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Existence of 1D vectorial Absolute

EXISTENCE OF 1D VECTORIAL ABSOLUTE MINIMISERS IN L^∞ UNDER MINIMAL ASSUMPTIONS

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Abstract. We prove the existence of vectorial Absolute Minimisers in the sense of Aronsson to the supremal functional $E_\infty(u, \Omega') = \|L(\cdot, u, Du

case of maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$) together with its associated system of equations begun in the early 2010s by the second author in a series of papers, see [K1]-[K6], [K8]-[K12

Theorem 1 generalises two respective results in the both the papers [BJW1] and [K9]. On the one hand, in [BJW1] Theorem 1 was established under the extra assumption $C_2 = C_3 = 0$ which forces $L_\infty(x,\eta,0) = 0$, for all $(x,\eta) \in \mathbb{R} \times \mathbb{R}^N$. Unfortunately this requirement is incompatible with important applications of (1.1) to problems of L^1-modelling of variational Data Assimilation (4DVar) arising in the Earth Sciences and especially in Meteorology (see [B, BS, K9]). An explicit model of L_∞ is given by

$$L_\infty(x,\eta,P) := \big|k(x) - K(\eta)\big|^2 + \big|P - V(x,\eta)\big|^2, \tag{1.7}$$

and describes the "error" in the following sense: consider the problem of finding spaces $W_b^{1,q_m}(\ ;\mathbb{R}^N)$ such that, for any s 1, u^m $* \ u^\infty$ weakly as $m \to \infty$ in $W^{1,s}(\ ;\mathbb{R}^N)$ along a subsequence. Moreover,

$$E_\infty(u^\infty;\ ) = C_\infty = \lim_{m\to\infty} C_m : \tag{2.2}$$

By approximate minimiser we mean that u^m satisfies

$$E_m(u^m;\ ) \quad C_m < 2^{\ m^2} : \tag{2.3}$$

Finally, for any A measurable of positive measure the following lower semi-continuity inequality holds

$$E_\infty(u^\infty;A) \quad \liminf_{m\to\infty} \tag{2.4}$$

as m ! 1 along a subsequence. Now, recalling that $^{m;}$ = u^m at the endpoints f 0; 1g, and since u^m is an approximate minimiser of (2.1) over $W_b^{1;m}$ (; R^N) for each m 2 N, by utilising minimality, the additivity of the integral and Hölder inequality, we get

$$E_m \; u^m ; (0;1) \qquad E_m \quad {}^{m;} \; ; (0;1) \;\; + 2 \quad m^2$$

and hence

$$E_m \; u^m ; (0;1) \; ^{\frac{1}{m}} \qquad E_m \quad {}^{m;} \; ; (0;1) \; ^{\frac{1}{m}} \; + 2 \quad ^m \tag{2.6}$$
$$E_1 \quad {}^{m;} \; ; (0;1) \;\; + 2 \quad ^m :$$

On the other hand, we have

$$E_1 \quad {}^{m;} \; ; (0;1) \;\; = \max \; \mathrm{n} \; E_1 \quad {}^{m;} \; ; (0; \;\;) \; ;$$
$$E_1 \quad {}^{m;} \; ; (\; ; 1 \qquad) \; ;$$
$$E_1 \quad {}^{m;} \; ; (1 \qquad ; 1) \; \mathrm{o}$$

and since $^{m;}$ = ^{1} on (; 1), we have

$$E_1 \quad {}^{m;} \; ; (0;1) \qquad \max \;\; E_1 \quad {}^{m;} \; ; (0; \;\;) \; ; E_1 \quad {}^1 \; ; (0;1) \; ; \tag{2.7}$$
$$E_1 \quad {}^{m;} \; ; (1 \qquad ; 1) \quad :$$

Combining (2.5)-(2.7) and (2.4), we get

$$E_1 \;\; u^1 \; ; (0;1) \qquad \liminf_{m ! 1} \quad \max \; \mathrm{n} \; E_1 \quad {}^{m;} \; ; (0; \;\;) \; ; E_1 \quad {}^1 \; ; (0;1) \; ;$$
$$E_1 \quad {}^{m;} \; ; (1 \qquad ; 1) \; \mathrm{o} \tag{2.8}$$
$$\max \; \mathrm{n} \; E_1 \quad {}^1 \; ; (0;1) \; ; E_1 \quad {}^{1 \; ;} \; ; (0; \;\;) \; ;$$
$$E_1 \quad {}^{1 \; ;} \; ; (1 \qquad ; 1) \; \mathrm{o} \; :$$

Let us now denote the di erence quotient of a function v : R ! R^N as $D^{1;t} v(x)$:= $\frac{1}{t}$[v(x + t) v(x)]. Then, we may write

$$D \;\; {}^{1 \; ;} \; (x) \; = \; D^{1;} \quad {}^1 \; (0), \qquad x \; 2 \; (0; \;\;);$$
$$D \;\; {}^{1 \; ;} \; (x) \; = \; D^{1;} \quad {}^1 \; (1), \qquad x \; 2 \; (1 \qquad ; 1),$$

Note now that

$$\text{(2.9)} \qquad E_1 \quad {}^{1 \; ;} \; ; (0; \;\;) \;\; = \; \max_{0 \;\; x} \quad 1 \; (1), \qquad x \; 2 \; (1 \qquad ; \qquad {}^{x}_{1}$$

The rest of the proof is devoted to establishing (2.10). Let us begin by recording for later use that

$$\begin{cases} \max\limits_{0 \ x} \ {}^{1;}(x) \quad {}^{1}(0) \ \to \ 0, & \text{as} \ \to \ 0; \\ \max\limits_{1 \ x \ 1} \ {}^{1;}(x) \quad {}^{1}(1) \ \to \ 0, & \text{as} \ \to \ 0. \end{cases} \tag{2.11}$$

Fix a generic $u \in W^{1;1}(\ ;\mathbb{R}^N)$, $x \in [0,1]$ and $0 < \varepsilon < 1/3$ and define

$$A_\varepsilon(x) := [x \quad \varepsilon, x + \varepsilon] \cap [0,1].$$

We claim that there exist an increasing modulus of continuity $\omega \in C(0,1\ )$ with $\omega(0^+) = 0$ such that

$$E_1 \quad u; A_\varepsilon(x) \qquad \operatorname*{ess\,sup}_{y \in A_\varepsilon(x)} L \quad x; u(x); Du(y) \qquad \omega(\varepsilon). \tag{2.12}$$

Indeed for a.e. $y \in A_\varepsilon(x)$ we have $|x$

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