School of Mathematical
and Physical Sciences

University of Reading

Department of Mathematics and Statistics

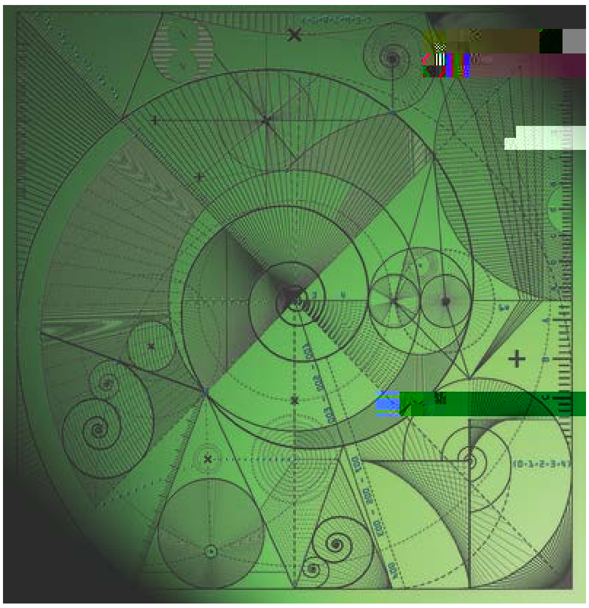
Preprint MPS-2016-02

15 March 2016

Accumulation of complex eigenvalues of an indefinite Sturm-Liouville operator with a shifted Coulomb potential

by

Michael Levitin and **Marcello Seri**



Accumulation of complex eigenvalues of an indefinite Sturm–Liouville operator with a shifted Coulomb potential

Michael Levitin Marcello Seri

16 September 2015
typeset September 21, 2015

Abstract

For a particular family of long-range potentials V, we prove that the eigenvalues of the indefinite Sturm–Liouville operator $A = \operatorname{sign}(x)(-\Delta + V(x))$ accumulate to zero asymptotically along specific curves in the complex plane. Additionally, we relate the asymptotics of complex eigenvalues to the two-term asymptotics of the eigenvalues of associated self-adjoint operators.

1 Introduction

Given a real-valued potential V such that

$$V \in L^\infty(\mathbb{R}); \quad \lim_{x\to\infty} V(x) = 0; \quad \limsup_{x\to\infty} x^2 V(x) < \frac{1}{4}; \tag{1}$$

consider a one-dimensional Schrödinger operator in $L^2(\mathbb{R})$

$$\begin{aligned} T := T_V &:= -\frac{d^2}{dx^2} + V(x);\\ \operatorname{Dom}(T) &:= \left\{ f \in L^2(\mathbb{R}) \mid f, f' \in AC(\mathbb{R}),\ Tf \in L^2(\mathbb{R}) \right\}. \end{aligned} \tag{2}$$

It is well known that in this case the spectrum $\operatorname{Spec}(T)$ is bounded from below, the essential spectrum $\operatorname{Spec}_{\mathrm{ess}}(T) = [0, \infty)$, and the negative spectrum $\operatorname{Spec}(T) \cap (-\infty, 0)$ consists of eigenvalues accumulating to zero from below.

Let $J := \operatorname{sign}(x)$ be the multiplication operator by ± 1 on $\mathbb{R}_\pm$. In what follows we consider the point spectrum of the operator

$$A := A_V := JT_V; \quad \operatorname{Dom}(A) = \operatorname{Dom}(T). \tag{3}$$

This operator is not self-adjoint (and not even symmetric) on $L^2(\mathbb{R})$, and its spectrum need not therefore be real. However, as $J^* = J^{-1} = J$, A can be treated as a self-adjoint operator in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$ with indefinite inner product

$$[f, g] := \langle Jf, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{g(x)}\operatorname{sign}(x)\,dx$$

MSC 2010: 34B24, 34L15, 47E05, 47B50, 33C15

Keywords: linear operator pencils, non-self-adjoint operators, Sturm–Liouville problem, Coulomb potential, complex eigenvalues, Kummer functions

ML: Department of Mathematics and Statistics, University of Reading, Whiteknights, PO Box 220, Reading RG6 6AX, UK; m.levitin@reading.ac.uk; `http://www.personal.reading.ac.uk/~ny901965/`

MS: Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK; m.seri@ucl.ac.uk; `http://academic.mseri.me/`

or equivalently as a J-self-adjoint operator [AzIo]. Operators of type (3) have been studied both in the framework of operator pencils, cf. [DaLe, Ma], and of indefinite Sturm–Liouville problems [BeKaTr, BeTr, KaTr, La].

