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INDUCED RANDOM -TRANSFORMATION

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Abstract. In this article we study the rst return map de ned on the switch region induced by

$K \quad : \quad [0; \frac{1}{\quad 1}] \; ! \quad [0; \frac{1}{\quad 1}]$ de ned by

$$K \ (!; x \ ) = \begin{cases} (!; T_{0} x); & \text{if } 0 \quad x < \frac{1}{}; \\ (\ !; T_{! \ 1} x); & \text{if } \frac{1}{} \end{cases}$$

k	$_k$	$_k$	$_k$
1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$1+2^{1=2}$
2	$1{:}7549{:}{:}{:}$	$1{:}8393{:}{:}{:}$	$1{:}8546{:}{:}{:}$
3	$1{:}8668{:}{:}{:}$	$1{:}9276{:}{:}{:}$	$1{:}9305{:}{:}{:}$
4	$1{:}9332{:}{:}{:}$	$1{:}9660{:}{:}{:}$	$1{:}9666{:}{:}{:}$
5	$1{:}9672{:}{:}{:}$	$1{:}9836{:}{:}{:}$	$1{:}9837{:}{:}{:}$

Figure 1. Tables of values for $_k$, $_k$ and $_k$

and let $_k$ denote the k-th multinacci number. Recall that the k-th multinacci number is the unique root of

$$x^{k+1} \quad x^k \quad x^{k\ 1} \qquad x \quad 1 = 0$$

contained in $(1;2)$.

Theorem 1.1. Let $\ 2\ (\ _k;\ _k]$ for some $k\ \ 2$; then for any $!\ 2\ $ and $(j_i)\ 2\ R^N$ there exists $x\ 2\ S$ such that $r_i(!;x\ ) = j_i$, for $i = 1;2;:::$. Moreover, if $\ 2\ (\ _k;\ _k]$ for all $k\ \ 2$; then there exists $!\ 2\ $ and $(j_i)\ 2\ R^N$ such that no $x\ 2\ S$ satisfies $r_i(!;x\ ) = j_i$, for $i = 1;2;:::$.

As we will see, the algebraic properties of $_k$ and $_k$ correspond naturally to conditions on the orbit of 1 and its reflection $\frac{1}{1}\ \ 1$. These points determine completely the dynamics of the greedy map G and lazy map L respectively, and hence it is not surprising that these points play a crucial role in our situation as well. For values of lying outside of the intervals $(\ _k;\ _k]$ it is natural to ask whether the following weaker condition is satisfied: given $(j_i)\ 2\ R^N$ does there exist $!\ 2\ $ and $x\ 2\ S$ such that $r_i(!;x\ ) = j_i$ for $i = 1;2;:::$. Let $_k$ denote the unique root of the equation

$$2x^{k+1} \quad 4x^k + 1 = 0$$

contained in $(1;2)$:

Theorem 1.2. Let $\ 2\ (\ _k;\ _k]$ for some $k\ \ 1$, then for any sequence $(j_i)\ 2\ R^N$ there exists $!\ 2\ $ and $x\ 2\ S$ such that $r_i(!;x\ ) = j_i$ for $i = 1;2;:::$.

If satisfies the hypothesis of Theorem 1.2 then the orbit of 1 and $\frac{1}{1}\ \ 1$ satisfy a cross over property. This cross over property is sufficient to prove Theorem 1.2. Note that $_k\ \ _k\ \ _k$ for each $k\ \ 1$. We include a tables of values for $_k$, $_k$ and $_k$ in Figure 1.

The second half of this paper is concerned with the maps $U_{;0}$ and $U_{;1}$: Before we state our results it is necessary to make a definition. Given a closed interval $[a;b]$, we call a map $T : [a;b]\ !\ [a;b]$ a generalized Lüroth series transformation (abbreviated to GLST) if there exists a countable set of bounded subintervals $f I_n g_{n=1}^1$ $(I_n = (l_n;r_n);[l_n;r_n=1$

This Lüroth expansion (a_n) can be seen to be generated by the map $T:[0,1]\to[0,1]$ where

$$T(x)=\begin{cases} n(n+1)x-n; & \text{if } x\in(\frac{1}{n+1},\frac{1}{n}] \\ 0; & \text{if } x=0 \end{cases}$$

GLST's were introduced in [2]. Our definition is slightly different to that appearing in this paper but all of the main results translate over into our context. Namely if $T:[a,b]\to[a,b]$ is a GLST then the normalised Lebesgue measure on $[a,b]$ is a T-invariant ergodic measure. Our main result for the maps $U_{\beta,0}$ and $U_{\beta,1}$ is the following theorem.

