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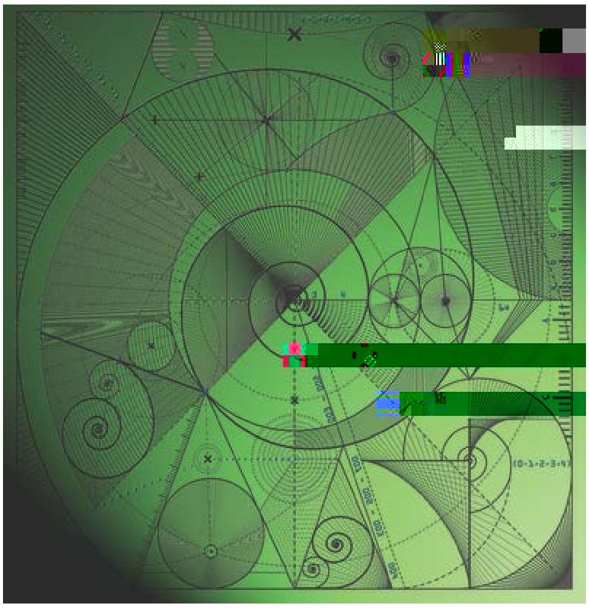

EXISTENCE, UNIQUENESS AND STRUCTURE OF SECOND ORDER ABSOLUTE MINIMISERS

NIKOS KATZOURAKIS AND ROGER MOSER

Abstract. Let R^n be a bounded open $C^{1;1}$ set. In this paper we prove the existence of a unique second order absolute minimiser u_1 of the functional

$$E_1 \ (u; O) := kF(\ ; \ u)k_{L^1 \ (O)}; \quad O \quad \text{measurable} \ ;$$

with prescribed boundary conditions for u and Du on @ and under natural assumptions on F. We also show that u_1 is partially smooth and there exists a harmonic function $f_1 \ 2 \ L^1( )$ such that

$$F(x; \ u_1 \ (x)) = e_1 \ \text{sgn} \ f_1 \ (x)$$

for all $x \ 2 \ f \ f_1 \ 6= 0 \ g$, where e_1 is the in mum of the global energy.

1. Introduction

For n 2 N, let R^n be a bounded open set and let also F : R ! R be a real function that is L( ) B (R)-measurable, namely, measurable with respect to the product -algebra of the Lebesgue subsets of with the Borel subsets of R. In this paper we consider variational problems for second order supremal functionals of the form

$$\text{(1.1)} \qquad E_1 \ (u; O) := \ F(\ ; \ u) \ _{L^1 \ (O)}; \quad O \quad \text{measurable};$$

where the admissible functions u range over the (Frechet) Sobolev space

$$\text{(1.2)} \qquad W^{2;1} \ (\ ) := \bigcap_{1<p< \ 1} \Big\{ u \ 2 \ W^{2;p}(\ ) : \quad u \ 2 \ L^1 \ (\ ) \Big\} :$$

The following is a natural notion of minimiser for variational problems of this type.

De nition 1 (Second order ond or(yp)olu1

Example. For = (1; 0) [(0; 1), consider the functional E_1 (u; O) = $ku''k_{L^1(O)}$ and let Q(x) := x $_{(\ 1;0)}$(x) x(x 1) $_{(0;1)}$(x), x 2 . Then Q is a global minimiser of E_1 (;) in $W^{2;1}_Q$ ( ) with E_1 (Q;) = 2 and is also a second order absolute minimiser. However, for any function 2 C^1_c (1; 0) with 0 < k $''k_{L^1(\ 1;0)}$ < 1, the perturbation Q + still satis es E_1 (Q + ;) = 2 and lies in $W^{2;1}_Q$ ( ), but does not minimise E_1 (; (1; 0)) over $W^{2;1}_Q$ ((1; 0)) because the only minimiser on (1; 0) with boundary data Q is the identity.