In both settings the literature is extensive, starting mostly with Soviet contributions in the 1960s, including those by Krein, Langer, Gohberg, Pontryagin and Shkalikov. We refer to [Ma, La] for reviews and bibliographies. In particular due to its many applications, for example in control theory, mathematical physics and mechanics, the field is still very active, with recent works on the theoretical,as well as numerical, aspects, (see e.g. [DaLe, ElLePo, HiTrVD, Ve] and references therein).

In the special case of indefinite Sturm–Liouville operators, it is well known that for positive potentials, $V \geq 0$, the spectrum of A_V is real and the operator A_V is similar to a self-adjoint operator [CuLa, CuNa, Py, Ko]. At a very basic level, this can be seen from the following abstract construction: if R and S are self-adjoint operators with $R > 0$, then, under mild restrictions, the spectrum of $R^{-1}S$ is the same as the spectrum of the self-adjoint operator $R^{-1/2}SR^{-1/2}$, and is therefore real.

The case $V \in L^1(\mathbb{R}; (1 + |x|)dx)$

Figure]: A numerical example showing accumulation to □ of complex eigenvalues (red diamonds) of the operator A , = c:□. The magenta and white circles on the negative real axis are the eigenvalues of T corresponding to the eigenfunctions which are even or odd with respect to zero, cf. [BeKaTr].

Proposition](e)(i) holds. Moreover we also prove (Theorem 6) that the complex eigenvalues of

$$T = T := T_V$$

accumulate to zero asymptotically along speci c curves in the complex plane, and that the explicit asymptotics of complex eigenvalues of T can be obtained from the asymptotics of eigenvalues of the self-adjoint operator

$$A = A := A_V \qquad (□)$$

(or, more precisely, from the eigenvalues of its restriction on either even or odd (with respect to zero) subspace). We also extend these results to the more general non-symmetric potentials

$$V_{\ ;\ +}(x) = \begin{cases} \dfrac{+}{] + jxj} & \text{if } x > □ \\ \dfrac{}{] + jxj} & \text{if } x < □ \end{cases} ; \quad + ; \quad 2\ R_+ : \qquad (6)$$

with account of multiplicities. Let $\lambda_n^{\#}(\alpha)$ denote the eigenvalues of $T^{\#}$, $\# = D$ or N, ordered increasingly. In what follows we often drop the explicit dependence on α.

It is well-known that $\lambda_n^{\#} < 0$ and $\lambda_n^{N} < \lambda_n^{D} < \lambda_{n+1}^{N}$ for all $n \in \mathbb{N}$, and also that $\lambda_n^{\#} \to 0$ as $n \to \infty$.

Before stating our main results, we need some additional notation.

Definition c. Let $\mathcal{F}$ denote the class of piecewise smooth functions $F : \mathbb{R}_+ \to \mathbb{R}$ which have a discrete set of singularities (with no finite accumulation points). At each singularity both one-sided limits of F are $\pm\infty$ and differ by sign. Assume for simplicity that 0 is not a singularity of F, and that $F(0) = 0$. For $F \in \mathcal{F}$ we denote by $\Phi_F(x)$ the continuous branch of the multi-valued $\operatorname{Arctan}(F(x))$ such that $\Phi_F(0) = 0$.

Remark . Away from the singularities of F, the function $\Phi_F(x)$ can be written in terms of the ordinary $\arctan(F(x))$ (which takes the values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) and the total signed index of F on $[0; x]$, which we denote by $Z_F(x)$, and which is defined as the total number of jumps from $+\infty$ to $-\infty$ on $[0; x]$ minus the total number of jumps in the opposite direction:

$$Z_F(x) := \#\left\{\tau \in (0;x] \,\middle|\, \lim_{t \to \tau-} F(t) = +\infty\right\} - \#\left\{\tau \in (0;x] \,\middle|\, \lim_{t \to \tau-} F(t) = -\infty\right\}. \tag{7}$$

Then

$$\Phi_F(x) = \arctan(F(x)) + \pi Z_F(x). \tag{8}$$

Obviously, $\Phi_{-F}(x) = -\Phi_F(x)$.

