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Preprint MPS-2016-04

15 April 2016

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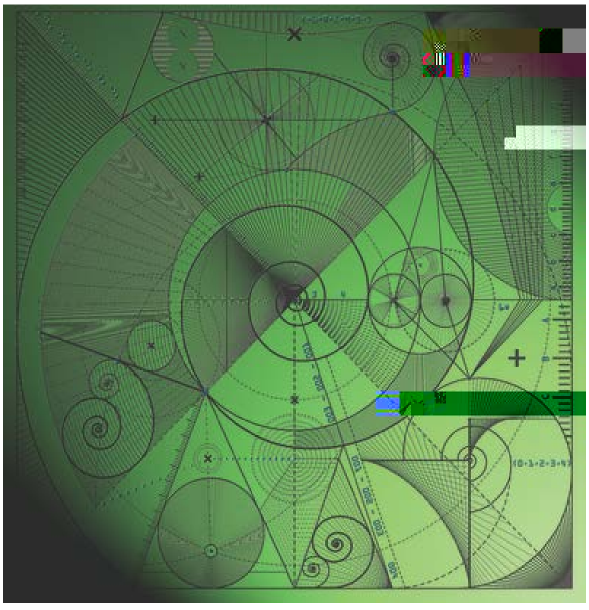

SOLUTIONS OF VECTORIAL HAMILTON-JACOBI EQUATIONS ARE RANK-ONE ABSOLUTE MINIMISERS IN L^{∞}

NIKOS KATZOURAKIS

Abstract. Given the supremal functional $E_{\infty}(u;$

$\mathbb{R}^n$, the respective PDE is the ∞-Laplace equation

$$\Delta_\infty u \,:=\, Du \otimes Du : D^2u \,=\, \sum_{i,j=1}^{n} D_i u \, D_j u \, D^2_{ij} u \,=\, 0. \tag{1.3}$$

Despite the importance for applications and the deep analytical interest of the area, the vectorial case of $N \geq 2$ remained largely unexplored until the early 2010s. In particular, not even the correct form of the respective PDE systems associated to L^∞ variational problem was known. A notable exception is the early vectorial contributions [BJW1, BJW2] wherein (among other deep results) L^∞ versions of lower semi-continuity and quasiconvexity were introduced and studied and the existence of Absolute Minimisers was established in some generality with H depending on u itself but for $\min\{n, N\}$

di erent sets of variations. In [K2] we proved the following variational characterisation in the class of classical solutions. A C^2 map u : $\mathbb{R}^n$! $\mathbb{R}^N$ is a solution to

(1.5) $$Du \quad Du : D^2u = 0$$

if and only if it is a Rank-One Absolute Minimiser on , namely when for all D b , all scalar functions g 2 $C^1_0(D)$ vanishing on @D and all directions 2 $\mathbb{R}^N$, u is a minimiser on D with respect to variations of the form u + g (Figure 1):

(1.6) $$kDuk_{L^1(D)} \quad Du + \quad Dg \; {}_{L^1(D)}:$$

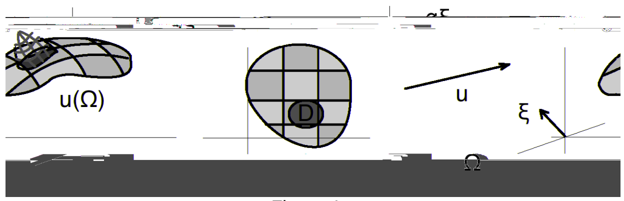

Figure 1.