On the other hand, if u 2 $W^{2;1}$ ( ) minimises E_1 (;) uniquely in $W^{2;1}$ ( ) with respect to its own boundary conditions, then we will show that u is actually the unique second order absolute minimiser. This is the situation that we will nd in the main results of this paper. We rst give a condition that guarantees that u is the unique minimiser for its boundary values on any subdomain. While it is not obvious that this condition can be met, we will subsequently prove that it is satis ed by exactly one function under given boundary conditions and very mild additional assumptions.

In the following, we will use the symbolisation \sgn" for the sign function, with the convention that sgn(0) = 0. We will assume also that

(1.3) is a bounded connected open subset of R^n, n 1,

and

(1.4) (F : R ! R is L( ) B (R)-measurable and for a.e. x 2 , 7! F(x;) is strictly increasing with F(x; 0) = 0.

Our rst main result therefore is:

Theorem 2 (Criterion for unique minimisers). Suppose(1.3)-(1.4) hold and consider (1.1) and a function u 2 $W^{2;1}$ ( ). If there exist a number e 0 and a function f 2 L^1( ) satisfying

(1.5) f = 0; on ;

such that

(1.6) F(; u) = e sgn(f); a.e. on ;

then

E_1 (u ; O) < E_1 u + ; O ;

for any open O and any 2 $W^{2;1}_0$ (O) u is− 794 Td962 0 Td [(794 Td 9.9626 Tf 16.81 0 Td [(()]TJ/F27 9.9

is by assumption strictly increasing, (1.6) is equivalent to the next representation formula for u :

$$u\ (x) = F(x;\ )^{\ 1}\ e\ \mathrm{sgn}\ f\ (x)\ \ ; \quad \text{a.e. } x\ 2\ \ :$$

Moreover, by using standard argument involving Green functions (see e.g. [GT, Ch. 2]), we could represent u in terms of F; f ; e ; u $j_{@}$; D u $j_{@}$.

Theorem 2 gives a connection between the variational problem and a PDE system of second order equations with a parameter consisting of (1.5) and (1.6). We will see later that under certain assumptions on , F and the boundary data, the system has in fact a solution (u ; f ; e) with f 6 0 if e > 0. It then follows that (u ; e) is unique and the system is equivalent to unique global minimality under prescribed boundary data. We may think of (1.5) and (1.6) as a PDE formulation of the L^1 variational problem. There does exist, however, a more conventional analogue of the \Euler-Lagrange equation" for (1.1). This is the fully nonlinear PDE of third order

$$\text{(1.7)} \qquad F(\ ;\ \ u)F\ (\ ;\ \ u)\ D\ jF(\ ;\ \ u)j^2\ \ ^2 = 0; \quad \text{on}\ \ :$$

A particular model case of (1.1)-(1.7) is what we call the \1 -Bilaplacian" and arises from the choice F(x;) = . Then, equation (1.6) becomes u = e sgn(f) and (1.7) becomes $ujD(j\ \ uj^2)j^2 = 0$. Due to the particular structure of the functional (1.1), however, in this case (1.7) becomes redundant since all the structural information of second order absolute minimisers can be obtained directly from the L^1 variational problem.

For our existence result, we will assume that F : R ! R satis es

$$\text{(1.8)} \qquad \begin{cases} F\ 2\ C^2(\ \ \ R); \\ F(x;0) = 0; \quad x\ 2\ \ ; \\ 9\ c > 0\ : \begin{cases} c \quad F\ (x;\ ) \quad \frac{1}{c}; & (x;\ )\ 2 \quad R; \\ F(x;\ )\ F\ \ (x;\ ) \quad \frac{1}{c}; & (x;\ )\ 2 \quad R; \end{cases} \end{cases}$$

where subscripts of F denote partial derivatives. The conditions of (1.8) imply that for any xed x 2 the partial function jF(x;)j is level-convex on R ((x;1)]TJ/F11 9.9626 Tf 8.302 0 Td [(u)]T 0 where $e_\infty = E_\infty(u_\infty, \Omega)$. Further, $f_\infty \neq 0$ if $e_\infty > 0$.