Our first result gives sharp two-term asymptotics of eigenvalues (accumulating to zero) of the self-adjoint op9 9] Tf-atrd

Figure c: $R_1(\;\;)$ and $R_0(\;\;)$

Before proceeding to the proofs, we want to discuss the statements of Theorem 6 in more detail.

Remark 7. (a) It is immediately seen from (]) and (]) that

$$(\ ) = \mathrm{Im} \quad (\ ) = \mathrm{Im} \quad ^{+}(\ ):$$

(b) If we introduce two functions : R_+ ! C by

$$(t) = t + \quad t^{3=2};$$

then

$$\mathrm{Im} \quad (t) = \mathrm{Im} \quad ^{+}(t) = \quad t^{3=2};$$

e. The Jost solutions of (16)

It is well known, see e.g. [OlLoBoCl,].7.] and].7.c], that the rst order asymptotic behaviour of the Kummer Hypergeometric Functions is given, as $|w| ! 1$, by

$$U(a;b;w) \quad w^{\ a}; \qquad \frac{}{c} < \arg(w) < \frac{}{c};$$

$$M(a;b;w) \quad \frac{e^{w} w^{b\ a}}{(a)} + \frac{e^{\ ia} w^{\ a}}{(b\ a)}; \qquad \frac{}{c} \quad \arg(w) < \frac{}{c}; \quad a; b \quad a \, 62 \ N [f \ g;$$

where ( ) stands for the usual Gamma function.

For 2 C n R_{+}, we have p f z 2 C j Re z < g, and therefore

$$U \ ] \ \frac{p}{c}; c; cy^{p} \qquad cy^{p} \ {}_{2}^{p} \ 1;$$

and

$$M \ ] \ \frac{p}{c}; c; cy^{p} \qquad \frac{(cy^{p})^{\frac{p}{2}+1}}{] \ \frac{p}{2}} e^{2y^{p}}$$

as $y ! 1$.

This in turn implies that the f and g

control their oscillations. A quick took at (c□) and (c6) shows that we require asymptotic formulas, as ! □+, for

$$U \left(\frac{}{\sqrt{c}} ; c ; c \sqrt{} \right) ; \quad c \in \{ □ ; \;] \} ; \qquad \text{(c7)}$$

Unfortunately, it is a di cult task | the corresponding formulas, are not, in fact, in the standard references. We rely, instead, on the results from the forthcoming book [Te] which we summarise and adapt in the Appendix.

□.c Asymptotic solutions of a transcendental equation

A crucial element of our analysis is the investigation of the large -roots of the equation

$$\tan(\;) = G(\; ; \;) \qquad \text{(c8)}$$

where is treated as a parameter, and where G depends analytically on in the vicinity of = 1 and, to leading order, is of class F as a function of . The required results are summarised in the following

Lemma 8. Let G(;) be an analytic function of around = 1 such that

$$G(\; ; \;) = G_0(\;) \;] + O(\; ^{1}) \; ; \qquad \text{as} \quad ! \, 1 \; ;$$

$G_0 \; 2 \; F$, and the O terms are regular in . Then the solutions $_n(\;)$, ordered increasingly, of the equation (c8), are given, as n ! 1 , by

$$_n(\;) = \frac{n}{} + \frac{]}{} \; _{G_0}(\;) + O(n \; ^{1}) : \qquad \text{(c9)}$$

The proof of Lemma 8 is in fact immediate as soon as we recall De nition c of and the fact that tan is -periodic.

Considering additional terms in the expansion of G one can get additional terms in the expansion of $_n$. This is in fact what we do in more detail in Section 6.c.

