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by

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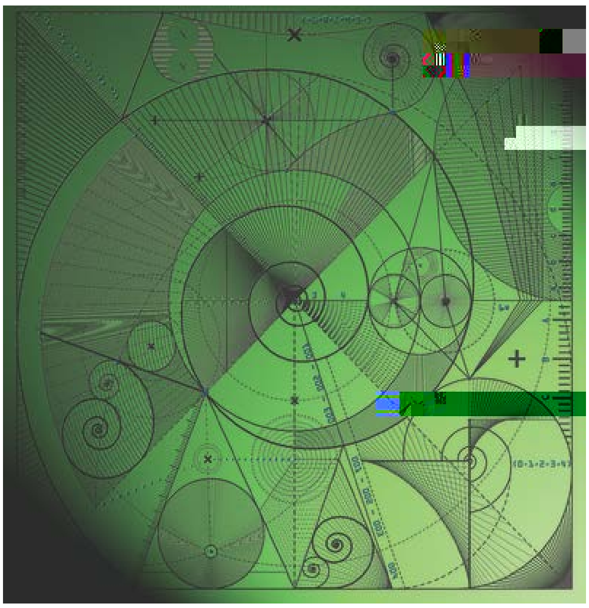

Inhomogeneous self-similar sets with overlaps

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which we refer to as the *inhomogeneous attractor* of the inhomogeneous IFS. Note that the attractors of classical (or homogeneous) IFSs are inhomogeneous attractors with condensation equal to the empty set. It turns out that F_C is equal to the union of $F_\emptyset$ and all images of C by compositions of maps from the defining IFS. This means that for countably stable dimensions, like the Hausdorff dimension $\dim_H$, one immediately obtains

$$\dim_H F_C = \max\{\dim_H F_\emptyset, \dim_H C\},$$

but establishing similar formulae for dimensions that are not countably stable is more challenging. As such it is natural to study the upper and lower box dimensions, $\overline{\dim}_B$ and $\underline{\dim}_B$. In what follows we will focus on inhomogeneous self-similar sets, i.e. inhomogeneous attractors where the defining contractions are similarities. For this class of attractors it was shown in [OS] that if the defining system satisfies an 'inhomogeneous strong separation condition', then the 'expected formula' also holds for upper box dimension, i.e.

$$\overline{\dim}_B F_C = \max\{\overline{\dim}_B F_\emptyset, \overline{\dim}_B C\}. \tag{1.1}$$

It was shown in [Fr2] that the analogous formula fails for lower box dimension, even if one has good separation properties, and that (1.1) remains valid even if the separation condition from [OS] is relaxed to the open set condition (OSC), which, in particular, does not depend on C. See [F2, Chapter 9] for the definition of the OSC. In [Fr3] the problem was addressed for inhomogeneous self-affine sets and in this context (1.1) does not generally hold for upper box dimension even if the OSC is satisfied.

In this paper we focus on the overlapping situation (i.e. without assuming the OSC) and prove that (1.1) does not hold in general by considering a construction based on number theoretic properties of certain Bernoulli convolutions (Section 2). In Section 3, relying on a specific spectral gap property of $SO(d)$ for $d \geq 3$, we provide another construction in $\mathbb{R}^d$ where $\underline{\dim}_B F_C = d - 1 - \varepsilon$ with $F_\emptyset$ and C being which proves (1.1) in many situations, most notably when the open set condition is satisfied.

Let $\mathcal{I}^* = \bigcup_{k \geq 1} \mathcal{I}^k$ denote the set of all finite sequences with entries in $\mathcal{I}$ and for $\mathbf{i} = (i_1, i_2, \dots, i_k) \in \mathcal{I}^*$ write

$$S_{\mathbf{i}} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}$$

and

$$c_{\mathbf{i}} = c_{i_1} c_{i_2} \cdots c_{i_k}$$

which is the contraction ratio of $S_{\mathbf{i}}$. The orbital set is defined by

$$\mathcal{O} = C \cup \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C)$$

and it is easy to see that $F_C = F_\emptyset \cup \mathcal{O} = \overline{\mathcal{O}}$ (cf. [S, Lemma 3.9]).

2 Inhomogeneous Bernoulli convolutions and failure of (1.1)

We begin by computing the box dimensions of a family of overlapping inhomogeneous self-similar sets based on Bernoulli convolutions. Fix $\lambda \in (0,1)$, let $X = [0,1]^2$ and let $S_0, S_1 : X \to X$ be defined by

$$S_0(x) = \lambda x \quad \text{and} \quad S_1(x) = \lambda x + (1 - \lambda, 0).$$

To the homogeneous IFS $\{S_0, S_1\}$ associate the condensation set

$$C = \{0\} \times [0,1]$$

and observe that $F_\emptyset = [0,1] \times \{0\}$ and so $\dim_B F_\emptyset = \dim_B C = 1$. We will denote the inhomogeneous attractor of this system by F_C^λ to emphasise the dependence on λ. In this section we construct several counterexamples to (1.1). Our first counterexample makes use of a well known class of algebraic integers known as *Garsia numbers*. We define a Garsia number to be a positive real algebraic integer with norm ± 2, whose conjugates are all of modulus strictly greater than 1. Examples of Garsia numbers include $\sqrt[n]{2}$ and $1.76929\dots$, the appropriate root of $x^3 - 2x - 2 = 0$. In [G] Garsia showed that whenever λ is the reciprocal of a Garsia number, then the associated Bernoulli convolution is absolutely continuous with bounded density.

