A Sub-Riemannian Santalo Formula with

A SUB-RIEMANNIAN SANTAL O FORMULA WITH APPLICATIONS TO ISOPERIMETRIC INEQUALITIES AND FIRST DIRICHLET EIGENVALUE OF HYPOELLIPTIC OPERATORS

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Abstract. Sub-Riemannian geometry is a generalization of Riemannian one, to include nonholonomic constraints. In this paper we prove a nonholonomic version of the classical Santalo formula: a result in integral geometry that descri bes the intric-type inequalities for a compact domain M with piecewise C^2 boundary. Moreover, we prove a universal (i.e. curvature independent) lower bound for the rst Diric hlet eigenvalue $_1(M)$ of the intrinsic sub-Laplacian,

$$_1(M) \quad k\ ^2$$

$$\overline{}\ \ L^2\ ;$$

in terms of the rank k of the distribution and the length L of the longest reduced sub-Riemannian geodesic contained in M. All our results are sharp for the sub-Riemannian structures on the hemispheres of the complex and quaternionic Hopf brations:

$$S^1 \ ! \ S^{2d+1} \ !^{\ p} \ CP^d; \qquad S^3 \ ! \ S^{4d+3} \ !^{\ p} \ HP^d; \qquad d \quad 1;$$

where the sub-Laplacian is the standard hypoelliptic operator of CR and quaternionic contact geometries, L = and k = 2d or 4d, respectively.

1. Introduction and results

Let $(M; g)$ be a compact Riemannian manifold with boundary $@M$ Santalo formula [19, 43] is a classical result in integral geometry that describes the Liouville measure of the unit tangent bundle UM in terms of the geodesic ow $_t : UM \ ! \ UM$. Namely, for any measurable function $F : UM \ ! \ R$ we have

$$(1) \qquad \int_{UM} F = \int_{@M} \left[\int_{U_q^+ @M} \left(\int_0^{`(v)} F(\ _t(v))\,dt \right) g(v; n_q)\ _q(v) \right] (q);$$

where is the surface form on $@M$ induced by the inward pointing normal vector n, $_q$ is the Riemannian spherical measure on U_qM, $U_q^+ @M$ is the set of inward pointing unit vectors at $q \, 2 \ @M$ and $`(v)$ is the exit length of the geodesic with initial vector v. Finally,

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and $r^p, R^p : M \to \mathbb{R}$ are

$$\frac{1}{R^p(q)} := \int_{U_q M} \frac{1}{r}$$

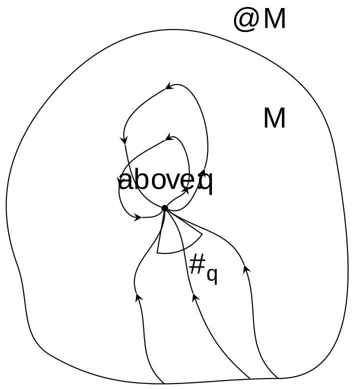

Figure 2. Visibility angle on a 2D Riemannian manifold. Only the geodesics with tangent vector in the dashed slice go to @M

where diam$_H$ denotes the horizontal diameter of the Carnot group.

In particular, if M is the metric ball of radius R, we obtain $_1$(M) k ^{2}=(2R)^2. Clearly (9) is not sharp, as one can check easily in the Euclidean case

1.5. Isoperimetric-type inequalities. In this section we relate the sub-Riemannian area and perimeter of M with some of its geometric properties. Since M is compact, the) sub-Riemann 10.9091 Tf 30.7199e

We can apply Proposition 7 to Carnot groups equipped with the Haar measure. In this case $\vartheta = \vartheta' = 1$ and $\ell = \mathrm{diam}^{\nabla}(M) = \mathrm{diam}_H(M)$. Moreover, ω is the Lebesgue volume of $\mathbb{R}^n$ and σ is the associated perimeter measure of geometric measure theory [16].

Corollary 9. *Let M be a compact n-dimensional submanifold with piecewise C^2 boundary of a Carnot group of rank k, with the Haar volume. Then,*

$$\frac{\sigma(\partial M)}{\omega(M)} \geq \frac{2\,|S^{k-1}|}{|S^k|\,\mathrm{diam}_H(M)},$$

where $\mathrm{diam}_H(M)$ is the horizontal diameter of the Carnot group.

