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Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to boundary integral equations on fractal screens

by

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traces, quotients, restriction of functions defined on a larger subset, ...). On Lipschitz open sets (defined e.g. as in [23, 1.2.1.1]), many of these different definitions lead to the same Sobolev spaces and to equivalent norms. But, as we shall see, the situation is more complicated for spaces defined on more general subsets of $\mathbb{R}^n$.

paper we consider the following definitions, which are equivalent only under certain conditions on and

s-nullity. In §3.3 we introduce the concept of s-nullity, a measure of the negligibility of a set in terms of Sobolev regularity. This concept will play a prominent role throughout the paper, and many of our key results relating different Sobolev spaces will be stated in terms of the s-nullity or intersection of an infinite sequence of simpler, nested "prefractal" sets. In §3.8 we determine which of the Sobolev spaces defined on the limiting set naturally emerges as the limit of the spaces defined on the approximating sets. This question is relevant when the different spaces on the limit set do not coincide, e.g. when $\widetilde{H}^s(\;) \subsetneqq H^s_{\overline{\;}}$. In this case the correct function space setting depends on whether the limiting set is to be approximated from "inside" (as a union of nested open sets), or from the "outside" (as an intersection of nested closed sets).

Boundary integral equations on fractal screens. §4 contains the major application of the paper, namely the BIE formulation of acoustic (scalar) wave scattering by fractal screens. We show how the Sobolev spaces $H^s(\;)$, $\widetilde{H}^s(\;)$, H^s_F all arise naturally in such problems, pulling together many of the diverse results proved in the other sections of the paper. In particular, we study the limiting behaviour as $j \to \infty$

we say that $(H; I)$ is a unitary realisation of

and the solution is bounded independently of the choice of V, by $\|u_V\|_H \le c^{-1}\|f\|_H$. Furthermore, given closed, nested subspaces $V_1 \subseteq V_2 \subseteq H$, Céa's lemma gives the following standard bound:

$$\|u_{V_1} - u_{V_2}\|_H \le \frac{C}{c} \inf_{v_1 \in V_1} \|v_1 - u_{V_2}\|_H. \tag{7}$$

Consider increasing and decreasing sequences of closed, nested subspaces indexed by $j \in \mathbb{N}$,

$$V_1 \subseteq \cdots \subseteq V_j \subseteq V_{j+1} \subseteq \cdots \subseteq H \quad \text{and} \quad H \supseteq W_1 \supseteq \cdots \supseteq W_j \supseteq W_{j+1} \supseteq \cdots;$$

and define the limit spaces $V_\infty := \overline{\bigcup_{j \in \mathbb{N}} V_j}$ and $W_\infty := \bigcap_{j \in \mathbb{N}} W_j$. Céa's lemma (7) immediately gives convergence of the corresponding solutions of (6) in the increasing case:

$$\|u_{V_j} - u_{V_\infty}\|_H \le \frac{C}{c} \inf_{v_j \in V_j} \|v_j - u_{V_\infty}\|_H \xrightarrow{j \to \infty} 0. \tag{8}$$

In the decreasing case the following analogous result applies.

Lemma 2.4. With $\{W_j\}_{j=1}^{\infty}$ and W_∞ defined as above, it holds that $\|u_{W_j} - u_{W_\infty}\|_H \to 0$ as $j \to \infty$.

Proof. The Lax–Milgram lemma gives that $\|u_{W_j}\|_H \le c^{-1}\|f\|_H$, so that $(u_{W_j})_{j=1}^{\infty}$ is bounded and has a weakly convergent subsequence, converging to a limit u_*. Further, for all $w \in W_\infty$, (6) gives

$$a(u_{W_\infty}, w) = \langle f, w \rangle = a(u_{W_j}, w) \to a(u_*, w),$$

as $j \to \infty$ through that subsequence, so that $u_* = u_{W_\infty}$. By the same argument every subsequence of $(u_{W_j})_{j=1}^{\infty}$ has a subsequence converging weakly to u_{W_∞}, so that $(u_{W_j})_{j=1}^{\infty}$ converges weakly to u_{W_∞}. Finally, we see that

$$c\|u_{W_j} - u_{W_\infty}\|_H^2 \le |a(u_{W_j} - u_{W_\infty}, u_{W_j} - u_{W_\infty})| = |\langle f, u_{W_j} \rangle - a(u_{W_j}, u_{W_\infty}) - a(u_{W_\infty}, u_{W_j} - u_{W_\infty})| \to 0$$

as $j \to \infty$, by the weak convergence of $(u_{W_j})_{j=1}^{\infty}$ and (6). ∎