Further, if rk(D u) const., u is a solution to

(1.7) $$jDuj^2[Du]^? \quad u = 0$$

if and only if u( ) has 1 -Minimal Area, namely when for all D b , all scalar functions h 2 $C^1(\overline{D})$ (not vanishing on @D) and all vector elds 2 $C^1(D;\mathbb{R}^N)$ which are normal to u( ), u is a minimiser on D with respect to normal free variations of the form u + h (Figure 2):

(1.8) $$kDuk_{L^1(D)} \quad Du + D(\,h\,) \; {}_{L^1(D)}:$$

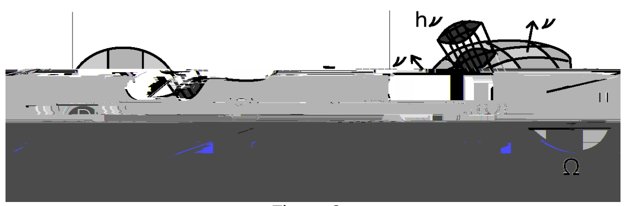

Figure 2.

We called a map 1 -Minimal with respect to functional $kD(\,)k_{L^1(\,)}$ when it is a Rank-One Absolute Minimiser on and u( ) has 1 -Minimal Area.

Perhaps the greatest di culty associated to (1.1) and (1.4

In this paper we consider the obvious generalisation of the rank-one minimality notion of (1.6) adapted to the functional (1.1). To this end, we identify a large class of rank-one Absolute Minimisers: for any $c \geq 0$, every solution $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the vectorial Hamilton-Jacobi equation

$$H\big(x, Du(x)\big) = c, \quad x \in \Omega, \tag{1.9}$$

actually is a rank-one absolute minimiser. Namely, for any $\Omega' \Subset \Omega$, any $\phi \in W^{1,\infty}_0(\Omega')$ and any $\xi \in \mathbb{R}^N$, we have

$$\operatorname*{ess\,sup}_{x\in\Omega'} H\big(x, Du(x)\big) \leq \operatorname*{ess\,sup}_{x\in\Omega'} H\big(x, Du(x) + \xi \otimes D\phi(x)\big).$$

For the above implication to be true we need the solutions to be in $C^1(\Omega;\mathbb{R}^N)$ and not just in $W^{1,\infty}_{loc}(\Omega;\mathbb{R}^N)$. This is not a technical difficulty: it is well known even in the scalar case that if we allow only for 1 non-differentiability point, strong solutions of the Eikonal equation $|Du| = 1$ are not absolutely minimising for the L^∞ norm of the gradient (e.g. the cone function $x \mapsto |x|$). However, due to regularity results which available in the scalar case, it suffices to assume everywhere differentiability (see [CEG, CC]).

Our only hypothesis imposed on H is that for any $x \in \Omega$ the partial function $H(x,\cdot) : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is rank-one level-convex. This means that for any $t \geq 0$, the sublevel sets $\{H(x,\cdot) \leq t\}$ are rank-one convex sets in $\mathbb{R}^{N\times n}$. A set $C \subseteq \mathbb{R}^{N\times n}$ is called rank-one convex when for any matrices $A, B \in C$ with $\mathrm{rk}(A - B) \leq 1$, the convex combination $\lambda A + (1-\lambda)B$ is in C for any $0 \leq \lambda \leq 1$. An equivalent way to phrase the rank-one level-convexity of $H(x,\cdot)$ is via the inequality

$$H\big(x, \lambda A + (1-\lambda)B\big) \leq \max\big\{H(x,A), H(x,B)\big\}, \quad \forall\, x,$$

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $n, N \in \mathbb{N}$ and $H : \Omega \times \mathbb{R}^{N\times n} \longrightarrow [0,\infty)$ a continuous function, such that for all $x \in \Omega$, $P \mapsto H(x,P)$ is rank-one level-convex, that is*

$$\{H(x,\cdot) \leq t\} \text{ is a rank-one convex in } \mathbb{R}^{N\times n}, \text{ for all } t \geq 0,\ x \in \Omega.$$