□.□ Approximation of Dirichlet eigenvalues

We can use the asymptotic approximation obtained in (A.8) to reduce (c□) to the simpler form

$$\cos \left(\frac{}{\sqrt{c}} \right) (J_1 (c \sqrt{}) + O(\;)) + \sin \left(\frac{}{\sqrt{c}} \right) (Y_1 (c \sqrt{}) + O(\;)) = □ : \qquad \text{(□□)}$$

This in turn can be rewritten as

$$\tan \left(\frac{}{\sqrt{c}} \right) = \frac{J_1 \left(c \sqrt{} \right)}{Y_1 \left(c \sqrt{} \right)} + O(\;) : \qquad \text{(□])}$$

Applying Lemma 8 with

$$= \frac{]}{\sqrt{c}} ; \qquad G_0(\;) = \frac{J_1 \left(c \sqrt{} \right)}{Y_1 \left(c \sqrt{} \right)} = \; R_1(\;) ;$$

we obtain, after a minor e ort,

$$\begin{aligned} _n &= \frac{2 \; 2}{e} (n \qquad _{G_0}(\;))^{2} + O(n \; ^{4}) \\ &= \frac{2}{en^2} \left( ] + \frac{c}{n} \; _{G_0}(\;) + O(n \; ^{2}) \right) \qquad \text{(□c)} \\ &= \frac{2}{en^2} \left( ] \quad \frac{c}{n} \; _{R_1}(\;) + O(n \; ^{2}) \right) \qquad \text{(□□)} \end{aligned}$$

as n ! +1 , thus proving the rst part of Theorem e.

.e Approximation of Neumann eigenvalues

The analysis for Neumann eigenvalues is slightly more complicated. Again we can use (A.8) to reduce (c6) to

$$\tan\left(\frac{\sqrt{\ }}{c}\right) = \frac{P(\ ;\ )}{Q(\ ;\ )} \qquad (e)$$

where

$$\begin{aligned} P(\ ;\ ) := & \sqrt{\ }\left(\sqrt{\ }\ 8\right)\left(\sqrt{\ } +]\right)J_1\left(c\sqrt{\ }\right) + (]\,]\sqrt{\ }\ 8\ )\left(c\sqrt{\ }\ \right)J_2\left(c\sqrt{\ }\right) \\ & 8\sqrt{\ }\sqrt{\ }\left(c\sqrt{\ }\ \right)J_3\left(c\sqrt{\ }\right) + O(\ ^{3=2}); \\ Q(\ ;\ ) := & \sqrt{\ }\left(\sqrt{\ }\ 8\right)\left(\sqrt{\ } +]\right)Y_1\left(c\sqrt{\ }\right) + (]\,]\sqrt{\ }\ 8\ )\left(c\sqrt{\ }\ \right)Y_2\left(c\sqrt{\ }\right) \\ & 8\sqrt{\ }\sqrt{\ }\left(c\sqrt{\ }\ \right)Y_3\left(c\sqrt{\ }\right) + O(\ ^{3=2}): \end{aligned}$$

Applying once again Lemma 8 with

$$= \frac{]}{c\sqrt{\ }}; \qquad G_0(\ ) = \frac{P(\ ;\ )}{Q(\ ;\ )} = \frac{J_0\left(c\sqrt{\ }\right)}{Y_0\left(c\sqrt{\ }\right)} = R_0(\ );$$

we quickly arrive at

$$_n = \frac{2}{en^2}\left(]+\frac{c}{n}\ G_0(\ ) + O(n^{\ 2})\right) = \frac{2}{en^2}\left(]\ \ \frac{c}{n}\ R_0(\ ) + O(n^{\ 2})\right) \qquad ()$$

as $n ! 1$, thus proving the second part of Theorem e.

6 Proof of the asymptotic results of the non-self-adjoint operator

6.] Eigenvalues and the Jost solutions

Lemma 9. The eigenvalues of() are the zeroes of the determinant

$$M(\ ) = M\ (\ ) = '^{\,0}(\ ;\ )'\ \ (\ ;\ ) + '^{\,0}\ (\ ;\ )'\ \ (\ ;\ ): \qquad (6)$$

Proof. Suppose that 2 C is an eigenvalue of A , and that g (x) 2 $L^2(R)$ is a corresponding eigenfunction. Then g solves the di erential equation

$$\frac{d^2}{dx^2}g\ (x)\ \ \frac{}{]+jxj}g\ (x) = \operatorname{sign}(x)\,g\ (x):$$

If g denote the restrictions of g on R_+ and R , then by integrability we must have

$$g_+(x) = C_+'\ (\ ;x\ ); \qquad g\ (\ x) = C\ '\ \ (\ ;x\ ); \qquad x\ 2\ R_+;$$

where ' (;x) is the Jost solution (ce).