3 Sobolev spaces

3.1.2 Sobolev spaces on $\mathbb{R}^n$

We define the Sobolev space $H^s(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ by

$$\boxed{H^s(\mathbb{R}^n) := J_{-s} L^2(\mathbb{R}^n) = \left\{ u \in S'(\mathbb{R}^n) : J_s u \in L^2(\mathbb{R}^n) \right\};}$$

equipped with the inner product $(u; v)_{H^s(\mathbb{R}^n)} := (J_s u; J_s v)_{L^2(\mathbb{R}^n)}$, whicJ

3.1.4 Sobolev spaces on closed and open subsets of R^n

$H^s(R^n)=H^s$

Lemma 3.2. Let be any non-empty open subset of $\mathbb{R}^n$, and $s \in \mathbb{R}$. Then

$$H_c^s = \widetilde{H}^s$$

The dual of	is isomorphic to	via the isomorphism
$H^{s}(R^{n})$	$H^{s}(R^{n})$	I^{s}
$\widetilde{H}^{s}(\,)$	$(H_{c}^{s})^{?}$	$\hat{I}_{s}$
	$H^{s}(\,)$	I_{s}
	$H^{s}(R^{n})=(H_{c}^{s})$	I_{s}
$H^{s}(\,)$	$\widetilde{H}^{s}(\,)$	I_{s}
H^{s}_{c}	$(\widetilde{H}^{s}(\,))^{?}$	$\tilde{I}_{s}$
$(H^{s}_{c})^{?}$	$\widetilde{H}^{s}(\,)$	$\hat{I}_{s}$
$\widetilde{H}^{s}(\,)^{?}$	H^{s}_{c}	$\tilde{I}_{s}$
$H^{s}_{0}(\,)$	$(\widetilde{H}^{s}(\,) \backslash H^{s}_{@})^{?};\widetilde{H}^{s}(\,)$	

$\longrightarrow D(R^n) \hookrightarrow S(R^n) \hookrightarrow L^2(R^n)$

$\longrightarrow H^s() \hookrightarrow H^s_{-} \quad H^s(R^n) = (H^s_c)^? \quad H^s_c \quad S(R^n)$

$H^s_0() \quad H^s() \quad D()$

$(R^n) \quad H^s(R^n) \quad \tilde{H}^s() \quad (\tilde{H}^s())^?$

where H^d is the d-dimensional Hausdor measure on R^n and $B_r(x)$ is the open ball of radius r centred at x. Condition (22

Theorem 3.11 ([25, Proposition 2.11]). Let F_1, F_2 be closed subsets of $\mathbb{R}^n$, and let $s \in \mathbb{R}$. Then the following statements are equivalent:

(i) $F_1 \ominus F_2$ is s-null.

(ii) $F_1 \setminus F_2$ and $F_2 \setminus F_1$ are both s-null.

(iii) $H^s_{F_1 \cap F_2} = H^s_{F_1} = H^s_{F_2} = H^s_{F_1 \cup F_2}$.

By combining Theorem 3.11 with the duality result of Theorem 3.3 one can deduce a corresponding result about spaces defined on open subsets. The following theorem generalises [34, Theorem 13.2.1], which concerned the case $\Omega_1 \subset \Omega_2 = \mathbb{R}^n$. The special case where $\mathbb{R}^n \setminus \Omega_1$ is a d-set was considered in [57]. (That result was used in [25] to prove item (xv) in Lemma 3.10 above.)

Theorem 3.12. Let Ω_1, Ω_2 be non-empty, open subsets of $\mathbb{R}^n$, and let $s \in \mathbb{R}$. Then the following statements are equivalent:

(i) Ω_1

Lemma 3.15 ([36, Theorems 3.29, 3.21]) Let $\Omega \subset \mathbb{R}^n$ be C^0 and let $s \in \mathbb{R}$. Then $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ (with $H^s(\Omega)$ present only for $s \geq 0$).

