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A new characterisation of ∞-harmonic and p-harmonic maps via affine variations in L^∞

by

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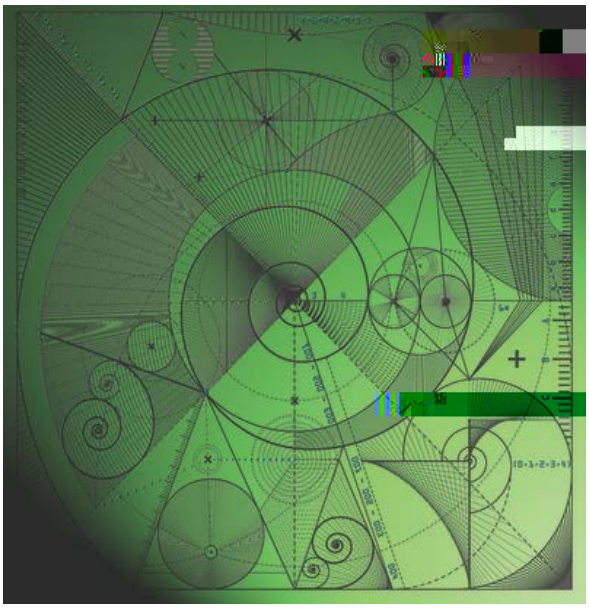

The system (1.1) is the fundamental equation which arises in vectorial Calculus of Variations in the space L^1, that is in connection to variational problems for the model functional

$$E_1(u;\Omega') := \|Du\|^2_{L^1(\Omega')}; \qquad \Omega' \Subset \Omega \ ; \ u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^N). \tag{1.3}$$

The scalar counterpart of (1.1) when $N = 1$ simplifies to

$$Du \otimes Du : D^2u = \sum_{i,j=1}^{n} D_i u D_j u D^2_{ij} u = 0$$

and first arose in the work of G. Aronsson in the 1960s ([A1, A2] and for a pedagogical introduction see [C, K8]) who pioneered the field of Calculus of Variations in the space L^1. The full system (1.1) first appeared in recent work of the author [K1] who initiated the study of the vectorial case in a series of papers ([K1]-[K7]). On the other hand, the p-Laplacian (1.2) is a classical model which arises in conventional Calculus of Variations for integral functionals, in particular as the Euler-Lagrange equation of

$$E_p(u;\Omega') := \|Du\|^p_{L^p(\Omega')}; \qquad \Omega' \Subset \Omega \ ; \ u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N). \tag{1.4}$$

A standard difficulty in both the scalar and the vectorial case of (1.1) is that it is nondivergence and since in general smooth solutions do not exist, the definition of generalised solutions is an issue. In the vectorial case, an additional difficulty is that the system has discontinuous coefficients even if the solution might be smooth (see [K2]). This happens because the projection $D[u(x)]^?$ "feels" the dimension of the tangent space $\mathbb{R}(Du(x)) \subseteq \mathbb{R}^N$.

In this paper we are concerned with the variational characterisation of appropriately defined generalised solutions to (1.1) and (1.2) in both the scalar and the vectorial case in terms of the supremal functional (1.3). The main results of this paper are contained in the statements of Theorems 9, 11 and 12 (and Corollaries 10, 13). Roughly speaking, these results claim that for $1 < p \leq \infty$ we have

$$\Delta_p u = 0 \text{ on } \Omega \quad \Leftrightarrow \quad \begin{cases} \text{For all } \Omega' \Subset \Omega \text{ and } A \in \mathcal{A}^p_{\Omega'}(u); \\ E_1(u;\Omega') \leq E_1(u+A;\Omega') \end{cases}$$

where $\mathcal{A}^p_{\Omega'}(u)$ is a designated set of affine mappings depending on u and on the subdomain Ω'. This result is quite surprising in that both the ∞-Laplacian (1.1) and the p-Laplacian (1.2) are associated to the respective supremal/integral functionals (1.3), (1.4) (and not both associated to (1.3)) when the classes of variations are compactly supported. In the scalar case, the appropriate notion of minimisers for (1.3) characterising ∞-Harmonic functions has been discovered by Aronsson and today we know