(II) Let

$$\Gamma_\infty := f_\infty^{-1}(\{0\}).$$

If $e_\infty > 0$, then u_∞ belongs to $C^{3,\alpha}(\Omega \setminus \Gamma_\infty)$ for any $\alpha \in (0,1)$ and Γ_∞ is a Lebesgue nullset. If $e_\infty = 0$, then u_∞ is a harmonic function.

(III) For $p \in \mathbb{N}$, let

$$e_p := \inf\left\{ E_p(u) : u \in W^{2,p}_{u_0}(\Omega) \right\}.$$

Then, for any p large enough there exists a global minimiser $u_p \in W^{2,p}_{u_0}(\Omega)$ of E_p satisfying $E_p(u_p) = e_p$. Moreover, $e_p \to e_\infty$ as $p \to \infty$ and there exists a subsequence $(p_\ell)_{\ell=1}^{\infty}$ such that $u_{p_\ell} \overset{*}{\rightharpoonup} u_\infty$ in the weak* topology of $W^{2,\infty}(\Omega)$ as $\ell \to \infty$. In addition,

$$\begin{cases} u_{p_\ell} \longrightarrow u_\infty, & \text{in } C^1(\overline{\Omega}); \\ D^2 u_{p_\ell} \rightharpoonup D^2 u_\infty, & \text{in } L^q(\Omega; \mathbb{R}^{n \times n}) \text{ for all } q \in (1,\infty); \\ \Delta u_{p_\ell} \longrightarrow \Delta u_\infty, & \text{a.e. on } \Omega \text{ and in } L^q(\Omega) \text{ for all } q \in (1,\infty); \end{cases}$$

as $\ell \to \infty$. Furthermore, $\Delta u_{p_\ell} \to \Delta u_\infty$ locally uniformly on $\Omega \setminus \Gamma_\infty$ if $e_\infty > 0$ and locally uniformly on Ω if $e_\infty = 0$.

(IV) Let

$$f_p := \frac{1}{e_p^{p-1}} \left| F(\cdot, \Delta u_p) \right|^{p-2} F(\cdot, \Delta u_p) F_\xi(\cdot, \Delta u_p) \tag{1.9}$$

if $e_p \neq 0$ and $f_p \equiv 0$ if $e_p = 0$. Then, the harmonic function f_∞ in (I) may be chosen such that $f_{p_\ell} \to f_\infty$ as $\ell \to \infty$ in the strong local topology of $C^1(\Omega)$.

By invoking Theorem 2, an immediate consequence is that the modes of convergence in Theorem 3(III) as $p \to \infty$ are actually full and not just subsequential. Also, known results on the regularity of nodal sets of solutions to elliptic equations [HS] imply that Γ_∞ is countably rectifiable, being equal to the union of countably many smooth submanifolds of Ω and a set of vanishing $(n-2)$-dimensional Hausdorff measure. This, however, uses only the fact that f_∞ is harmonic and it seems plausible that the full statement in (I) could give further information.

The optimal regularity of Γ_∞ is an open question which we do not attempt to answer here. Certainly, full regularity of the set Γ_∞ cannot be expected as there are limitations: in [KP2] it was noted that Γ_∞ in general may not be a smooth submanifold, as for certain data u_0 the intersection of transversal lines in Ω was observed in numerical experiments. In most cases the set Γ_∞ is necessarily non-empty and divides Ω into two distinct parts, whilst the equations $F(\cdot, \Delta u_\infty) = \pm e_\infty$ do not permit any solutions in $W^{2,\infty}_{u_0}(\Omega)$ for most boundary data, even if $n = 1$. Therefore, the Laplacian of u_∞ will have a jump on Γ_∞ and, in terms of u_∞, no more regularity than $W^{2,\infty}(\Omega)$ can be expected (see the numerical and explicit solutions in [KP2]).