This inequality is not sharp even in the Euclidean case, but it is very easy to compute the horizontal diameter for explicit domains. For example, if M is the sub-Riemannian metric ball of radius R, then $\mathrm{diam}_H(M) = 2R$.

1.6. Remark on the change of volume. Fix a sub-Riemannian structure (N, D, g), a compact set M with piecewise C^2 boundary and a complement V such that (H1) holds. Now assume that, for some choice of volume form ω, also (H2) is satisfied, so that we can carry on with the reduction procedure and all our results hold. One can derive the analogous of Propositions 2, 3, 4, 7 for any other volume $\omega' = e^{\varphi}\omega$, with $\varphi \in C^1(M)$. In all these results, it is sufficient to multiply the r.h.s. of the inequalities by the volumetric constant $0 < \kappa \leq 1$ defined as $\kappa := \frac{\min e^{\varphi}}{\max e^{\varphi}}$, and indeed replace ω with $\omega' = e^{\varphi}\omega$ in Propositions 2 and 3, σ with $\sigma' = e^{\varphi}\sigma$ in Proposition 7, and the sub-Laplacian Δ_{ω} with [3] $\Delta_{\omega'}$.

Furthermore, the

Remark 4. This construction gives a canonical way to define a measure on U^*M and its fibers in the general sub-Riemannian case, depending only on the choice of the volume ω on the manifold M. It turns out that this measure is also invariant under the Hamiltonian flow. Notice though that in the sub-Riemannian setting, fibers have infinite volume.

3.3. **Invariance.** Here we focus on the case of interest where $E \subseteq T^*M$ is a rank k vector sub-bundle and $E' \subset E$ is a corank 1 sub-bundle as defined in Section 3.2. We stress that E' is not necessarily a vector sub-bundle, but typically its fibers are cylinders or spheres.

Recall that the sub-Riemannian geodesic flow $\phi_t : T^*M \to T^*M$ is the Hamiltonian flow of $H : T^*M \to \mathbb{R}$. Moreover, in our picture, $M \subset N$ is a compact submanifold with boundary ∂M of a larger manifold N, with $\dim M = \dim N = n$.

Definition 2. *A sub-bundle $E \subseteq T^*M$ is* invariant *if $\phi_t(\lambda) \in E$ for all $\lambda \in E$ and t such that $\phi_t(\lambda) \in T^*M$ is defined. A volume form $\eta \in \Omega^{n+k}(E)$ is* invariant *if $\mathcal{L}_{\vec{H}}\eta = 0$.*

Our definition includes the case of interest for Santaló formula, where sub-Riemannian geodesics may cross $\partial M \neq \emptyset$. In other words, E is invariant if the only way to escape from E through the Hamiltonian flow is by crossing the boundary $\pi^{-1}(\partial M)$. Moreover, if η is an invariant volume on an invariant sub-bundle E, then $\phi_t^*\eta = \eta$.

Lemma 12 (Invariant induced measures)**.** *Let $E \subseteq T^*M$ be a rank k*

Writing $\nabla_X Y u(x) = \nabla(Yu)(x) \cdot X(x)$, where $v \cdot w$ is the Euclidean scalar product of $v, w \in \mathbb{R}^n$, proves the claim.

The above implies that the bracket $[X, Y]$ is tangent to ∂M a.e. on $C(\partial M)$ for any $X, Y \in \Gamma(D)$. In particular, this contradicts the bracket-generating assumption (11).

4.2. (Sub-)Riemannian Santaló formula. For any covector $\lambda \in U_q^* M$, the exit length $\ell(\lambda)$ is the first time $t \geq 0$ at which the corresponding geodesic $\gamma(t) = \pi \circ \phi_t(\lambda)$ leaves M crossing its boundary, while $\tilde{\ell}(\lambda)$ is the smallest between the exit and the cut length along $\gamma(t)$. Namely

$$\ell(\lambda) = \sup\{t \geq 0 \mid \gamma(t) \in M\},$$

$$\tilde{\ell}(\lambda) = \sup\{t \leq \ell(\lambda) \mid \gamma|_{[0,t]} \text{ is minimizing}\}.$$