When Ω is not C^0

Proposition 3.18. Suppose that $\Omega \subsetneq \operatorname{int}(\overline{\Omega})$ and that $\operatorname{int}(\overline{\Omega})$ is C^0. Then:

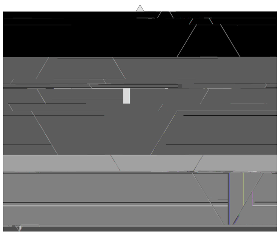
(i) $\widetilde{H}^s(\Omega) \neq H^s_{\overline{\Omega}}$ for all $s < -n/2$.

(ii) If $\operatorname{int}(\overline{\Omega}) \setminus \Omega$ is a subset of the boundary of a Lipschitz open set, with $\operatorname{int}(\overline{\Omega}) \setminus \Omega$ having non-empty relative interior in $\partial\Omega$, then $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ if and only if $s \geq -1/2$. (A concrete example in one dimension is where Ω is an open interval with an interior point removed. An example in two dimensions is where Ω is an open disc with a slit cut out. Three-dimensional examples relevant for computational electromagnetism are the "pseudo-Lipschitz domains" of [3, Definition 3.1].)

(iii) If $0 < d := \dim_H(\operatorname{int}(\overline{\Omega}) \setminus \Omega) < n$ then $\widetilde{H}^s(\Omega) \neq H^s_{\overline{\Omega}}$ for all $s < -(n-d)/2$ and $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ for all $s > -(n-d)/2$.

(iv) If $\operatorname{int}(\overline{\Omega}) \setminus \Omega$ is countable then $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ if and only if $s \geq -n/2$.

(v) If $\widetilde{H}^t(\Omega) \neq H^t_{\overline{\Omega}}$ for some $t \in \mathbb{R}$ then $\widetilde{H}^s(\Omega) \neq H^s_{\overline{\Omega}}$ for all $s < t$. (Whether the assumption that $\operatorname{int}(\overline{\Omega})$ is C^0 is necessary here appears to be an open question. Lemma 3.16(iii for



Lemma 3.28. Let $\Omega \subset \mathbb{R}^n$ be non-empty and open, and let $s \in \mathbb{R}$. Then $H^s_0(\Omega) = H^s(\Omega)$ if and only if $\widetilde{H}^{-s}(\Omega) \cap H^{-s}_{\partial\Omega} = \{0\}$.

Proof. This follows from Theorem 3.3 and Lemma 3.7, which together imply that, by duality, $H^s_0(\Omega) = H^s(\Omega)$ if and only if $(\widetilde{H}^{-s}(\Omega) \cap H^{-s}_{\partial\Omega})^{\perp,\widetilde{H}^{-s}(\Omega)} = \widetilde{H}^{-s}(\Omega)$, which holds if and only if $\widetilde{H}^{-s}(\Omega) \cap H^{-s}_{\partial\Omega} = \{0\}$. $\square$

Corollary 3.29. Let $\Omega \subset \mathbb{R}^n$

For $2 \leq n \in \mathbb{N}$ the bounded C^0 open set of [25, Lemma 4.1(vi)] satisfies s_0

(vi) For $s \quad 0$, $j \quad : \widetilde{H}^s(\ ) \to H^s_0(\ )$ is injective and has dense image; if $s \in \mathbb{N}_0$ then it is a unitary isomorphism;

(vii) $j \quad : \widetilde{H}^s(\ ) \to H^s(\ )$ is bijective if and only if $j \quad : \widetilde{H}^{\ s}(\ ) \to H^{\ s}(\ )$ is bijective;

(viii) $j \quad : \widetilde{H}^{\ s}(\ ) \to H^{\ s}(\ )$ is injective if and only if $j \quad : \widetilde{H}^s(\ ) \to H^s(\ )$ has dense image; i.e. if and only if $H^s_0(\ ) = \ H^s(\ )$;

(ix) The following are equivalent:

$j \quad : \widetilde{H}^s(\ ) \to H^s_0(\ )$ is a unitary isomorphism;

