{1.11}$$

The vectors a, ξ may vary on different connected components of Ω_1.

(iii) On S, $|Du|^2$ is constant and also $\mathrm{rk}(Du) = 1$. Moreover if $S = \partial\Omega_1 \cap \partial\Omega_2$ (that is if both the 1D and 2D phases coexist) then $u : S \to \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$u = a + \xi f, \quad |Df|^2 = C^2 > 0, \quad a \in \mathbb{R}^N, \quad \xi \in \mathbb{S}^{N-1}. \tag{1.12}$$

The main result of this paper is to generalise these results to higher dimension $N \geq n \geq 2$. The principle result in this paper in the following extension of theorem 1.1:

Theorem 1.2 (Phase separation of n-dimensional ∞-Harmonic mappings). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $u : \Omega \to \mathbb{R}^N$, $N \geq n \geq 2$, be an ∞-Harmonic map in $C^2(\Omega; \mathbb{R}^N)$, that is a solution to the ∞-Laplace system (1.1). Then, there exist disjoint open sets $\{\Omega_r\}_{r=1}^n \subseteq \Omega$, and a closed nowhere dense set S such that $\Omega = S \cup \bigcup_{i=1}^n \Omega_i$ such that:

(i) On Ω_n we have $\mathrm{rk}(Du) \equiv n$ and the map $u : \Omega_n \to \mathbb{R}^N$ is an immersion and solution of the Eikonal equation:

$$|Du|^2 = C^2 > 0. \tag{1.13}$$

The constant C may vary on different connected components of Ω_n.

(ii) On Ω_r we have $\mathrm{rk}(Du) \equiv r$, where r is integer in $\{2, 3, 4, \ldots, (n-1)\}$, and $|Du(\gamma(t))|$ is constant along trajectories of the parametric gradient flow of $u(\gamma(t, x, \xi))$

$$\begin{cases} \dot\gamma(t, x, \xi) = \nu^\top Du\big(\gamma(t, x, \xi)\big), & t \in (-\varepsilon, 0) \cup (0, \varepsilon); \\ \gamma(0, x, \xi) = x. \end{cases} \tag{1.14}$$

where $\xi \in \mathbb{S}^{N-1}$, and $\nu \in N\big(Du(\gamma(t, x, \xi))^\top\big)$.

(iii) On Ω_1 we have $\mathrm{rk}(Du) \equiv 1$ and the map $u : \Omega_1 \to \mathbb{R}^N$ is given by an essentially scalar ∞-Harmonic function $f : \Omega_1 \to \mathbb{R}$:

$$u = a + \xi f, \quad \Delta_\infty f = 0, \quad a \in \mathbb{R}^N, \quad \xi \in \mathbb{S}^{N-1}. \tag{1.15}$$

The vectors a, ξ may vary on different connected components of Ω_1.

(iv) On S, when $S \cap \partial\Omega_p \cap \partial\Omega_q = \emptyset$ for all p and q such that $2 \leq p < q \leq n-1$, then we have that $|Du|^2$ is constant and also $\mathrm{rk}(Du) \equiv 1$. Moreover on

$$\partial\Omega_1 \cap \partial\Omega_n \subseteq S,$$

(when both 1D and nD phases coexist), we have that $u : S \to \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

$$u = a + \xi f, \quad |Df|^2 = C^2 > 0, \quad a \in \mathbb{R}^N, \quad \xi \in \mathbb{S}^{N-1}. \tag{1.16}$$

PHASE SEPARATION OF

Lemma 2.3 (cf. [14]). Let $u : \mathbb{R}^n \supseteq \ \to \mathbb{R}^N$ be in $u \in C^2(\ ; \mathbb{R}^N)$. Consider the gradient flow

$$
(2.6)\qquad \begin{cases} \dot{\ }(t; x;\ ) = \dfrac{|Du|^2}{|{}^{\top} Du|^2}\, {}^{\top} Du\,(\ (t; x;\ ))\,; & t \neq 0 \\ \ (0; x;\ ) = x; \end{cases}
$$

for $x \in$, $\in S^{N-1} \setminus [\![Du]\!]^{\perp}$

u must coincide and hence, we obtain $u = f + a$ for $2 S^{N \ 1}; a \ 2 \ R^N$ and $f \ 2 \ C^2(\ _1; R)$ where and a may vary on di erent connected components of $_1$. The theorem follows.

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Department of Mathematics , College of Science, University of Basra, Basra, Iraq
AND Department of Mathematics and Statistics, University of Reading, Whiteknights,,
PO Box 220, Reading RG6 6AX, UK

E-mail address : h.a.h.abugirda@pgr.reading.ac.uk