Theorem 1.3. There exists a set $M\subseteq(1,2)$ of Hausdorff dimension 1 and Lebesgue measure zero such that:

(1) If $\beta\in M$ then both $U_{\beta,0}$ and $U_{\beta,1}$ are GLSTs.
(2) If $\beta\notin M$ then both $U_{\beta,0}$ and $U_{\beta,1}$ are not GLSTs.

What is more we can describe the set M explicitly.

Before we move on to our proofs of Theorems 1.1, 1.2 and 1.3 we provide a worked example. Namely we consider the case where $\beta=\frac{1+\sqrt{5}}{2}$. This case exhibits some of the important features of our later proofs.

Example 1.1. When $\beta=\frac{1+\sqrt{5}}{2}$ then $S=[\frac{1}{\beta},1]$. Let $C_j=\{\omega\in\Omega:\omega_1=j\}$, $j=0,1$, then for any $\omega\in C_0$, $r_1(\omega,1)=\infty$, and $r_1(\omega,\frac{1}{\beta})=1$, while for any $\omega\in C_1$, we have $r_1(\omega,1)=1$ and $r_1(\omega,\frac{1}{\beta})=\infty$. If $x\in(\frac{1}{\beta},1)$, then $r_1(\omega,x)\geq 2$ for all $\omega\in\Omega$.

Let

$$B_i^0:=\{x\in S:U_{\beta,0}(x)=(T_1^{i-1}\circ T_0)(x)\} \tag{1}$$

and

$$B_i^1:=\{x\in S:U_{\beta,1}(x)=(T_0^{i-1}\circ T_1)(x)\} \tag{2}$$

where $i\geq 2$. A simple calculation shows that

$$B_i^0=\Big[\sum_{n=2}^{i+1}\frac{1}{\beta^n},\sum_{n=2}^{i+2}\frac{1}{\beta^n}\Big]^{i}=(T_1^{i-1} \tag{3}$$

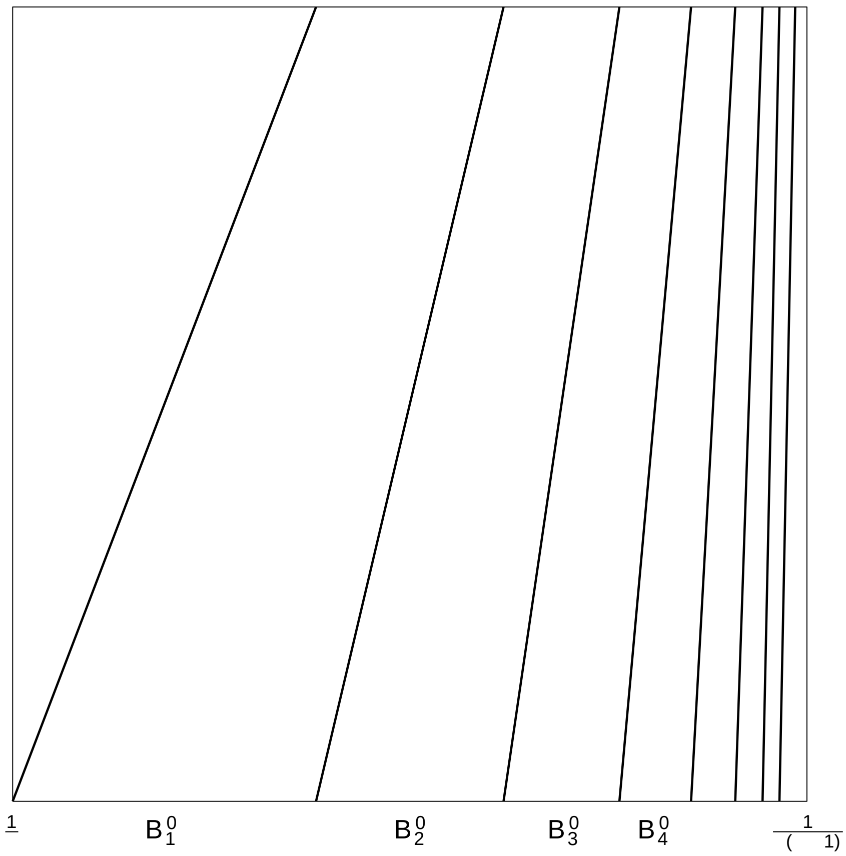

Figure 2. The graph of $U_{;0}$ when $= \frac{1+\sqrt{5}}{2}$

Where in the above $_{B_i^0}$ denotes the characteristic function on B_i^0: Note that the result stated above holds with $r_j((0)^1;x)$ replaced with $r_j((1)^1;x)$:

By Theorem 1.1 we know that there exists $!\ 2$ and $(j_i)_{i=1}^1\ 2\ R^N_{\frac{1+\sqrt{5}}{2}}$ for which no x satisfies $r_i(!;x) = j_i$ for following properties are important consequences of the above. First of all it is straightforward to see that for $\beta \in (\beta_k, \beta_k]$ we have $R_\beta = \{k+1, k$

every $x \in S$ is eventually mapped outside of S. This implies that equation (11) cannot hold and we have proved our result.

Combining Propositions 2.1, 2.2 and 2.3 we conclude Theorem 1.1.

2.2. Proof of Theorem 1.2. We now prove Theorem 1.2 our proof is similar to Theorem 1.1 in that we make use of a nested interval construction. However, with our proof we do not explicitly construct the desired $!$; we can only show existence, as such our proof takes on an added degree of abstraction.

Let us start by examining the consequences of $\in (\ _k;\ _k]$ for some $k \geq 1$: For $k \geq 2$ we ignore the intervals $(\ _k;\ _k]$ as their proof is covered by Theorem 1.1. For in the remaining parameter space the following inclusions hold

$$(12) \qquad \bigcup_{k \geq 1} (T_1^k \circ T_0) \left(\frac{1}{\ } \quad 2 \quad \frac{1}{\ }; \frac{1}{2(\ \ -1)} \right]$$

numbers, thus M has Hausdor dimension 1 and Lebesgue measure zero. It is worth noting that if 2 f 2 (1; 2) : $U_{;0}(\frac{1}{})$ 2 f $\frac{1}{}$; $\frac{1}{(\quad 1)}$gg then card (1) = @0. The important observation to make from the de nition of M is that the following statement holds

$$2\ M \quad () \qquad \frac{1}{} \text{ and } \frac{1}{(\qquad 1)} \text{ are never mapped into the interior of } S:$$

This property will be su cient to prove that both $U_{;0}$ and $U_{;1}$ are GLSTs. Our proof of Theorem 1.3 is split over the following propositions.

Proposition 3.1. If =2 M then $U_{;0}$ and $U_{;1}$ are not GLSTs.

Proof. If =

We begin with the most simple case, we assume that $U_{\beta,0}(x) = (T_1^{n_1}\circ T_0)(x)$, i.e. $T_0(x)\in D_{n_1}$. Importantly, since $\beta\in M$ we know that $1\notin D^0_{n_1}$. Thus $T_0(S)\cap D_{n_1} = [1,\frac{1}{\beta-1}]\cap D_{n_1} = D_{n_1}$. Therefore $T_0^{-1}(D_{n_1})\subseteq S$ and any y in this interval satisfies $U_{\beta,0}(y) = (T_1^{n_1}\circ T_0)(y)$. This implies that

$$\{y\in S : U_{\beta,0}(x) = (T_1^{n_1}\circ T_0)(y)\} = (T_1^{n_1}\circ T_0)^{-1}(S). \tag{21}$$

It remains to show that equation (20) holds in the general case. Obviously

$$\{y\in S : U_{\beta,0}(y) = (T_{\omega_i}^{n_i}\circ\cdots\circ T_1^{n_1}\circ T_0)(y)\} \subseteq (T_{\omega_i}^{n_i}\circ\cdots\circ T_1^{n_1}\circ T_0)^{-1}(S). \tag{22}$$

So we have to show that the opposite inclusion holds, for this we examine the formula for $U_{\beta,0}$ more closely. We assume $U_{\beta,0}(x) = (T_{\omega_i}^{n_i}\circ\cdots\circ T_1^{n_1}\circ T_0)(x)$ for some $i\geq 2$. Since $i\geq 2$ we have $T_0(x)$ is contained in a connected component of $[1,\frac{1}{\beta-1})\setminus\cup_{n=1}^{\infty}D_n$. Let us denote this interval by I_1. We also let

$$E := \Big\{T_0^{-n}\Big(\frac{1}{\beta}\Big), T_0^{-n}\Big(\frac{1}{\beta(\beta-1)}\Big), T_1^{-n}\Big(\frac{1}{\beta}\Big), T_1^{-n}\Big(\frac{1}{\beta(\beta-1)}\Big), G_\beta^n(1), G_\beta^n\Big(\frac{1}{\beta-1}-1\Big) : n\geq 0\Big\}.$$

Here G_β is the greedy map defined earlier. Since $\beta\in M$ no element of E is contained in the interior of a C_n, a D_n, or S.