We also introduce the following subsets of the unit cotangent bundle $\pi : U^* M \to M$:

$$U^+ \partial M = \{\lambda \in U^* M|_{\partial M} \mid \langle \lambda, \mathbf{n} \rangle > 0\},$$

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Remark 5. Even if M is compact and hence $\tilde{\ell} < +\infty$, in general $\tilde{U}^*M \subsetneq U^*M$. Moreover, if also $\ell < +\infty$ (that is, all geodesics reach the boundary of M in finite time), then $U^*M = \tilde{U}^*M$. Thus, our statement of Santaló formula contains [19, Theorem VII.4.1].

Remark 6. If ∂M is only Lipschitz and $C(\partial M)$ has positive measure, the above Santaló formulas still hold by removing on the left hand side from U^*M and $\tilde{U}^*M$ the set $\{e^{t}(\lambda) \mid \pi(\lambda) \in C(\partial M) \text{ and } t \geq 0\}$. Nothing changes on the right hand side as $\sigma(C(\partial M)) = 0$ by definition of σ_n.

Proof. Let $A \subset [0,+\infty) \times U^+\partial M$ be the set of pairs (t,λ) such that $0 < t < \ell(\lambda)$. By Lemma 15 it follows that A is measurable. Let also $Z = \pi^{-1}(\partial M) \cap U^*M$ which clearly has zero measure in U^*M. Define $\Psi : A \to U^*M \setminus Z$ as $\Psi(t,\lambda) = e^{t}(\lambda)$. This is a smooth diffeomorphism, whose inverse is $\Psi^{-1}(\lambda) = (e^{-\ell(-\lambda)}(\lambda), \ell(-\lambda))$. In particular, U^*M is measurable. Then, using Lemma 17 (see below),

$$\int_{U^*M} F\mu = \int_{\Psi(A)} F\mu = \int_A \Psi^*(F\mu) = \tag{17}$$
$$= \int_{\partial M}\left[\int_{U_q^+\partial M}\left(\int_0^{\ell(\lambda)} F(e^{t}(\lambda))\,dt\right)\langle\lambda, n_q\rangle\,\eta_q(\lambda)\right]\sigma(q),$$

by Fubini Theorem. Analogously, with $\tilde{A} = \{(t,\lambda) \mid 0 < t < \tilde{\ell}(\lambda)\}$ and $\tilde{Z} = Z \cup \{e^{\tilde{\ell}(\lambda)}(\lambda) \mid \lambda \in U^+\partial M\}$ the map $\Psi : \tilde{A} \to \tilde{U}^*M \setminus \tilde{Z}$ is a diffeomorphism with the same inverse. Then, the same computations as (17) replacing A with $\tilde{A}$ and Z with $\tilde{Z}$ yield (16).

Lemma 17. *The following local identity of elements of $\Lambda^{2n-1}(\mathbb{R} \times U^+\partial M)$ holds*

$$\Psi^*\mu|_{(t,\lambda)} = \langle\lambda, n_q\rangle\, dt \wedge \eta \wedge \sigma, \qquad \lambda \in U^+\partial M;$$

+ Tf5(i)-1.5(l)-382.453(i)-1.33606(d)5 .Td [(+)5]TJ2.453(i)-1.3360688621(h)-5.4449R57 4d [(+)-1.69395]TJ

where, in canonical coordinates (p,x) on T^*M

$$\mu = \iota_{\vec{e}}\,\Theta; \qquad \eta = \frac{1}{\omega(x)}dp; \qquad \Theta = \sigma^n/n!; \qquad \omega = \omega(x)dx.$$

Proof. For any $(t,\lambda) \in \mathbb{R} \times U^+\partial M$ let $\{\partial_t, v_1, \ldots, v_{2n-2}\}$ be a set of independent vectors

Putting together (18), (19), and (20) completes the proof of the statement.

4.3. **Reduced Santaló formula.** The following reduction procedure replaces the non-compact set U^*M in Theorem 16 with a compact subset that we now describe.

