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by

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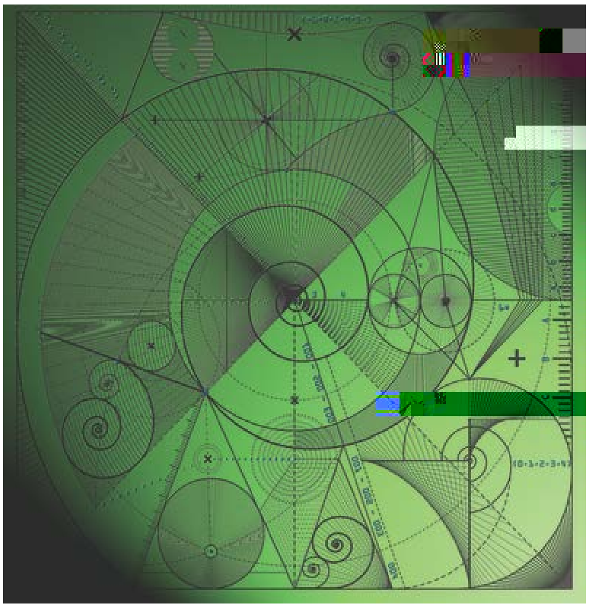

A nite difference scheme for a class of nonlinear diffusion problems that preserves scaling symmetry

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Abstract

A 1-D moving-mesh nite difference scheme based on local conservation is constructed for a class of second-order nonlinear diffusion problems with moving boundaries that (a) preserves scaling properties and (b) is exact at the nodes for initial conditions sampled from similarity solutions. Details are presented and the exactness property con rmed for two moving boundary problems, the porous medium equation and a simplistic glacier equation.

The scheme is also tested for non self-similar initial conditions by computing relative errors in the approximate solution (in the L_1 norm) and the approximate boundary position, indicating superlinear convergence.

Keywords: Nonlinear diffusion, moving-meshes, scale-invariance, similarity, conservation, nite differences, porous medium equation, glacier equation.

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1. Introduction

Partial differential equations (PDEs) govern many physical processes which occur in branches of applied mathematics. However, due to the complexity of these equations the solution cannot always be determined analytically and numerical approximation becomes fundamental both for extracting quantitative solutions and for achieving a qualitative understanding of the behaviour of the solution.

In this paper we consider one-dimensional second-order nonlinear diffusion equations of the general form

$$u_t = (uq)_x \qquad (a(t) < x < b(t)) \tag{1}$$

for a function $u(x;t)$, where q is of the form $f\, p(u)_x\, g^s$ and s is an odd integer, posed on nite moving domains. Typical boundary conditions for this problem consist of a Dirichlet condition on u and a ux condition on uv, where v is the boundary velocity, at each moving boundary. Here we shall assume that $u = 0$ at the moving boundaries. In general the position of the boundary depends on the solution.

Many PDE problems that arise in practical applications possess symmetries involving simultaneous scaling of the variables t, x, and

The main thrust of this paper is the construction of a scale-invariant moving mesh scheme for nonlinear diffusion problems of the form (1) that is exact for initial conditions that coincide with a self-similar scaling solution (thus preserving a scaling symmetry) and accurate for general initial conditions. The layout of the paper is as follows. In section 2 we recall the scaling properties of a general PDE problem of the form (1) and the construction of self-similar solutions. Details are given for two nonlinear diffusion equations of the form (1); a porous medium equation (PME) and a simplified glacier equation (SGE). A moving-mesh finite difference scheme based on conservation of the local integral of u is then described in section 3 which propagates solutions exactly when the initial condition coincides with a self-similar solution. Numerical calculation confirms that relative errors in the approximate solution and approximate boundary position are zero to within rounding error. Section 4 contains numerical results from the numerical algorithm when the initial condition does not coincide with a similarity solution. Both the PME and SGE are used to assess the accuracy of the numerical method for a non self-similar initial condition by computing the relative errors in the approximate solution and the approximate boundary position for varying numbers of mesh points.