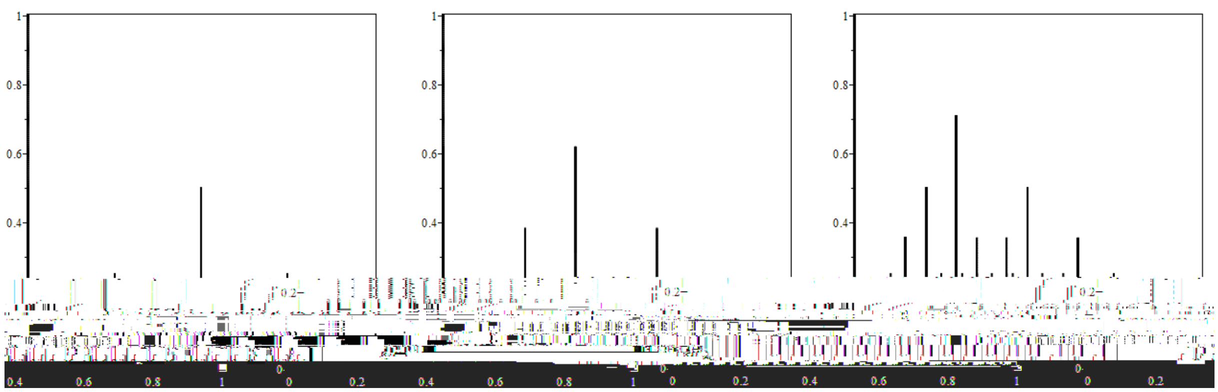

Figure 1: Three plots of F_C^λ, where λ is chosen to be 1/2 (where there are no overlaps), the reciprocal of the golden mean (which is Pisot), and the reciprocal of $\sqrt{2}$ (which is Garsia).

Theorem 2.1. *If $\lambda \in (1/2, 1)$ is the reciprocal of a Garsia number, then*

$$\dim_B F_C^\lambda = \frac{\log(4\lambda)}{\log 2} > 1.$$

We defer the proof of Theorem 2.1 to Section 2.2 below. For every $\lambda \in (1/2, 1)$ which is the reciprocal of a Garsia number, the set F_C^λ provides a counterexample to (1.1) for the upper (and lower) box dimension, but it is also worth noting that this example is `sharp' in that, given the data:

and s = log 2=log , we prove that this is as large as $\overline{\dim}_B F_C$ can be. For more details, see Corollary 4.9 and Remark 4.10.

Our second source of counterexamples to (1.1) is a much larger set. As the following statement shows, F_C typically provides a counterexample to (1.1) whenever lies in a certain subinterval of (1=2; 1).

Theorem 2.2. For Lebesgue almost every 2 (1=2; 0:668) we have

$$\dim_B F_C = \frac{\log(4\ )}{\log 2}:$$

The appearance of the quantity 0668 is a consequence of transversality arguments used in [BS]. Our proof of Theorem 2.2 will rely on counting estimates appearing in this paper and will be given in Section 2.3. We note that the value log(4)=log 2 also appears as the dimension of a related family of sets. In particular, for 2 (1=2; 1), let A be the (homogeneous) self-a ne set associated to the IFS consisting of a ne maps $T_0; T_1 : X$! X de ned by

$$T_0(x; y) = (\ x; y= 2) \quad \text{and} \quad T_1(x) = (\ x + 1 \quad ; y= 2 + 1=2):$$

It follows from standard dimension formulae for self-a ne sets that the box dimension of A is given by log(4)=log 2 for every 2 (1=2; 1), see for example [Fr1, Corollary 2.7]. Also, for every 2 (0; 1=2], we have $\dim_B F_C = \dim_B A = 1$, but this case is not so interesting because the IFS de ning F_C does not have overlaps. The relevance of this comparison is purely aesthetic, noting that the projection of this IFS onto the rst coordinate gives the Bernoulli convolution and onto the second coordinate gives a simple IFS of similarities yielding C as the attractor.

Figure 2: Three plots of A , where is chosen to be the reciprocal of the golden mean, the reciprocal of p 2, and 4=5.

The key reason that the sets F_C provide counterexamples to (1.1) is that the set $F_;$ is trapped in a

$\overline{\dim}_B E_{;} \leqslant 1 + s(";\ )$. By choosing " sufficiently small (after fixing), $s(";\ )$ can be made arbitrarily small, in particular to guarantee

$$\max\{\overline{\dim}_B E_{;}\,;\ \overline{\dim}_B C\} \leqslant 1 + s(";\ ) < \frac{\log(4\ )}{\log 2} \leqslant \overline{\dim}_B E_C$$

and so (1.1) fails despite the fact that E_C is not contained in a subspace. For more discussion on possible mechanisms for violating (1.1), see Section 5.