To carry out this procedure we fix a transverse sub-bundle $\mathcal{V} \subset TM$ such that $TM = \mathcal{D} \oplus \mathcal{V}$. We assume that $\mathcal{V}$ is the orthogonal complement of $\mathcal{D}$ w.r.t. to a Riemannian metric g such that $g|_{\mathcal{D}}$ coincides with the sub-Riemannian one and the associated Riemannian volume coincides with ω. In the Riemannian case, where $\mathcal{V}$ is trivial, this forces $\omega = \omega_R$, the Riemannian volume. In the genuinely sub-Riemannian case there is no loss of generality since this assumption is satisfied for any choice of ω.

Definition 3. The *reduced cotangent bundle* is the rank k vector bundle $\pi : T^*M^r \to M$ of covectors that annihilate the vertical directions:

$$T^*M^r := \{\lambda \in T^*M \mid \langle \lambda, v \rangle = 0 \text{ for all } v \in \mathcal{V}\}.$$

The *reduced unit cotangent bundle* is $U^*M^r := U^*M \cap T^*M^r$.

Observe that U^*M^r is a corank 1 sub-bundle of T^*M^r, whose fibers are spheres $\mathbb{S}^{k-1}$. If T^*M^r is invariant in the sense of Definition 2, we can apply the construction of Section 3.3. The Liouville volume Θ on T^*M induces a volume on T^*M^r as follows.

Let $X_1, \dots, X_k$ and $Z_1, \dots, Z_{n-k}$ be local orthonormal frames for $\mathcal{D}$ and $\mathcal{V}$, respectively. Let $u_i(\lambda) := \langle \lambda, X_i \rangle$ and $v_j(\lambda) := \langle \lambda, Z_j \rangle$ smooth functions on T^*M. Thus

$$T^*M^r = \{\lambda \in T^*M \mid v_1(\lambda) = \dots = v_{n-k}(\lambda) = 0\}.$$

For all $q \in M$ where the fields are defined, $(u; v) : T_q^*M \to \mathbb{R}^n$ are smooth coordinates on the fiber and hence $\partial_{u_1}, \dots, \partial_{u_k}, \partial_{v_1}, \dots, \partial_{v_{n-k}}$ are vectors on $T_\lambda(T_q^*M) \subset T_\lambda(T^*M)$ for all $\lambda \in \pi^{-1}(q)$. In particular, the vector fields $\partial_{v_1}, \dots, \partial_{v}$ o formula.

Lemma 18 (Explicit reduced vertical measure). Let $q_0 \in M$ and x a set of canonical coordinates $(p; x)$ such that q_0 has coordinates x_0 and

$\{\partial_{x_1}, \ldots, \partial_{x_k}\}_{q_0}$ is an orthonormal basis of D_{q_0},
$\{\partial_{x_{k+1}}, \ldots, \partial_{x_n}\}_{q_0}$ is an orthonormal basis of V_{q_0}.

In these coordinates $\omega|_{x_0} = dx|_{q_0}$. Then $\mu^r_{q_0} = \mathrm{vol}_{\mathbb{R}^k}$ and $\eta^r_{q_0} = \mathrm{vol}_{\mathbb{S}^{k-1}}$. In particular,

$$\int_{U_{q_0}M^r} \eta^r_{q_0} = |\mathbb{S}^{k-1}|, \qquad \forall q_0 \in M,$$

where $|\mathbb{S}^{k-1}|$ denotes the Lebesgue measure of $\mathbb{S}^{k-1}$ and $\mathrm{vol}_{\mathbb{R}^k}$, $\mathrm{vol}_{\mathbb{S}^{k-1}}$ denote the Euclidean volume forms of $\mathbb{R}^k$ and $\mathbb{S}^{k-1}$.

We now state the reduced Santaló formulas. The sets U

q reach the whole Euclidean plane $q^{\perp}\{z = 0\}$ (the left-translation of $\mathbb{R}^{2d} \subset \mathbb{R}^{2d+1}$). At $q = (x; z)$ this is the plane orthogonal to the vector $\left(\frac{1}{2}Jx; 1\right)$ w.r.t. the Euclidean metric.

5.2.