Let $u \in C^1(\Omega;\mathbb{R}^N)$ be a solution to the vectorial Hamilton-Jacobi PDE

$$H(\cdot, Du) = c \quad \text{on } \Omega,$$

for some $c \geq 0$. Then, u is a rank-one Absolute Minimiser of the functional

$$E_\infty(u,\Omega') = \operatorname*{ess\,sup}_{x\in\Omega'} H\big(x, Du(x)\big), \quad \Omega' \Subset \Omega, \quad u \in W^{1,\infty}_{loc}(\Omega;\mathbb{R}^N).$$

In addition, the following marginally stronger result holds true: for any $\Omega' \Subset \Omega$, any $\psi \in W^{1,\infty}_0(\Omega')$ and any $\xi \in \mathbb{R}^N$, we have

$$E_\infty(u,\Omega') \leq \inf_{B\in\mathcal{B}(\Omega;\Omega')} E_\infty\big(u + \xi\psi, B\big)$$

where $\mathcal{B}(\Omega;\Omega')$

We illustrate the idea by assuming rst in addition that 2 $W^{1;1}_u$ (^{0}; R^N) \ C^1(^{0}; R^N). In this case, the point x is a critical point of (u) and we have D (u) (x) = 0. Hence,

$$\begin{aligned} D(\quad u)(x) &= [\ ]^> D(\quad u)(x) + [\ ]^? D(\quad u)(x) \\ &= \quad D \quad (\quad u)\ (x) + D\ [\ ]^? (\quad u)\ (x) \\ &= 0 \end{aligned}$$

because $[]^?$ $[\ ]^?$ u on ^{0}. Thus,

$$\begin{aligned} E_1\ (u; \ ^0) = \ c &= H\,(x; Du(x)) \\ &= H \quad x; \quad D(\quad u)(x) + [\ ]^? Du(x) \end{aligned}$$

and hence

$$\begin{aligned} E_1\ (u; \ ^0) &= H \quad x; \quad D(\quad)(x) + [\ ]^? D \ (x) \\ &= H\,(x; D \ (x)) \\ &\quad \operatorname{ess\,sup}_{y 2 B \ (x)} H\,(y; D \ (y)) \\ &= E_1 \quad ; B \ (x) \end{aligned} \tag{2.2}$$

for any B (x) B (u); ^{0} , whence the conclusion ensues.

Now we return to the general case of 2 $W^{1;1}_u$ (^{0}; R^N). We extend by u on n ^{0} and consider the sets

$$_k := \begin{cases} x\ 2 \quad ^0 : \operatorname{dist}(x; @ \ ^0) > \dfrac{d_0}{k} \ ; & k\ 2\ N; \\ ; ; & k = 0 ; \end{cases} \tag{2.3}$$

where $d_0 >$ 0 is a constant small enough so that $_1$ 6= ; . We set

$$V_k := \ _k \ n \ \overline{_{k \ 1}}; \quad k\ 2\ N \tag{2.4}$$

and consider a partition of unity ($_k)^1_{k=1}$ C^1_c (^{0}) over ^{0} so that 0) :(398 w 0 0 m 22.295 0 l S Q BT /

= k

j j = 1, we have

$$k \;\; " \qquad k_{C(\overline{V_l})} = \sup_{V_l} \; \overset{1}{X}_{k=1} \; k \quad "=k \,) \quad \overset{1}{X}_{k=1} \; k \;\; !$$

$$\sup_{V_l} \; \overset{1}{X}_{k=1} \; k \quad "=k$$

(2.7)

$$= \sup_{V_l} \; \overset{l+1}{X}_{k=l\;\;1} \; k \quad "=k$$

$$3 \max_{k=l\;\;1;l;l+1} \quad "=k \qquad {}_{C(\overline{}{}^{0})};$$

whilst, for l = 1 we similarly have

(2.8) $$k \;\; " \qquad k_{C(\overline{V_1})} \quad 2 \max_{k=1;2} \quad "=k \qquad {}_{C(\overline{}{}^{0})}:$$