The paper ends with concluding remarks.

2. Background

The work of Budd et al [10, 11, 12, 13] has underlined the importance of preserving the geometric structures of the underlying PDE problem in constructing a moving-mesh method. In this section scale-invariance and similarity solutions are recalled and illustrated in the context of two nonlinear diffusion equations, a porous medium equation and a simplified glacier equation.

2.1. Scale-invariance

A PDE problem of the form (1) exhibits scale-invariance if the scaling transformation

$$t = \hat{t}; \quad x = \hat{x}; \quad u = \hat{u}; \quad q = \hat{q} \tag{2}$$

maps the variables (t, x, u, q) to another set $(\hat{t}, \hat{x}, \hat{u}, \hat{q})$ for some arbitrary positive (group) parameter such that equation (1) remains the same in the transformed coordinates.

Substituting the scaling transformation (2) into the PDE (1), it is easy to show that the powers and satisfy $1 = +$ (leading to $= 1$). A further relation between the scaling powers depends on the particular form of the function $p(u)$ and will be described for each example in section 2.3.

The total integral (mass)

$$= \int_{a(t)}^{b(t)} u(\,; t\,)\, d \tag{3}$$

has rate of change

$$\frac{d}{dt} = \int_{a(t)}^{b(t)} u_t \, d \; + u(b(t); t)\dot{b} \; - \; u(a(t); t)\dot{a}$$

$$= \int_{a(t)}^{b(t)} (uq) \; d \; + u(b(t); t)\dot{b} \; - \; u(a(t); t)\dot{a}$$

$$= u(b(t); t)\{ q(b(t); t) + \dot{b}\} \quad u(a(t); t)\{ q(a(t); t) + \dot{a}\} = 0$$

by the $u = 0$ boundary condition. Hence the total mass is constant in time. After substitution from (2),

$$= \int_{a(\hat{t})}^{b(\hat{t})} \hat{u}(\hat{}; \hat{t}) \; d(\hat{}) = {}^{+} \int_{a(\hat{t})}^{b(\hat{t})} u(\hat{}; \hat{t})\, d\hat{}$$

where the moving boundaries $a(t)$ and $b(t)$ transform in the same way as x, and thus is constant in time if and only if $+ = 0$.

2.2. Self-similar solutions

A systematic approach in which the scaling transformation (2) may be used to construct exact solutions to scale-invariant PDE problems is as follows. Solutions are sought such that $u(x;t)$ is a function of x and t, which allows the number of independent variables of the differential equation to be reduced by one [8]. These solutions, termed similarity solutions or self-similar solutions, have contributed some of the greatest insights into nonlinear ows [8, 15]. Such symmetries are structurally important and are useful since the resulting equation may be more easily solved than the original problem.

In order to construct such solutions we de ne a `similarity' transformation which is invariant under the action of (2). Introduce the so-called similarity variables

$$= \frac{u}{t}; \qquad = \frac{q}{t}; \qquad = \frac{x}{t}: \tag{4}$$

By assuming functional relationships of the form

$$= f( ); \qquad = g( ); \tag{5}$$

(where f and g are suf ciently differentiable functions) and substituting (2) into equation (1), a time-independent ODE satis ed by ( ) and ( ) is obtained. From (4) and (5), in terms of x and t,

$$u(x;t) = t\ f\ \frac{x}{t}\ ; \qquad q(x;t) = t\ g\ \frac{x}{t}\ : \tag{6}$$

For a xed parameter the solutions may be described in terms of the moving coordinate

$$b(;t) = t \tag{7}$$

and the functions

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$_t$ = $(v)_x$ (continuity equation)
v = p_x = (Darcy's law)
p = (equation of state)

where is the density, v is the velocity (given by Darcy's law), is the viscosity, is the permeability of the

A self-similar scaling solution, given in [20, 17], is therefore

$$u(x,t) = \frac{1}{t^{1/11}} \left(\frac{7}{4\sqrt{3}\,11} \right)^{3/7} \left(1 - \left(\frac{x}{t^{1/11}} \right)^{4/3} \right)^{3/7}_{+} \qquad (14)$$

where the notation $+$ denotes the positive part of the solution, thus determining the extent $b(t)$ of the domain. The position $b(t)$ of the boundary is given by $b(t) = t^{1/11}$ and its velocity $v = (1/11)\,t^{10/11}$, in accordance with (9).