5.2.1. Riemannian submersions. A Riemannian submersion $\pi : (\tilde{M}; \tilde{g}) \to (M; g)$ is triv-

The orthogonal complement $D := V^{\perp}$ with the restriction $g|_D$ of the round metric define the standard sub-Riemannian structure on the complex Hopf fibrations. In real coordinates, as subspaces of $\mathbb{R}^{2d+2}$, the hemisphere and its boundary are

$$M = S^{2d+1}_{+} := \left\{ \sum_{i=0}^{d} x_i^2 + y_i^2 = 1 \mid x_0 \geq 0 \right\}, \quad \partial M = \left\{ \sum_{i=0}^{d} x_i^2 + y_i^2 = 1 \mid x_0 = 0 \right\}.$$

A different set of coordinates we will use is the following

$$(\theta, w_1, \dots, w_d) \mapsto \left(\frac{e^{i\theta}}{\sqrt{1+|w|^2}}, \frac{w_1 e^{i\theta}}{\sqrt{1+|w|^2}}, \dots, \frac{w_d e^{i\theta}}{\sqrt{1+|w|^2}} \right),$$

where $\theta \in (-\pi, \pi)$ and $w = (w_1, \dots, w_d) \in \mathbb{C}^d$. In particular $(w_1, \dots, w_d)$ are inohomgeneous coordinates for $\mathbb{CP}^d$ given by $w_j = z_j/z_0$ and θ is the fiber coordinate. The north pole corresponds to $\theta = 0$ and $w = 0$. The hemisphere is characterized by $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and its boundary by $\cos(\theta) = 0$.

Example 4 (Quaternionic Hopf fibrations). Let $\mathbb{H}$ be the field of quaternions. If $q = x + iy + jz + kw$, with $x, y, z, w \in$

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Consider the subset $D = U^+\partial M \cap \{\ell < +\infty\}$. Let $f\ (t) := f(\ _t(\ ))$. For $\ \in D$ we have $f\ (0) = f\ (\ell(\ )) = 0$ and the one-dimensional Poincaré inequality (25) gives

$$(27) \qquad \int_0^{\ell(\ )} \langle\ _t(\ ), \nabla_H f\rangle^2 dt = \int_0^{\ell(\ )} f'(t)^2 dt \quad \frac{2}{\ell^2(\ )}\int_0^{\ell(\ )} f\ (t)^2 dt.$$

Indeed we can replace ℓ with L, which is $\ _t$-invariant. Then

$$\frac{|S^{k-1}|}{k}\int_M |\nabla_H f(q)|^2 \omega(q) \qquad 2\int_{\partial M}$$

Proof of Proposition 4. With $L := \sup_{\gamma \in U M^r} L(\gamma)$, the Hardy inequality (4) can be further simplified into

$$\int_M |\nabla_H f|^2 \omega \geq \frac{k^2}{L^2} \int_M f^2 \omega.$$

By the min-max principle (13), whenever any $f \in C_0^\infty(M)$ such that $\int_M f^2 \omega = 1$, we have

$$\lambda_1(M) \geq \int_M |\nabla_H f|^2 \omega \geq \frac{k^2}{L^2}.$$

Proof of Proposition 5. Fix a north pole q_0 and the hemisphere M

Definition 7. The sub-Riemannian diameter and reduced diameter are:

$$\mathrm{diam}(M) := \sup\{ d(x;y) \mid x;y \in$$

Denote $@_v = (@_{v_1};\dots;@_{v_{n-k}})$ and $@_u = (@_{u_1};\dots;@_{u_k})$ and $@_x = (@_{x_1};\dots;@_{x_n})$. Recall that $^r(@_u;@_x) = (@_u;@_v;@_x) = (1)^{k(n k)} (@_v;@_u;@_x)$. Then, using twice $(L_{Ħ})(w) = Ħ((w)) (L_{Ħ}(w))$ for any `-form and `-uple w, we obtain

$$(L_{Ħ} \ ^r)(@_u;@_x) = Ħ(\ ^r(@_u;@_x)) \ ^r(L_{Ħ}(@_u;@_x))$$
$$= (1)^{k(n k)} \Big[Ħ((@_v;@_u;@_x)) (@_v;L_{Ħ}(@_u;@_x)) \Big]$$
$$= (1)^{k(n k)} \Big[(L_{Ħ})(@_v;@_u;@_x) + (L_{Ħ}(@_v);@_u$$ Ħ((90784 -20 10.909Td [(@)@.9

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