We will systematically use the Alexandro 1-point compacti cation of the space $R_s^{Nn^2}$. Its topology will be the one which makes it homeomorphic to the sphere of dimension $N\,n(n+1)/2$ (via the stereographic projection which identi es the north pole with $f1g$). We will denote it by

$$\overline{R}_s^{Nn^2} := R_s^{Nn^2} [f1g \quad :$$

Then, the space $R_s^{Nn^2}$ will be viewed as a metric vector space, isometrically contained into its 1-point compacti cation.

Young Measures. Let R^n be open. The Young measures can be identi ed with a subset of the unit sphere of a certain L^1 space of measure-valued maps and this provides very useful properties, like compactness.

De nition 1. The set of Young Measures from to $\overline{R}_s^{Nn^2}$ is the subset of the unit sphere of the space $L_w^1 \quad ; M \ \overline{R}_s^{Nn^2}$ which contains probability-valued maps:

$$Y \quad ; \overline{R}_s^{Nn^2} := \left\{ \# 2 L_w^1 \quad ; M \ \overline{R}_s^{Nn^2} \quad : \#(x) 2 P \ \overline{R}_s^{Nn^2} \ ; \text{ for a.e. } x 2 \right\} :$$

The space $L_w^1 \quad ; M \ \overline{R}_s^{Nn^2}$ is a dual Banach space and consists of measure-valued maps $3\ x\ 7! \ \#(x) 2 M \ \overline{R}_s^{Nn^2}$ which are weakly* measurable, in the sense that for any Borel set $U \ \overline{R}_s^{Nn^2}$, the function $3\ x\ 7! \ [\#(x)](U) 2 R$ is measurable. The norm of the space is given by

$$k\#k_{L_w^1 \ (\ ; M \ (\overline{R}_s^{Nn^2}))} := \operatorname*{ess\,sup}_{x2} k\#(x)k \ \overline{R}_s^{Nn^2}$$

where \k k" denotes the total variation. For background material on these spaces we refer e.g. to [FL, Ed, V] and to [K9]-[K12]. The L_w^1 space above is the dual space of the space $L^1 \quad ; C^0 \ \overline{R}_s^{Nn^2}$ of Bochner integrable maps. The points of this L^1 space are the Carath eodory functions $: \quad \overline{R}_s^{Nn^2} ! \quad R$ which satisfy

$$k \ k_{L^1(\ ; C^0(\overline{R}_s^{Nn^2}))} := \int \quad (\ x; \ )_{\ C^0(R_s^{Nn^2})} dx < 1 :$$

It is well known that the unit ball of L_w^1 is sequentially weakly* compact. Hence, for any bounded sequence $\#^m)_1^1 \quad L_w^1$, there is a limit map $\#$ and a subsequence of m's along which $\#^m \ * \# \ \text{as} \ m ! 1$.

Remark 2 (Properties of Y.M.). We note the following facts about Young measures (proofs can be found e.g. in [FG]):

i) [Functions as Y.M.] The set of measurable maps $U : \quad R^n ! \quad R_s^{Nn^2}$ can be identi ed with a subset of the Young measures via the embedding $U 7! \ _U$, $_U(x) := \ _{U(x)}$.

ii) [Weak* compactness of Y.M] The set of Young measures is convex and sequentially compact in the weak* topology induced from L_w^1 .

The next lemma is a minor variant of a classical result (see [FG, FL, K9]) but it plays a fundamental role in our setting because it guarantees the compatibility of classical/strong solutions with D-solutions.