3. A moving domain

In general the extent of the domain of the solution of (1) depends on the solution itself, and so the approach taken to solve the equation is crucial. A standard approach is to solve for u on a fixed domain and then adjust the boundary according to the boundary conditions by interpolation. Another way is to solve for u and the boundary position simultaneously. A useful device is to stretch the domain in proportion to the (unknown) boundary position and solve a modified PDE, although this procedure may affect the structure of the PDE [5]. A more physical way of deforming the domain is based on a local conservation of mass, which determines a nodal velocity v (in terms of the solution u) and has the advantage that the subsequent recovery of u is algebraic [1, 24]. This approach is summarised below.

The Eulerian equation of conservation (continuity) for a conserved quantity u is

$$u_t + (uv)_x = 0 \qquad (15)$$

where v is the Eulerian velocity. Equation (15) is scale-invariant under (2) when v scales as ^{1}. Combining (15) with the scale-invariant PDE (1),

$$(uq)_x + (uv)_x = 0;$$

yielding (given q and a boundary or anchor condition on v) the velocity

$$v(x,t) = \quad q \qquad (16)$$

at all points of the domain (provided that $u \neq 0$). For the nonlinear diffusion equations (1) the velocity (16) is

$$v(x,t) = \{ p(u)_x \}^s : \qquad (17)$$

If u is constant (in time) at the moving boundary $x = b(t)$, say, then for all t

$$\frac{Du}{Dt} = 0 = u_t + v_b u_x = (uq)_x + v_b u_x$$

where v_b is the boundary velocity, from which

$$v_b = \{ (uq)_x / u_x \} \qquad (18)$$

if $u_x \neq 0$. From (18) the boundary velocity depends on the solution, which is o2 302.881 cm51 9.9626 Tdaryo2 3((th=u

3.1. A moving-mesh finite difference scheme

Consider a one-dimensional mesh with time-dependent mesh points

$$a(t) = x_0(t) < x_1(t) \ldots; < x_N(t) = b(t)$$

where $a(t)$ and $b(t)$ are the (moving) boundaries.

3.1.1. Generating the mesh velocities

The velocity is taken to be a finite difference approximation of (17) ([24]). In the case where $s = 1$ a convenient second-order centred accurate approximation v_j^n at any time t^n consists of a barycentric average of the two first-order approximations to $p(u)_x$ in adjacent cells (see e.g. [26, 6]). Thus the mesh-velocity v_j at any point x_j is calculated as

$$v_j = -\frac{\dfrac{p(u_{j+1}) - p(u_j)}{(x_{j+1} - x_j)^2} + \dfrac{p(u_j) - p(u_{j-1})}{(x_j - x_{j-1})^2}}{\dfrac{1}{x_{j+1} - x_j} + \dfrac{1}{x_j - x_{j-1}}} \qquad (22)$$

with truncation error

$$T_j = \frac{1}{6}(x_j - x_{j-1})(x_{j+1} - x_j)\, p(u)_{xxx}\Big|_{x=\theta_i} \qquad (23)$$

where θ_i is an intermediate value. It is straightforward to confirm that the formula (22) is scale-invariant under the transformation (2).

In the case of similarity the instantaneous velocity is proportional to x by (9) and equal to $-p(u)_x$ when $s = 1$ by (17). Thus $p(u)_x$ is proportional to x, the truncation error (23) vanishes, and the general second-order formula (22) is exact in this case.

Remark 1. The same result is obtained by evaluating the derivative of the quadratic interpolating polynomial through $p(u_{j-1})$, $p(u_j)$ and $p(u_{j+1})$ at $x = x_j$, as we now show.