Let us also note further that (I)–(IV) above have been obtained in [KP2] for $n = 1$ and in some other special cases (although were not stated in this explicit fashion), whilst the qualitative behaviour emerging here was observed numerically for $n = 2$ and $F(x, \xi) = \xi$.

When combined, Theorem 2 and Theorem 3 imply in particular the following.

Corollary 4. Under the hypotheses of Theorem 3, there exists a unique global minimiser u_∞ of $E_\infty(\cdot,\Omega)$ in $W^{2,\infty}_{u_0}(\Omega)$, which is a second order absolute minimiser and a strong solution to the Dirichlet problem for (1.7):

$$\begin{cases} F(\cdot;\Delta u)F_p(\cdot;\Delta u)\,\big|D\big(F(\cdot;\Delta u)\big)\big|^2 = 0; & \text{in } \Omega;\\ u = u_0; & \text{on } \partial\Omega;\\ Du = Du_0; & \text{on } \partial\Omega; \end{cases}$$

More precisely, u_∞ is thrice differentiable a.e. on Ω and satisfies the PDE in the pointwise sense.

The study of supremal functionals and of their associated equations is known as Calculus of Variations in the space L^∞. Second order variational problems in L^∞ have only relatively recently been studied and are still poorly understood. It is remarkable that for our specific problem, we obtain not just unique absolute minimisers, but also a fair amount of detailed information about their structures, with relatively simple means. On the other hand, our methods take advantage of the special structure of the problem and are unlikely to work in general, although they allow the following modest generalisation: all of the preceding results hold for the seemingly more general case where the Laplacian is replaced by the projection $A:D^2u = \sum_{i,j} A_{ij}D^2_{ij}u$ on a fixed positive symmetric matrix $A \in \mathbb{R}^{n\times n}_+$. This gives rise to the following functional:

$$E_\infty(u;\mathcal{O}) = \big\|F(\cdot;A:D^2u)\big\|_{L^\infty(\mathcal{O})}; \quad \mathcal{O} \Subset \Omega \text{ measurable}.$$

However, this case can easily be reduced to the case we study herein via the change of variables $x \mapsto \Lambda O^\top x$ for a diagonal $n \times n$ matrix Λ and an orthogonal matrix $O \in O(n)$ arising from the spectral representation $A = O\Lambda^2O^\top$.

Some of the techniques that underpin Theorem 3 have been successfully deployed to problems somewhat different to (1.1) (with dependence on u in addition to Δu) [MS, S1], which suggests that further generalisation might be possible. In order to keep the presentation simple, however, we do not explore this possibility any further in this work.

We conclude this introduction by placing the L^∞ problem we study herein into the wider context of Calculus of Variations. Variational problems for first order functionals of the form

$$(1.10)\qquad (u,\mathcal{O}) \mapsto \operatorname*{ess\,sup}_{x\in\mathcal{O}} H\big(x,u(x),Du(x)\big), \quad u \in W^{1,\infty}(\Omega),\ \mathcal{O} \subseteq \Omega,$$

together with the associated PDEs, first arose in the work of Aronsson in the 1960s [A1]–[A3]. The first order case is very well developed and the relevant bibliography is very extensive. For a pedagogical introduction to the theme which is accessible to non-experts, we refer to the monograph [K8] (see also [C]). The vectorial case of (1.10) for maps with dependence on pure second derivatives only. Some preliminary investigations (relevant to the second order case of energy density $H(; u; u^0; u^{00})$ when $n = 1$) had previously been performed via different methods by Aronsson and Aronsson-Barron in [A4, AB].