Lemma 3. Let $U \ ; U^1 : \quad R^n ! \quad R_s^{Nn^2}$ be measurable maps, 2 N. Then, up the passage to a subsequence, the following equivalence holds true:

$$_U \ * \ _{U^1} \ \text{in} \ Y \quad ; \overline{R}_s^{Nn^2} \quad () \quad U \ ! \ U^1 \ \text{a.e. on} \ :$$

The notion of D-Solutions to fully nonlinear 2nd order systems. Herein we consider the special case of once di erentiable solutions to second order systems

not use this version herein, however. Note rst that (2.2) can be rephrased as the following di erential inclusion for the support:

$$\operatorname{supp}(D^2u(x)) \quad \left\{ X \in R_s^{N\,n\,2} : F\ Du(x); X = 0 \right\} [f1g \quad ; \quad \text{a.e. } x \in \quad :$$

Then, for any compactly supported $\in C_c^0\ R_s^{N\,n\,2}$ o in nity and for a.e. $x \in$, the continuous function

$$\overline{R}_s^{N\,n\,2} \ni X \mapsto (\ X)\, F\ Du(x); X \in R^N$$

is well-de ned on the compacti cation and also vanishes on the support of any di use hessian measure. As a consequence, we have the statement

$$\int_{\overline{R}_s^{N\,n\,2}} (\ X)\, F\ Du(x); X \ d[D^2u(x)](X) = 0; \quad \text{a.e. } x \in \quad ; \tag{2.3}$$

for any $\in C_c^0\ R_s^{N\,n\,2}$ and any di use hessian$D^2u \in Y \quad ; \overline{R}_s^{N\,n\,2}$. It can be easily seen that the converse is true as well (see [K9]) and hence (2.3) is a restatement of (2.2).

For more details on the material of this section (e.g. analytic properties, equivalent formulations of De nition 5, etc) we refer to [K9]-[K12].

The notion of Feeble Viscosity Solutions to fully nonlinear 2nd order equations. The de nitions of this paragraph are taken from [K5] (see also [JJ, JLM] where the \feeble" counterparts of the \usual" viscosity notion rst appeared) but here we apply them only to the case of thep-Laplacian for 1 on(app)-27(eared[(on]TJ 0 sd[(onasd[(onind

If $u \in C^1(\ )$, then any pair $(\ P; X\ )$ in $J_0^{2,}\ u(x)$ satisfies $P = Du(x)$. In this case we will use the notation

$$D^{2,}\ u(x) := \left\{ X \in R_s^{n^2} \quad (Du(x); X) \in J_0^{2,}\ u(x) \right\}$$

and we will call $D^{2,}\ u(x)$ the set of feeble 2nd order sub/super derivatives of u at $x \in$.

3. Two elementary lemmas

In this brief section we isolate a couple of very simple technical results which contain an essential common part of the proofs of the main results in both the scalar and the vectorial case.

Lemma 7. Let R^n be open and $u \in C^1(\ ; R^N)$. Given ^0b , we set

$$^0(u) := \left\{ x \in \overline{\ \ ^0} \quad |Du(x)| = \|Du\|_{L^1(\ ^0)} \right\}$$

Let further $A : R^n \ ! \ R^N$ be an affine map.

a) Suppose that for some ^0b and any > 0, u satisfies

$$\|Du\|_{L^1(\ ^0)} \quad \|Du + DA\|_{L^1(\ ^0)}:$$

Then, we have

$$\max_{z \in \overline{\ \ ^0}} |Du(z_0$$

Lemma 8. Let R^n be open and $u \in C^1(\ ; R^N)$. Given $^0 b$, let $^0(u)$ be as in Lemma 7. Let further $A : R^n \ ! \ R^N$ be an a ne map. We set

$$h(t) := \ Du + tDA \ ^2_{L^1(\ ^0)} \ k \ Du k^2_{L^1(\ ^0)}; \quad t \quad 0:$$