As an eigenfunction should be continuously di erentiable at zero, we obtain

$$\begin{cases} C_+'\ (\ ;\ )\ \ C\ '\ \ (\ ;\ ) = ; \\ C_+'^{\,0}(\ ;\ ) + C\ '^{\,0}\ (\ ;\ ) = ; \end{cases}$$

which has a non-trivial solution if and only if M ( ) = . □

Remark]. (a)

(b) By [BeTr, Proposition e.6] one can instead look for the eigenvalues of ( ) as the zeroes of the m-function

$$m(\ ) = \frac{'^{0}(\ ;\ )}{'(\ ;\ )} + \frac{'^{0}(\ ;\ )}{'(\ ;\ )}: \qquad (\ 7)$$

The use of half-line m-functions is natural and has been already suggested elsewhere, and described in great generality for inde nite Sturm-Liovuille problems with turning point at in [KaTr] (see also references therein).

(c) In what follows we assume that is in the upper half plane C_+ and look for the eigenvalues on the rst quadrant. The nal result will follow by symmetry (see Proposition](a) and Proposition](f)).

6.c The determinant

We can use (ce) and the known relations [OlLoBoCl, x] .] between Kummer hypergeometric functions to rewrite (6) as

$$M(\ ) = \frac{2\sqrt{\ }\, e^{\sqrt{\ }\ \sqrt{\ }}}{8\ ^{5=2}} \ \sqrt{\ } + c\ U\ \frac{\sqrt{\ }}{c};\ ];\ c\sqrt{\ }$$

where we have dropped the lower order terms.

Simplifying, writing S = T K , and collecting terms in K , we get

K + ([] + i)j 2 1 p — (p — ([] + i)) 6ei 2 e([] + i) p — + c

+ T y1 8i 2 + (+]6i) p —] (j 2 (8]] p —) + 8j 3 p —p —)

j 1]6ij 3 p —p — e([] + i) 2 p — + ([] i)

+ cj 2 6e 3]c8([] i) 2p —]i + ([] + i) 3=2

+ T (8 p —) ([] + i) p — (p — ([] + i)) (8 + i p —) y1

+ (+ ci p —) (8 p —p — y 3 + y2 ( ]] p — + 8i))

K y1 8j 3 p —p — 8i 2 + ([]6 + i) p —] (e)

+ j 2 6e 3 (e c]6i) 2p — ([]76 +]i) +]] 3=2

+ ([] + i)T p — (p — ([] + i)) 6e 2 e([] i) p — ci y1

+ 8([] i) p —p — y 3 e([] + i) 2 p — + ([] i)

+ y2 6e([] + i) 3 c6 2p — +]([] i) +]]i 3=2

+ j 1 (8 + i p —) (k c p —) (y2 (8]] p —) + 8 p —p — y 3)

+ ([] i) p — (p — ([] + i)) (8 p —) y1 = :

In what follows, we essentially replicate the reasoning in Lemma 8, but working to a higher

We want to derive a similar expansion for $\tanh\left(\frac{}{2\sqrt{\ + \ ^{3/2} + \ ^{2}}}\right)$. We use

$$\tanh(t_1 - t_2) = \frac{\sinh(t_2) + \cosh(t_2)\tanh(t_1)}{\cosh(t_2) - \sinh(t_2)\tanh(t_1)}$$

with $t_1 := \frac{}{2\sqrt{}}$ and $t_2 := \frac{}{4} + \frac{3\ ^{2}}{4}\sqrt{} + O(\ )$.