Importantly $I_1 = (a_1,b_1)$ where $a_1,b_1\in E$. In this case either

$$(a_1,b_1) = \Big(1, T_1^{-n_1}\Big(\frac{1}{\beta}\Big)\Big) \text{ or } (a_1,b_1) = \Big(T_1^{-(n_1-1)}\Big(\frac{1}{\beta(\beta-1)}\Big), T_1^{-n_1}\Big(\frac{1}{\beta}\Big)\Big).$$

Therefore

$$T_1^k(I_1)\cap S = \emptyset \text{ for } 1\leq k\leq n_1-1 \text{ and } T_1^{n_1}(I_1)\subseteq\Big(\frac{2-\beta}{\beta-1},\frac{1}{\beta}\Big).$$

The endpoints of $T_1^{n_1}(I_1)$ are elements of E and are therefore not contained in the interior of any C_n. Either $(T_1^{n_1}\circ T_0)(x)\in C_n$ for some n or maybe $(T_1^{n_1}\circ T_0)(x)\in T_1^{n_1}(I_1)\setminus\cup_{n=1}^{\infty}C_n$. If $(T_1^{n_1}\circ T_0)(x)\in T_1^{n_1}(I_1)\setminus\cup_{n=1}^{\infty}C_n$ then let the connected component it is contained in be denoted by I_2. Let $I_2 = (a_2,b_2)$ then again $a_2,b_2\in E$. In which case

$$T_0^k(I_2)\cap S = \emptyset \text{ for } 1\leq k\leq n_2-1 \text{ and } T_0^{n_2}(I_2)\subseteq\Big(\frac{1}{\beta(\beta-1)},1\Big) \tag{23}$$

The endpoints of $T_0^{n_2}(I_2)$ are again contained in E and therefore do not intersect the interior of any D_n. The point x has either been mapped into a D_n or is contained in a connected component of $T_0^{n_2}(I_2)\setminus\cup_{n=1}^{\infty}D_n$. If it is contained in a connected component of $T_0^{n_2}(I_2)\setminus\cup_{n=1}^{\infty}D_n$ then we repeat the previous steps. Eventually x is mapped into either C_{n_i} or D_{n_i} and our algorithm terminates. Without loss of generality we assume x is eventually mapped into D_{n_i}. The above algorithm yields a finite sequence of intervals $(I_j)_{j=1}^{i-1}$ which satisfy the following properties:

(1) $I_1\subseteq T_0(S)$.
(2) For $1\leq j\leq i-1$
$$T^k_{\omega_j}(I_{n_j})\cap S=\emptyset \text{ for } 1\leq k\leq n_j$$
(3) For $1\leq j\leq i-2$ we have $I_{j+1}\subseteq T^{n_j}_{\omega_j}(I_j)$
(4)
$$D_{n_i}\subseteq T^{n_{i-1}}_{\omega_{i-1}}(I_{i-1}).$$

Where in the above $\omega_j=0$ if j is even and $\omega_j=1$ if j is odd. These properties have the following consequences:

(5) $(T^{n_i}_{\omega_i})^{-1}(S)\subseteq I_{i-1}$
(6) For $1\leq j\leq i-1$
$$(T^{n_i}_{\omega_i}\circ\cdots\circ T^k_{\omega_j})^{-1}(S)\cap S=\emptyset \text{ for } 1\leq k\leq n_j.$$
(7) For $1\leq j\leq i-1$
$$(T^{n_i}_{\omega_i}\circ\cdots\circ T^{n_j}_{\omega_j})^{-1}(S)\subseteq I_{n_j}$$
(8)
$$(T^{n_i}_{\omega_i}\circ\cdots\circ T_1^{n_1}\circ T_0)^{-1}(S)\subseteq S.$$

Property (8) states that $(T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)^{-1}(S) \subseteq S$. Moreover, properties (5), (6) and (7) imply that every $y \in (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)^{-1}(S)$ satisfies $U_{\omega;0}(y) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(y)$