Then, h is convex, $h(0) = 0$ and also the lower right Dini derivative of h at zero satis es

$$\underline{D}h(0^+) := \liminf_{t! \ 0^+} \frac{h(t) \quad h(0)}{t} \quad \max_{y \in \ ^0(u)} \ 2Du(y) : DA \ :$$

Proof of Lemma 8. E ectively, this is an application of Danskin's theorem [D], but we may also prove it directly. By setting

$$H(t; y) := \ Du(y) + tDA \ ^2$$

we have

$$h(t) = \max_{y \in \ ^0} H(t; y) \quad \max_{y \in \ ^0} H$$

By applying Lemma 7 to u and A, we have

$$
\begin{aligned}
0 \quad & \max_{z2\ \text{"}(x)} \quad Du(z) \quad DA \\
= \quad & \max_{z2\ \text{"}(x)} \quad Du(z) \qquad X_x : Du(x)) \\
= \quad & \max_{z2\ \text{"}(x)} \qquad X_x : Du(x) \quad Du(z) \\
! \quad & \qquad X_x : Du(x) \quad Du(x) \ ;
\end{aligned}
$$

as " ! 0. Hence,Du

a ne variations. In view of the well known $C^{1,\alpha}$ regularity results for p-Harmonic mappings [U], the hypothesis that solutions are C^1 is actually super uous.

Theorem 11 (p-Harmonic functions). Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in C^1(\Omega)$. Given $\beta_0 > 0$, let $\Omega_0(u)$ be as above and consider the sets of a ne functions

$$A^{\beta_0,p}_{\Omega_0}(u) := \left\{ A : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \begin{array}{l} D^2 A \equiv 0 \text{ and there exist } \lambda \in \mathbb{R}, \\ x \in \Omega_0(u) \text{ and } X_x \in D^{2,\alpha} u(x) \text{ s.th.:} \\ DA \equiv (p-2) X_x + (I : X_x) I \, Du(x), \end{array} \right\} \subseteq \mathbb{R},$$

where $p \in (1, \infty)$. Then, the following statements are equivalent:
(a) We have

$$\operatorname{div} |Du|^{p-2} Du$$

Fix also " > 0 and let "(x) be as in Lemma 7 and note that for any > 0, A 2 A$^{+;p}_{"(x)}$(u). Hence, by arguing as in Theorem 9 we have that

$$\begin{aligned} 0 \quad & Du(x) \quad DA \\ &= Du(x) \quad (p \quad 2)X_x Du(x) + (I : X_x)Du(x) \\ &= \quad (p \quad 2)Du(x) \quad Du(x) + jDu(x)j^2 I \quad : X_x : \end{aligned}$$

Hence,u is a feeble viscosity solution on .

Conversely, x any ^{0}b and x 2 ^{0}(u). If $J^{2;+}_0$ u(x) 6= ; , then any A 2 A$^{+;p}_0$(u) can be written as

$$A(z) = a + \quad (p \quad 2)X_x + (I : X_x)I \quad : Du(x) \quad z; \quad z 2 R^n;$$

for somea 2 R, 0 and some (Du(x); X$_x$) 2 $J^{2;+}_0$ u(x). Let h be the function of Lemma 8 for such anA. By applying Lemma 8, we have

$$\begin{aligned} \underline{D}h(0^+) \quad & 2Du(x) \quad DA \\ &= 2 \quad (p \quad 2)Du(x) \quad Du(x) : X_x + jDu(x)j^2 I : X_x \\ & 0; \end{aligned}$$

since by assumptionu is a subsolution on in the feeble viscosity sense. By using that h(0) = 0 and that h is convex, we deduce as in Theorem 9 thath(t) 0 for t 0 and hence

$$kDuk_{L^1(\ ^0)} \quad k Du + DA k_{L^1(\ ^0)}$$

for any A 2 A$^{+;p}_0$(u) and any ^{0}b . Thus, (b) , (c). The case of supersolutions follows analogously and hence the theorem ensues.