As $\tanh(\ = 2\sqrt{}\ ) = 1$ for $\ \to 0$ modulo exponentially small terms, we get (again up to exponentially small errors)

$$\tanh\left(\frac{}{2\sqrt{\ + \ ^{3/2} + \ ^{2}}}\right) = \sinh(t_2 \qquad \tanh$$

7 Generalizations and other remarks

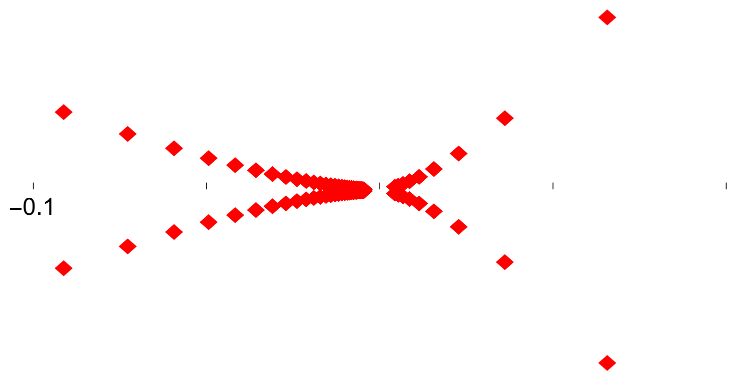


Figure □: Approximated eigenvalues of $A(\ _+;\ \ )$ for $\ =]:□$, $\ _+ = □$.

The procedure used to prove Theorem 6 can be repeated in a completely similar way to obtain a result for the operator

$$A(\ _+;\ \ ) = JT_V;\quad V(x) = \begin{cases} \frac{+}{1+|x|} & \text{if } x > □ \\ \frac{}{1+|x|} & \text{if } x < □ \end{cases};\qquad _+;\ \ \in \mathbb{R}_+ :$$

In this case the m-function is of the form

$$M(\ ) = \frac{'^{\,0}_{\ ;\ +}(□)}{'_{\ ;\ +}(□)} + \frac{'^{\,0}_{\ ;\ }(□)}{'_{\ ;\ }(□)}:$$

The curves in the upper (resp. lower) half plane are no more symmetric w.r.t. iR, however for the left quadrants and right quadrants we can extend Theorem 6. The only di erence is that now the and $^+$ are now functions of both $_+$ and .

Let ; $\in \mathbb{R}_+$. Set

$$f\ (\ ;\ \ ) := \frac{J_1^2(c\sqrt{-}) + J_0^2(c\sqrt{-})}{J_1^2(c\sqrt{-}) + J_0^2(c\sqrt{-})}\,□\, i\quad \sqrt{-}\,□\,J_0(c\sqrt{-})J_1(c\sqrt{-}) + Y_0(c\sqrt{-})Y_1(c\sqrt{-})\,□\,□$$
$$\sqrt{-}\,(J_0(c\sqrt{-})J_1(c\sqrt{-}) + Y_0(c\sqrt{-})Y_1(c\sqrt{-}));$$

$$f_+(\ ;\ \ ) := \frac{J_1^2(c\sqrt{-}) + J_0^2(c\sqrt{-})}{J_1^2(c\sqrt{-}) + J_0^2(c\sqrt{-})}\,□\, i + \sqrt{-}\,□\,J_0(c\sqrt{-})J_1(c\sqrt{-}) + Y_0(c\sqrt{-})Y_1(c\sqrt{-})\,□\,□$$
$$+\ \sqrt{-}\,(J_0(c\sqrt{-})J_1(c\sqrt{-}) + Y_0(c\sqrt{-})Y_1(c\sqrt{-})):$$

Then the two factor multiplying the term Re $^{3=2}$ are given by

$$(\ _+;\ \ ) := \frac{e}{} \begin{cases} \ ^1 \arctan(]=f\ (\ _+;\ \ )) & \text{if Re } > □ \\ _+{}^1 \arctan(]=f\ (\ \ ;\ _+)) & \text{if Re } < □ \end{cases};$$

$$^+(\ _+;\ \ ) := \frac{e}{} \begin{cases} \ ^1 \arctan(]=f_+(\ _+;\ \ )) & \text{if Re } > □ \\ _+{}^1 \arctan(]=f_+(\ \ ;\ _+)) & \text{if Re } < □ \end{cases}:$$

One can immediately see that the asymmetry appearing w.r.t. iR is re ected in the asymmetric dependence on $_+$ and .

It is interesting to observe that for Re $>$ the e ect of is much stronger than the one of $_{+}$ (the latter appears only in the cotangent term, its contribution is bounded, while the former additionally appears as an inverse prefactor). The situation is opposite when Re $<$.