For general values of the odd integer s (including $s = 1$) the velocity is $v = -\{p(u)_x\}^s$ by (17). Because the velocity is proportional to x in the case of similarity by (9), it follows that $p(u)_x$ is proportional to $x^{1/s}$. Then by integration (taking the origin of x at a point where $p(u)$ vanishes) the function $p(u)$ is proportional to $x^{1+1/s}$ and hence $\{p(u)\}^s$ is a monomial $Q(x)$ of degree $1 + s$. The velocity in terms of $Q(x)$ is then

$$v = -\{p(u)_x\}^s = -\left\{Q(x)^{1/s}\right\}_x^s = -\left((1/s)\,Q(x)^{1/s-1}Q_x\right)^s = -(1/s)^s Q(x)^{1-s}(Q_x)^s. \qquad (24)$$

The evaluation of $Q(x_j) = \{p(u_j)\}^s$ at $x = x_j$ is straightforward. Moreover, since $Q(x)$ is a monomial of degree $1 + s$ the evaluation of Q_x at $x = x_j$ is exact if it is calculated by differentiating the interpolating polynomial of degree $1 + s$ through three adjacent values of $Q(x_j)$.

PME

For the PME we have $s = 1$ and $p(u) = -(u^m)_x/m$ with $v = -(u^m)_x/m$. The velocity can therefore be calculated either from (22) or from (24) with $Q(x_j) = -(u_j)^m/m$ and the derivative Q_x found by differentiating the quadratic interpolating polynomial through adjacent values of $-u_j^m/m$.

SGE

For the SGE $s = 3$ and $p(u) = (3/7)u^{7/3}$ with $v = -\{p(u)_x\}^3$. The velocity can therefore be calculated from (24) with $Q(x_j) = (3/7)^3(u_j)^7$ and the derivative Q_x found by differentiating the quadratic interpolating polynomial through adjacent values of $(3/7)^3(u_j)^7$.

3.2. Advancing x(t)

The mesh point locations $x_j(t)$ can now be obtained via time integration of the ODE system

$$\frac{dx_j}{dt} = v(x_j; t); \qquad (j = 1; \ldots; N-1)$$

We seek a time-stepping scheme which is scale invariant and exact for self-similar solutions. Often used is the explicit Euler time-stepping scheme,

$$x_j^{n+1} = x_j^n + t v_j^n ; \qquad j = 1 ; :::; N \quad 1 \tag{25}$$

which although scale invariant is not exact for self-similar solutions.

Observe from (7) that the function $y=n$

3.4. The numerical algorithm

In summary, a scale-invariant moving mesh algorithm for the approximate solution of nonlinear diffusion equations of the form (1) is as follows:

Given initial data with mesh points x_j^0 and values u_j^0, evaluate the c_j's from (29) at the initial time. Then for each time step:

(1) Compute the mesh velocities v_j using (22) (when $s = 1$) or (24) (for any s).

(2) Move the mesh from t^n to t^{n+1} to obtain x_j^{n+1} using the time-stepping scheme (27).

(3) Update the values u_j^{n+1} values at the next time step from equation (30).

Remark 3. The solution is propagated exactly when the initial condition is sampled from a self-similar solution initially. Any vector of nodal values sampled from a self-similar solution is a xed point of the scheme.

4. Numerical results

When the moving-mesh algorithm of section 3.4 is implemented in Matlab for the examples described in section 2.3 (the PME (10) for various positive values of m) and the SGE (12)) the scheme propagates initial self-similar solutions exactly at the nodes (to within rounding error), as expected.

Where the time-stepping scheme (step 2 of the algorithm) is replaced by the forward Euler scheme corresponding to putting $= 1$ in (27) (as is common with many authors) the scheme reverts to the nite difference scheme described in [24] where tests on the PME with $m = 1$ indicate second order convergence in the l^1 norm of the solution error and in the position of the boundary. For $m = 2; 3$ the convergence rate reduced to superlinear, apparently due to the in nite slope of the exact solution at the boundary in these cases (c.f. [1]).

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