Apart from the intrinsic mathematical interest, the motivation to study higher order L^1 minimisation problems comes from several diverse areas. In applied disciplines like Data Assimilation in the geosciences, PDE-constrained optimisation in aeronautics, etc. (see e.g. the model problem in [K9] and references therein, as well as the classical monograph [L]), a prevalent underlying problem is the construction of approximate solutions to second order ill-posed PDE problems. For instance, in the modelling of aquifers, one needs to solve a Poisson equation $u = f$ coupled with a pointwise constraint of the form $K(u) = k$ for given functions $f; K; k$. By minimising the error function $j\ u\ f j^2 + jK(u)\ kj^2$ in L^1, one can obtain uniformly (absolutely) best approximations.

Minimisation problems in L^1 similar to the above have also been studied in the context of differential geometry and in questions related to the Yamabe problem. In particular, the second author, together with Schwetlick [MS], and subsequently Sakellaris [S1] considered the problem of minimising the scalar curvature of a Riemannian metric on a given manifold and in a given conformal class. When formulated in terms of differential operators, this gives rise to a functional similar in structure to (1.1). This work uses different boundary conditions, however, and no attempt is made to prove uniqueness or find second order absolute minimisers. Nevertheless, some of the tools in the proofs of the above results originate in the above quoted papers.

We close with some remarks about generalised solutions to the equations governing the \extremals" of Calculus of Variations in L^1. In the scalar first order case, the theory of viscosity solutions of Crandall-Ishii-Lions (see [CIL, C, K8]) proved to be an apt framework within which the generally non-smooth solutions to the so-called Aronsson equation, which is a second order PDE, can be studied rigorously. However, viscosity solutions are of purely scalar nature and fail to work in either the vectorial or the higher order case (where we have either a second order system with discontinuous coefficients, or a fully nonlinear third order PDE). In the recent papers [K9, K10] a new theory of generalised solutions has been introduced which is based on a probabilistic representation of derivatives which do not exist classically and in the papers [AK, AyK, CKP, K11, K12, K13, KP, KP2, KP3] several results have been obtained in this framework. However, in the setting of the present paper, the particular structure of the problem at hand allows to prove directly existence of strong solutions to the fully nonlinear PDE (1.7).

2. Proofs

In this section we establish the proofs of Theorems 2 and 3 and of Corollary 4.

Proof of Theorem 2. Fix $\ 2 W_0^{2;1}(\ )$ with $\ 6\ 0$ on $\ $. Since f is a harmonic function in $L^1(\ )$, it follows that

$$f \qquad = 0: \tag{2.1}$$

We set

$$:= f^{\ 1}(f\,0g):$$

By standard results on the nodal set of solutions to elliptic equations [HS] and the connectedness of , it follows that if f 6 0 then is a Lebesgue nullset and if f 0 then = .

Let us rst consider the case f 6 0. Note that cannot vanish almost everywhere on (as this would imply that 0 by uniqueness of solutions of the Dirichlet problem for the Laplace equation). Therefore, we deduce that f 6= 0 on a subset of positive Lebesgue measure in . Hence, (2.1) implies that there exist measurable sets with L^n( ) > 0 such that

f > 0; a.e. on ;

where L^n

which establishes the desired bound. Finally, (2.5) is a consequence of (1.8), which gives $|\xi||F_\xi(x;\xi)| \le c|\xi|$, and of (2.3), which gives $c|\xi| \le c^2|F(x;\xi)|$.

We will construct the solutions to our problem by approximation with minimisers of L^p functionals. Therefore, we need to understand the behaviour of the latter.

Proposition 6. *Suppose that $F \in C^2(\Omega \times \mathbb{R})$ satisfies (1.8). Then for any $p > c^{3}+1$ there exists a minimiser u_p of E_p over the space $W^{2,p}_{u_0}(\Omega)$. Moreover, u_p is a weak solution to the Dirichlet problem for the Euler-Lagrange equation associated with the functional E_p:*

$$\begin{cases} \Delta\Big(F(\cdot,\Delta u)^{p-2}F(\cdot,\Delta u)\,F_\xi(\cdot,\Delta u)\Big) = 0, & \text{in } \Omega, \\ u = u_0, & \text{on } \partial\Omega, \\ Du = Du_0, & \text{on } \partial\Omega. \end{cases}$$