By the standard properties of molli ers, we have that the function

(2.9) $$! \,(t) := \sup_{0< \;<t} \qquad {}_{C(\overline{}{}^{0})}; \qquad 0 < t < d \;\; {}_0;$$

is an increasing continuous modulus of continuity with $!\,(0^+) = 0$. By (2.7)-(2.9), we have that

(2.10) $$k \;\; " \qquad k_{C(\overline{V_l})} \qquad \begin{cases} 3! \;\; \dfrac{"}{l\;\;\;1} \; ; & l \quad 2; \\ 2!\,("); & l = 1: \end{cases}$$

Since the C^1 regularity of " is obvious (becauseu by assumption is such and $[\;]^?$ $[\;]^?$ u), the claim has been established.

Note now that since u 2 $W_0^{1;1}$ (^{0}; R^N), the set B (

our continuity assumption and the $W^{1;1}$ regularity of imply that there exists a positive increasing modulus of continuity $!_1$ with $!_1(0^+) = 0$ such that on the ball $B_{=2}(x_0)$ we have

(2.13)
$$
\begin{aligned}
H(\ ;D^{"}) = H &\Big(\ ;\ \sum_{k=1}^{1}\ _k\ D(\ )^{"=k} \\
&+ \sum_{k=1}^{1} D\ _k\ (\ )^{"=k} + [\ ]^{?} D\Big) \\
H &\Big(\ ;\ \sum_{k=1}^{1}\ _k\ D(\ )^{"=k} + [\ ]^{?} D\Big) \\
&+ !_1\Big(\sum_{k=1}^{1} D\ _k\ (\ )^{"=k}\Big) \\
&=: A + B.
\end{aligned}
$$

By further restricting $" < = 2$, we may arrange

(2.14)
$$\bigcup_{x \in B_{=2}(x)} B_{"}(x) \quad B\ (x_0)$$

and by (2.4)-(2.5), there exists $K(\ ) \in \mathbb{N}$ such that

(2.15)
$$B\ (x_0) \quad \bigcup_{k=1,\ldots,K(\ )} \overline{V_k}.$$

This implies that for any $x \in B\ (x_0)$,

(2.16)
$$\sum_{1}^{1}\ _k(x) = \sum_{1}^{K(\ )+1}\ _k(x) = 1$$

forming a convex combination. We now recall for immediate use right below the following Jensen-like inequality for level-convex functions (see e.g. [BJW1, BJW2]): for any probability measure on an open set $U \quad \mathbb{R}^n$ and any -measurable function $f : U \quad \mathbb{R}^n \to [0; 1\ )$, we have

(2.17)
$$\int_U f(x)\, d\ (x) \qquad \operatorname*{ess\,sup}_{x \in U} f(x)\ ;$$

when $: \mathbb{R}^n \to \mathbb{R}$ is any continuous level-convex function. Further, by our rank-one level-convexity assumption on H and if is as above, for any $x \in$ and $\in \mathbb{R}^N$ with $|\ | = 1$, the function

(2.18)
$$(\ p) := H\big(x;\quad p + [\ ]^{?} D\ (x)\big)\ ;\qquad p \in \mathbb{R}^n;$$

is level-convex. Indeed, given $p, q \in \mathbb{R}^n$ and $t \quad 0$ with $(\ p); (\ q) \quad t$, we set

$$
\begin{cases}
P := \quad p + [\ ]^{?} D\ (x); \\
Q := \quad q + [\ ]^{?} D\ (x).
\end{cases}
$$

Then, $P - Q = \quad (p - q)$ and hence $\mathrm{rk}(P - Q) \quad 1$. Moreover, $H(x;P) = (\ p) \quad t$ and $H(x;Q) = (\ q) \quad t$ which gives

$$\big(\ p + (1 - \ )q\big) = H\big(x;\ P + (1 - \ )Q\big) \qquad t$$

for any $\ 2\ [0;1]$, as desired.