5. The vectorial case N 2

In this section we extend the results of the previous section to the full case of the 1 -Laplace

nullset and choose from each equivalence class the representative which is rede ned as $_{f\,0g}$ at points where $D^2u(x)$ does not exist. Moreover, let u be a x map in $C^1(\ ; R^N)$. Since $Du(x)$ exists for all $x\ 2\ $, by perhaps a further re-de nition of every D^2u on a Lebesgue nullset, it follows that u is D-solution to the system if and only if for (any xed such representative of) any di use hessian, we have

$$A\ \ Du(x)\ \ : X_x\ = 0; \quad \text{for all } x\ 2\ \ \text{and}\ \ X_x\ 2\ \text{supp}\ \ D^2u(x)\ :$$

(We remind that at points $x\ 2\ $ for which $D^2u(x) = \ _{f1g}$ and hence supp $D^2u(x)$ $= \ ;$

Now we argue similarly for the normal component of the system. Suppose that for any ^{0}b and any $A\ 2\ A^{?;1}_{0}\ (u)$, we have

$$kDuk_{L^{1}(\ ^{0})}\quad k\ Du + DA\ k_{L^{1}(\ ^{0})}:$$

We x as before $x\ 2$ and $X_{x}\ 2\ \mathrm{supp}\ D^{2}u(x)$. If $Du(x) = 0$, then the system $jDuj^{2}[Du]^{?}\ u = 0$ is trivially satis ed at x. If $Du(x) \neq 0$, then we choose any direction normal to $Du(x)$, that is

$$n_{x}\ 2\ R\ Du(x)^{\ ?}\quad R^{N};$$

which means that $n_{x}^{>}\ Du(x) = 0$. We note that if $Du(x) : R^{n}\ !\quad R^{N}$ is surjective, then we can nd only the trivial n_{x}

since by assumption $Du \quad Du : D^2u = 0$ on in the D-sense. In view of the fact that $h(0) = 0$ and h is convex, it follows that

$$h(t) \quad h(0) + \underline{D}h(0^+)\,t \quad 0; \quad t \quad 0;$$

and hence

$$kDuk_{L^1(\ ^0)} \quad k\ Du + DAk_{L^1(\ ^0)}; \quad A \ 2\ A^{>}_{0}{}^{;1}(u); \quad {}^0b \quad :$$

The case of $A \ 2\ A^{?}_{0}{}^{;1}$ is completely analogous: any such nonconstant A satisfies $A(x)\,?\,R(Du(x))$ and $DA \ 2\ L^{X_x}\ A(x)$ for some $X_x \ 2$ supp $(D^2u(x))$ and some $x \ 2 \ ^0(u)$. By applying Lemma 8 again, we have

$$\underline{D}h(0^+) \quad \max_{y2\ ^0(u)} \ 2Du(y) : DA$$
$$2Du(x) : DA:$$

If $Du(x) \ 6 \ 0$, then by the definition of $L^{X_x}\ A(x)$ we have

$$\begin{aligned}\underline{D}h(0^+) \quad & 2DA : Du(x)\\ = \quad & 2(n_x \quad I) : X_x\\ = \quad & 2n_x^{>} \quad [Du(x)]^{?} \quad I \ : X_x\\ = \quad & 0\end{aligned}$$

because by assumption $jDuj^2[Du]^{?} \ u = 0$ on in the D-sense. If $Du(x) = 0$, then again $\underline{D}h(0^+) \quad 0$. In either cases, we obtain

$$h(t) \quad h(0) + \underline{D}h(0^+)\,t \quad 0; \quad t \quad 0;$$

and hence

$$kDuk_{L^1(\ ^0)} \quad k\ Du + DAk_{L^1(\ ^0)}; \quad A \ 2\ A^{?}_{0}{}^{;1}(u); \quad {}^0b \quad :$$

The theorem has been established.

Proof of Corollary 13

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