Furthermore, there exist a (global) minimiser u_∞ of the functional $E_\infty(\cdot,\Omega)$ over the space $W^{2,\infty}_{u_0}(\Omega)$ such that $E_p(u_p) \to E_\infty(u_\infty,\Omega)$ as $p \to \infty$. Also, there exists a subsequence $(p_\ell)_1^\infty$ such that

$$\begin{aligned} u_{p_\ell} &\to u_\infty, && \text{in } C^1(\overline{\Omega}), \\ D^2u_{p_\ell} &\overset{*}{\rightharpoonup} D^2u_\infty, && \text{in } L^q(\Omega;\mathbb{R}^{n\times n}), \text{ for all } q \in (1,\infty), \end{aligned}$$

as $\ell \to \infty$.

Proof of Proposition 6. By (2.3)–(2.4) of Lemma 5, for $p > c^{3}+1$ the functional E_p is convex in $W^{2,p}_{u_0}(\Omega)$ and

$$E_p(u) \ge c\big(\mathcal{L}^n(\Omega)\big)^{-1/p}\|\Delta u\|_{L^p(\Omega)},$$

for any $u \in W^{2,p}_{u_0}(\Omega)$. Since $u - u_0 \in W^{2,p}_0(\Omega)$, by the Calderon-Zygmund L^p estimates (e.g. [GT, GM]) and the Poincare inequality we have a positive constant $c_0 = c_0(p;$

and hence $(\Delta u_p)_{p \ge p_0}$ is bounded in $L^k(\Omega)$. By the previous arguments and (2.6), we conclude that $(u_p)_{p \ge p_0}$ is bounded in $W^{2,k}_{u_0}(\Omega)$ for any $k \in \mathbb{N}$. By a standard diagonal argument, weak compactness and the Morrey theorem, there exists

$$u_\infty \in \bigcap_{1<k<\infty} W^{2,k}_{u_0}(\Omega)$$

such that the desired convergences hold true along a subsequence as $p_\ell \to \infty$. When we pass to the limit as $\ell \to \infty$ in (2.7), the weak lower semicontinuity of the L^k norm implies

$$\|\Delta u_\infty\|_{L^k(\Omega)} \le \frac{(\mathcal{L}^n(\Omega))^{1/k}}{c} E_\infty(u_0;\Omega).$$

Letting $k \to \infty$ we obtain $\Delta u_\infty \in L^\infty(\Omega)$. Thus, $u_\infty \in W^{2,\infty}_{u_0}(\Omega)$, as desired. It remains to show the convergence of $E_{\to\infty}$

In other words, given any boundary condition u_0 2 $W^{2;1}$ ( ) such that u_0 is continuous up to the boundary, we can nd another function w in the same space with the same boundary data, the Laplacian of which is as small as desired in a neighbourhood of the boundary.

Proof of Lemma 7. Since @ is $C^{1;1}$-regular, the result of the appendix establishes that the distance function dist(; @) belongs to $C^{1;1}(\overline{_{2r_0}})$ for some $r_0 > 0$ small enough.

Let d be an extension of dist(; @) from $_{r_0}$ to $\overline{}$ which is in the space $W^{2;1}$ ( ). Extend u_0 and d by zero on R^n n . Let ($)_{>0}$ $C^1_c(R^n)$ be a standard mollifying family (as e.g. in [E]). We set

$$v \;:=\; u_0 \quad \frac{d^2}{2} \qquad u_0 \;: \tag{2.9}$$

Then v u_0 2 $W^{2;1}_0$ ( ) since d = 0 on @ and

$$Dv \;=\; Du_0 \quad d \;\frac{d}{2}D \qquad u_0 \;+\; \qquad u_0\; Dd \;;$$

$$D^2v \;=\; D^2u_0 \quad Dd \quad Dd \qquad u_0 \quad d\;\frac{d}{2}D^2 \qquad u_0$$

$$+\,D \qquad u_0 \quad Dd \,+\, Dd \quad D \qquad u_0 \;+\; \qquad u_0\; D^2d \;:$$