$j \quad _{H^s(\ )} = k \ k_{H^s(\mathbb{R}^d)}$ for all $\ \in D(\ )$;

$D(\ ) \quad (H^s_{\ c})^?$;

(x) If   is bounded, or ^{c} is bounded with non-empty interior, then the three equivalent statements in (ix) hold if and only if

Proposition 3.33. Suppose that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, where $\{\Omega_j\}_{j=1}^{\infty}$ is a nested sequence of non-empty open subsets of $\mathbb{R}^n$ satisfying $\Omega_j \subset \Omega_{j+1}$ for $j = 1, 2, \ldots$. Then Ω is open and

$$\widetilde{H}^s(\Omega) = \overline{\bigcup_{j=1}^{\infty} \widetilde{H}^s(\Omega_j)}. \tag{29}$$

Proof. We will show below that

$$D(\Omega) = \bigcup_{j=1}^{\infty} D(\Omega_j). \tag{30}$$

Then (29) follows easily from (30) because

$$\widetilde{H}^s(\Omega) = \overline{D(\Omega)} = \overline{\bigcup_{j=1}^{\infty} D(\Omega_j)} = \overline{\bigcup_{j=1}^{\infty} \overline{D(\Omega_j)}} = \overline{\bigcup_{j=1}^{\infty} \widetilde{H}^s(\Omega_j)}.$$

To prove (30), we first note that the inclusion

(

have exactly one solution, and moreover the sequence $(u_{V_{j,j}})_{j=1}^{\infty}$ converges to $u_{\widetilde{H}^s(\Omega)}$ in the $H^s(\mathbb{R}^n)$ norm, because the sequence $(V_{j,j})_{j=1}^{\infty}$ is dense in $\widetilde{H}^s(\Omega)$. (Here we use Proposition 3.33 and (8).)

As a concrete example, take $\Omega \subset \mathbb{R}^2$ to be the Koch snowflake [20, Figure 0.2], Ω_j the prefractal set of level j (which is a Lipschitz polygon with $3 \cdot 4^{j-1}$ sides), $s = 1$ and $a(u,v) = \int_{B_R} \nabla u \cdot \nabla \overline{v}\, dx$ the sesquilinear form for the Laplace equation, which is continuous and coercive on $\widetilde{H}^s(B_R)$, where B_R is any open ball containing $\overline{\Omega}$. The $V_{j,k}$ spaces can be taken as nested sequences of standard finite element spaces defined on the polygonal prefractals. Then the solutions $u_{V_{j,j}} \in V_{j,j}$ of the discrete variational problems, which are easily computable with a finite element code, converge in the $H^1(\mathbb{R}^2)$ norm to $u_{\widetilde{H}^1(\Omega)}$, the solution to the variational problem on the right hand side in (31).

4 Boundary integral equations on fractal screens

This section contains the paper's major application, which has motivated much of the earlier theoretical analysis. The problem we consider is itself motivated by the widespread use in telecommunications of electromagnetic antennas that are designed as good approximations to fractal sets. The idea of this form of antenna design, realised in many applications, is that the self-similar, multi-scale fractal structure leads naturally to good and uniform performance over a wide range of wavelengths, so that the antenna has effective wide band performance [

here Γ is as defined in §1):

$$\text{Find } u \in C^2(\mathbb{R}^3 \setminus \overline{\Gamma}) \cap W^1_2(\mathbb{R}^3 \setminus \overline{\Gamma}) \text{ such that } \Delta u + k^2 u = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Gamma} \text{ and}$$
$$u = f \in H^{1/2}(\Gamma) \text{ on } \Gamma \text{ (Dirichlet) or}$$
$$\frac{\partial u}{\partial n} = g \in H^{-1/2}(\Gamma) \text{ on } \Gamma \text{ (Neumann)} .$$