Now, by using (2.4)-(2.5), (2.14)-(2.16) and the level-convexity of the function of (2.18), for any $x\ 2\ B_{=2}(x_0)$ we have the estimate

$$A(x) = H\left(x;\ \left[\sum_{k=1}^{K(\ )+1} \ _k(x)\ \ D(\ \ )\ \ ^{"=k}\ (x)\right] + [\ \ ]^{?} D\ \ (x)\right)$$

$$(2.19)\qquad = \left(\sum_{k=1}^{K(\ )+1} \ _k(x)\ \ D(\ \ )\ \ ^{"=k}\ (x)\right)$$

$$\max_{k=1;:::;K(\ )+1} \ \ D(\ \ )\ \ ^{"=k}\ (x)$$

$$= \max_{k=1;:::;K(\ )+1} \left(\int_{B_{"=k}(x)} D(\ \ )(y)\ \ ^{"=k}(|x\ \ y|)\, dy\right) :$$

Since for any x and $";k$, the map

$$:=\ ^{"=k}(|x\ \ \ |)\, L^n$$

is a probability measure on the ball $B_{"=k}(x)$ which is absolutely continuous with respect to the Lebesgue measure L^n, in view of (2.17), (2.19) gives

$$A(x)\ \ \max_{k=1;:::;K(\ )+1} \left(\operatorname*{ess\,sup}_{y\,2\,B_{"=k}(x)} D(\ \ )(y)\right)$$

$$(2.20)\qquad = \max_{k=1;:::;K(\ )+1} \left(\operatorname*{ess\,sup}_{y\,2\,B_{"=k}(x)} H\ \ x;\ \ \ D(\ \ )(y) + [\ \ ]^{?} D\ \ (x)\right)$$

$$\operatorname*{ess\,sup}_{y\,2\,B_{"}(x)} H\ \ x;\ \ \ D(\ \ )(y) + [\ \ ]^{?} D\ \ (x)\ :$$

By the continuity of H and Du, there is a positive increasing modulus of continuity $!_2$ with $!_2(0^+) = 0$ such that

$$\begin{cases} \ H(x;P)\ \ H(y;Q)\ \ \ !_2\ \ |x\ \ y| + |P\ \ Q|\ ; \\ \ Du(x)\ \ Du(y)\ \ \ !_2\ \ |x\ \ y|\ ; \end{cases}$$

for all $x;y\ 2\ B\ (x_0)$ and $|P|;|Q|\ \ \ k\ D\ \ k_{L^1(\ \ ^0)} + 1$. By using that $[\ \ ]^{?}\ \ \ \ [\ \ ]^{?}\ u$ on $\ ^0$, (2.20) and the above give

$$A(x)\ \ \operatorname*{ess\,sup}_{y\,2\,B_{"}(x)} H\ \ x;\ [\ \ ]^{>} D\ \ (y) + [\ \ ]^{?} D\ \ (x)$$

$$\operatorname*{ess\,sup}_{y\,2\,B_{"}(x)} \Big(\ H\ \ y;\ [\ \ ]$$

>88(of)]TJ/F11 9.9626 Tf 89.385 0 Td [(H)]TJ/F8 9.9626 Tf 11.961 0 Td [(and)-288(D)]TJ/F11 9.9626 Tf 26.532 0 T

By (2.14), (2.21) gives

$$
\begin{aligned}
A(x) \quad & \operatorname*{ess\,sup}_{y \in B_{\varepsilon}(x)} H\ \ y;\, D\ \ (y) \ \ + \sup_{y \in B_{\varepsilon}(x)} \omega_2 \ \ |x \quad y| + \ \ Du(y) \quad Du(x) \\
& \operatorname*{ess\,sup}_{y \in B\ \ (x_0)} H\ \ y;\, D\ \ (y) \ \ + \omega_2 \ \ \varepsilon + \omega_2(\varepsilon) \ \ ;|
\end{aligned}
$$

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