The expressions for (; $_{+}$) are more involved than the ones for ( ) but, as expected, they simplify to (e) and (ee) for $_{+}$ = . As that case, it is possible to use the standard results on Bessel functions to show that the two constants have non-zero real and imaginary part for any $>$.

To answer the general question posed in [Be]] for a wider class of potentials one would need good estimates of the Jost functions in a complex half ball containing the origin and the positive and negative real axis. To our knowledge, the best result of this kind is contained in a paper by Yafaev [Ya]. In that work, however, the author needed to exclude two cones containing the real axis for his estimates to hold. Additionally he could get only the rst term in the asymptotic expansion, whereas for our result we would need at least the rst two.

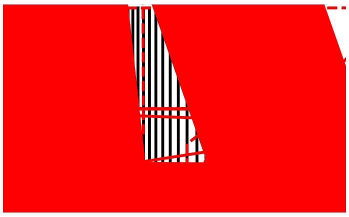


Figure 6: Plot of real part (left) and imaginary part (right) of $U\left(\frac{}{2^{p}}; c; c^{p}\right)$ (black) and its approximation given by (A.8) (dashed red) for small values of and $= c{:}$.

and Re(az) = Re(=a).

Observe that for $\in R_+$, Re(az) > i Re(a) > .

The coe cients A_0 and B_0 also have explicit expressions that can be derived using some symmetry properties and L'Hôpital rule, see [Te, (c7.e.7e)]):

$$A_0 = \frac{}{c\sin(\ )}\ {}_c\ \frac{c}{}\tan\ \cos(c\ )$$

$$B_0 = \frac{}{c\sin(\ )}\ {}_c\ \frac{c}{}\tan\ \frac{\sin(c\ )}{}$$

where $= \frac{1}{2}iw_0$.

The computation of A_n and B_n for n is quite involved, however we will need only A_1. One can exploit the procedure to compute A_0 and B_0, and the recursive de nition of the coe cients to get a Taylor approximation in negative powers of a for c 2 f ;]g. We get

$$\text{if } c = ;\quad A_0^0 =] + O(a\ ^2);\quad A_1^0 = \frac{}{]6} + O(a\ ^2);\quad B_0^0 = ; \tag{A.6}$$

$$\text{if } c = \ ];\quad A_0{}^1 =] + O(a\ ^4);\quad A_1{}^1 = \frac{]]}{]6} + O(a\ ^2);\quad B_0{}^1 = \frac{p\,\_}{ca} + O(a\ ^3): \tag{A.7}$$

With these, (A.c) can be re-written

$$U\left(a; c; \frac{}{a}\right)\ \left(\frac{p\,\_}{a}\right)^{\frac{1\ c}{2}}(a+])\,e^{\frac{}{2a}}\left(C_{c\ 1}(a;\ )\,(A_0^c + A_1^c) + C_{c\ 2}(a;\ )\,B_0^c + O(a\ ^2)\right) \tag{A.8}$$

where

$$C\ (a;\ ) := \cos(\ a\ )J\ (c^{p}\,\_) + \sin(\ a\ )Y\ (c^{p}\,\_) \tag{A.9}$$

and A_0^c, A_1^c and B_0^c are obtained dropping the error term in the appropriate coe cient in (A.6) and (A.7).

Remark A.]. Here the error is in fact O($=a^2$), xi85d.yTdu[(B)]TJ/F20 7.J/F5.18626 Tf 30>t precision] Tf 97.c] ce]

In our case

$$t_1 \quad c \;\; i + c \quad + O(\;\; ^2):$$

For jaj], arg t_1 is in the upper complex half plane. In particular this allows a and z to be in the closure of the rst and fourth quadrant.

B Aknowledgements

The research of M. Seri has been supported by the EPSRC grant EP/J□]68c9/]. We would like to thank Jussi Behrndt for bringing the problem to our attention and stimulating discussions, and Adri Olde Daalhuis, Fritz Gesztesy, and Niko Temme for useful comments and references provided. We also acknowledge the hospitality of the Isaac Newton Institute for Mathematical Sciences in Cambridge, where this work was completed during the programme Periodic and Ergodic Spectral Problems.