Where $U_+ := \{x \in \mathbb{R}^3 : x_3 > 0\}$ and $U_- := \mathbb{R}^3 \setminus \overline{U_+}$ are the upper and lower half-spaces, by $u = f$ on Γ we mean precisely that $\gamma^\pm u|_\Gamma = f$, where $\gamma^\pm$ are the standard trace operators $\gamma^\pm : H^1(U_\pm) = W^1_2(U_\pm) \to H^{1/2}(\Gamma_\infty)$. Similarly, by $\partial u/\partial n = g$ on Γ we mean precisely that $\partial_n^\pm u|_\Gamma = g$, where $\partial_n^\pm$ are the standard normal derivative operators $\partial_n^\pm : W^1_2(U_\pm;\Delta) \to H^{-1/2}(\Gamma_\infty)$; here $W^1_2(U_\pm;\Delta) = \{u \in W^1_2(U_\pm) : \Delta u \in L^2(U_\pm)\}$, and for definiteness we take the normal in the x_3-direction, so that $\partial u/\partial n = \partial u/\partial x_3$.

These screen problems are uniquely solvable: one standard proof of this is via BIE methods [44]. The following theorem, reformulating these screen problems as BIEs, is standard (e.g. [44]), dating back to [51] in the case when Γ is C^∞ (the result in [51] is for $k > 0$, but the argument is almost identical and slightly simpler for the case $\Im(k) > 0$). The notation in this theorem is that $[u] := \gamma^+ u - \gamma^- u \in \widetilde{H}^{1/2}(\Gamma) \subset H^{1/2}(\Gamma_\infty)$ and $[\partial_n u] := \partial_n^+ u - \partial_n^- u \in \widetilde{H}^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma_\infty)$

[25, Proposition 3.4, Remark 3.14]), the third from (34), and the second equality follows because $a_D(\; ; \;) = a_{dom}(S \; ; S \;)$, for all $\; \in H^{1/2}(\;)$ (cf. the proof of [17, Theorem 2]).

We are interested in sequences of screen problems, with a sequence of screens $_1; \; _2; \ldots$ converging in some sense to a limiting screen. We assume that there exists $R > 0$ such that the open set $_j^R := \{x \in \; _1 : |x| < R\}$ for every $j \in \mathbb{N}$. Let a

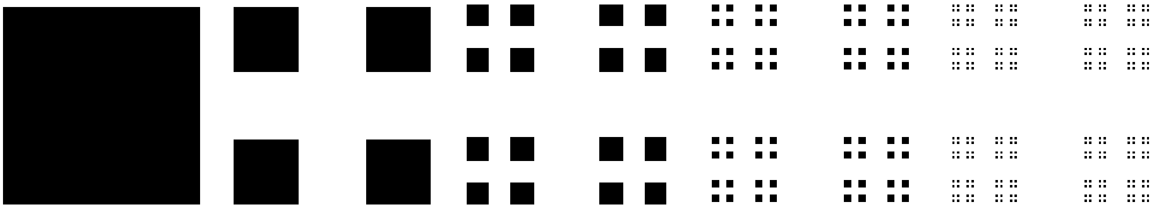


Figure 5: The rst ve terms in the recursive sequence of prefactals converging to the standard two-dimensional middle-third Cantor set (or Cantor dust).

where $E = \cap_{j=0}^{1} E_j$ is the middle- Cantor set and E^2 is the associated two-dimensional Cantor set (or \Cantor dust"), which has Hausdor dimension $\dim_H(E^2) = 2\log 2 = \log(1= \ ) \ 2\ (0;2)$. It is known that E^2 is s-null if and only if s (dim$_H(E^2)$ n)=2 (see [25, Theorem 4.5], where E^2 is denoted $F^{(2)}_{2\log 2=\log(1= \ );1}$). Theorem 4.3 applied to this example shows that if 1=4 < < 1=2 then there exists f_1 2 $H^{1=2}$($_1$) such that the limiting solution 2 $H_F^{1=2}$ to the sequence of screen problems is non-zero. On the other hand, if 0 < 1=4 then the theorem tells us that the limiting solution = 0.

It is clear from Theorem 4.3 that whether or not the solution to the limiting sequence of screen problems is zero depends not on whether the limiting set F

By Proposition 3.33, $V := \overline{\bigcup_{j \in \mathbb{N}} V_j} = \widetilde{H}^{1/2}(\ )$. The first sentence of the following proposition is immediate from (8), and the second sentence is clear.

Proposition 4.5.

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