By using that

$$\mathrm{tr}(Dd \quad Dd) \;=\; jDdj^2 \;=\; 1 \quad \text{on} \quad {}_{r_0};$$

for $0 < r < r_0$ we deduce

$$k \quad v\, k_{L^1}$$

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Proof of Theorem 3. Let us begin by setting

$$e_1 := \inf \ E_1(u; \ ) : u \in W^{2;1}_{u_0}(\ ) \ :$$

If $e_1 = 0$, then everything in Theorem 3

$_r$ of the boundary $@$. Since $u_p \quad w \ 2 \ W_0^{2;p}(\ )$, it is an admissible test function and by (2.12), integration by parts gives

$$f_p \quad (u_p \quad w) = 0:$$

Hence, by the above together with (1.9), (1.8), (2.2) and (2.5), we obtain

$$
\begin{aligned}
f_p \quad w = & \quad f_p \quad u_p \\
= & \ \frac{1}{e_p^{p \quad 1}} \quad F(\ ; \quad u_p) \ ^{p \quad 2} F(\ ; \quad u_p) \, F \ (\ ; \quad u_p) \quad u_p \\
= & \ \frac{1}{e_p^{p \quad 1}}
\end{aligned}
$$

for $x \in \setminus _\infty$. On any compact set $K \setminus _\infty$, we have the uniform convergence $f_{p_\ell} \to f_\infty$ as $\ell \to \infty$, whereas F is uniformly bounded from above and below by (1.8). Hence, by restricting ourselves along the subsequence p_ℓ and letting $\ell \to \infty$ we obtain uniform convergence of the right-hand side of (2.17) to $^{1}(\,, e_1 \,\mathrm{sgn}(f_\infty))$ on K. But since we already know that $u_p \rightharpoonup u_\infty$ weakly in $L^2(\,)$, it follows that

$$(2.18) \qquad x,\; u_\infty(x) = {}^{1}\big(x, e_1 \,\mathrm{sgn}\, f_\infty(x)\big), \quad x \in K.$$

As a consequence,

$$F\big(x,\; u_\infty(x)\big) = e_1 \,\mathrm{sgn}\, f_\infty(x), \quad x \in K.$$

Now let us recall that $L^n(\,_\infty) = 0$. This is a consequence of general regularity results for nodal sets of solution to elliptic equations [HS]. The statement of item (I) then follows.

In order to prove item (II), we note that (2.18) implies that $u_\infty \in C^2(\setminus _\infty)$. The desired statement then follows from standard Schauder theory [GT].

For item (III), rst recall the subsequential cg 0 15b,7g4 Tdn18(tice)-46.9738 a $(u_2$ C_1

The C^2 regularity of the distance function for a C^2 boundary $\partial\Omega$ is a classical result, see e.g. [GT, Appendix 14.6]. On the other hand, the case of C^1 regularity of the distance function when the boundary $\partial\Omega$ is C^1 holds under the extra hypothesis that the distance is realised at one point; see e.g. [F].

In order to prove the desired $C^{1,1}$ regularity of the distance function near $\partial\Omega$ when the boundary itself is a $C^{1,1}$ manifold (which we utilised in Lemma 7), we first note the following fact: suppose that $r > 0$ is such that $1/r$ is larger than the essential supremum of the curvature of $\partial\Omega$. If $x \in \Omega$ and $y \in \partial\Omega$ with $|x$

Now note that

$$Dd(\ (x^0; t)) = N(x^0)$$

whenever $t > 0$ is small enough. Hence if denotes the projection onto $R^{n \quad 1}$ f 0g, then we obtain the formula

$$Dd = N \qquad ^1$$

near x_0. The right-hand side is of class $C^{0;1}$, and thus d is of class $C^{1;1}$ near x_0. A compactness argument then proves the above statement.

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