References

[AzIo] T. Ya. Azizov, I. S. Iokhvidov, Linear operators in spaces with an inde nite metric, John Wiley,]989

[Be□7] J. Behrndt, On the spectral theory of singular inde nite Sturm{Liouville operators, J. Math. Anal. Appl. □□**e** (c□□7),]e□9{]ee9

[Be]□] J. Behrndt, An open problem: accumulation of nonreal eigenvalues of inde nite Sturm{Liouville operators, Integral Equations Operator Theory **77** (c□]□), no. □, c99{□□]

[BeKaTr] J. Behrndt, Q. Katatbeh, C. Trunk, Accumulation of complex eigenvalues of inde nite Sturm{Liouville operators, J. Phys. A **e]** (c□□8), no. ce, cee□□□

[BePhTr] J. Behrndt, F. Philipp, C. Trunk, Bounds for the non-real spectrum of di erential operators with inde nite weights, Math. Ann. □□**7** (c□]□),]8□{c]□

[BeTr] J. Behrndt, C. Trunk, On the negative squares of inde nite Sturm{Liouville operators, Journal of Di erential Equations **c□8** (c□□7), e9]{□]9

[CuLa] B. Curgus, H. Langer, A Krein space approach to symmetric ordinary di erential operators with an inde nite weight function, J. Di erential Equations **79** (]989), □]{6]

[CuNa] B. Curgus, B. Najman, The operator $(\mathrm{sgn}\, x)\frac{d^2}{dx^2}$ is similar to a self-adjoint operator in $L^2(R)$. Proc. Amer. Math. Soc. **]c□** (]99□),]]c□{]]c8

[DaLe] E. B. Davies, M. Levitin, Spectra of a class of non-self-adjoint matrices, Linear Algebra Appl. **ee8** (c□]e), □□{8e

[ElLePo] D. M. Elton, M. Levitin, I. Polterovich, Eigenvalues of a one-dimensional Dirac operator Anal. mh8

[KaKoMa] I. M. Karabash, A. S. Kostenko, M. M. Malamud, The similarity problem for J-nonnegative Sturm–Liouville operators, J. Differential Equations 246 (2009), no. 3, 964–997

[KaMa] I. M. Karabash, M. M. Malamud, Indefinite Sturm–Liouville operators $(\operatorname{sgn} x)(\frac{d^2}{dx^2}+q(x))$ with finite-zone potentials, Oper. Matrices 1 (2007), no. 3, 301–368

[KaTr] I. M. Karabash, A functional model, eigenvalues, and finite singular critical points for indefinite Sturm–Liouville operators, Oper. Theory Adv. Appl. 203 (2010), 247–287

[KaTr] I. M. Karabash, C. Trunk, Spectral properties of singular Sturm–Liouville operators with indefinite weight $\operatorname{sgn} x$, Proc. Royal Soc. Edinb.: Sect. A 139 (2009), no. 3, 483–503

[Ko] A. Kostenko, The similarity problem for indefinite Sturm–Liouville operators and the HELP inequality, Adv. Math. 246 (2013), 368–413

[La] H. Langer, Krein space, in: Encyclopedia of Mathematics, available at http://www.encyclopediaofmath.org/index.php?title=Krein_space&oldid=18988

[Ma] A. S. Markus, Introduction to the Spectral Theory of Polynomial Operator Pencils, Amer. Math. Soc., Providence, RI, 1988

[OlLoBoCl] F. W. J. Olver, D. W. Lozier, Ronald F. Boisvert, Charles W. Clark (eds.), NIST handbook of mathematical functions, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010, available at http://dlmf.nist.gov

[Py] S. G. Pyatkov, Maximal semidefinite invariant subspaces for J-dissipative operators, Oper. Theory Adv. Appl. 221 (2012), 549–570

[Te] N. M. Temme, Asymptotic Methods for Integrals, World Scientific, Singapore, 2015

[vH] H. van Haeringen, The bound states for the symmetric shifted Coulomb potential, J. Math. Phys. 19 (1978), no. 10, 2165–2170

[Ve] O. Verdier, Reductions of operator pencils, Math. Comp. 83 (2014), 189–214

[Ya] D. R. Yafa.184 -13o1o wva.184 -13o1o wva.387 9msH.38-t1(H.38-2a2.516 L549 T(D.)-306y- 07(