2.1 Notational remark

For real-valued functions A and B, we will write $A(x) \lesssim B(x)$ if there exists a constant $c > 0$ independent of the variable x such that $A(x) \leqslant cB(x)$, $A(x) \gtrsim B(x)$ if there exists a constant $c^0 > 0$ independent of the variable x such that $A(x) \geqslant c^0 B(x)$ and $A(x) \quad B(x)$ if $A(x) \lesssim B(x)$ and $A(x) \gtrsim B(x)$. In our setting, x is normally some > 0 from the definition of box dimension or some $k \in \mathbb{N}$ and the comparison constant $c; c^0$ can depend on fixed quantities only, like and the defining parameters in the IFS.

2.2 Proof of Theorem 2.1

Before we get to the proof we state a useful separation property that holds for the reciprocals of Garsia numbers and demonstrate the relevance to our situation via (2.1) below.

Lemma 2.3 (Garsia [G]). *Let $\in (1/2; 1)$ be the reciprocal of a Garsia number and $(i_k)_{k=1}^n ; (i_k^0)_{k=1}^n \in \{0; 1\}^n$ be distinct words of length n. Then*

$$(1 \quad) \sum_{k=1}^{n} i_k \ ^{k\ 1} \quad (1 \quad) \sum_{k=1}^{n} i_k^0 \ ^{k\ 1} \geqslant \frac{K}{2^n}:$$

For some strictly positive constant K that only depends on :

This lemma is due to Garsia [G]. For a short self-contained proof of this fact we refer the reader to [B1, Lemma 3.1]. We note that for any $I = (i_1; : : : ; i_n) \in \{0; 1\}^n$ we have

$$S_I(C) = \{S_I(0;0)\} \quad [0;\ ^n] = \left\{ (1 \quad) \sum_{k=1}^{n} i_k \ ^{k\ 1} \right\} \quad [0;\ ^n]: \tag{2.1}$$

Combining Lemma 2.3 with (2.1), we see that whenever is the reciprocal of a Garsia number the images of C will be separated by a factor $K \ 2 \ ^n$. This property is the main tool we use in our proof of Theorem 2.1.

Proof of Theorem 2.1. Fix > 0 and decompose the unit square into horizontal strips of height $^k \quad ^{k+1}$ for k ranging from 0 to $k(\,;\ )$, defined to be the largest integer satisfying $^{k(\,;\ )+1} >$. Observe that the only part of F_C which intersect the interior of the kth vertical strip is

$$\bigcup_{l=0}^{k} \ \bigcup_{I \in \{\ 1;2\}^l} S_I(C)$$

which is a union of vertical lines. Within the kth vertical strip, each line intersects on the order of $^k =$ squares from the -mesh i26 Tf 7.5]TJ/ [(k) Tf 4.981 0 Td [(g)]TJ/FI7pgF11 9.9626 Tf 7N t 7N T5ip,T5ip,T5ip, where the ^{1} comes from the intersections below the $k(\; ; \;)$th strip. It follows from Lemma 2.3 and subsequent discussion that

$$N \; (\; (\; k)) \qquad \min\{ 2^{k} ; \quad ^{1} \}$$

which is the maximum value possible and where the `comparison constants' are independent of and k, but do depend on , which is xed. Let $k_0(\;)$ be the largest integer satisfying $2^{k_0(\;)} < \quad ^{1}$. It follows that

$$
\begin{aligned}
N \quad F_C \qquad & \quad ^{1} + \sum_{k=0}^{k_0(\;)} \quad ^{k} = \quad 2^{k} + \sum_{k=k_0(\;)+1}^{k(\; ; \;)} \quad ^{k} = \quad ^{1} \\
= \qquad & \quad ^{1} + \quad ^{1} \sum_{k=0}^{k_0(\;)} (2 \;)^{k} + \quad ^{2} \sum_{k=k_0(\;)+1}^{k(\; ; \;)} \quad ^{k} \\
& \quad ^{1} + \quad ^{1} (2 \;)^{k_0(\;)} + \quad ^{2} \quad ^{k(\; ; \;)} \quad ^{k_0(\;)} \\
& \quad ^{1} + \quad ^{1} \quad ^{\log(2 \;)= \log 2} + \quad ^{2} \quad ^{\log \; = \; \log 2} \\
& \quad ^{1 \; \log(2 \;)= \log 2}
\end{aligned}
$$

which yields

$$\overline{\dim}_B F_C \; = \; \underline{\dim}_B F_C \; = \; 1 + \log(2 \;)=\log 2 \; = \; \frac{\log(4 \;)}{\log 2}$$

as required. □

Theorem 2.5 (Theorem 2.1 from [BS]). *There exists $C_1 > 0$ such that*

$$\int_J \frac{\#R_2(s;\beta;n)}{2^n}\,d\beta \leq C_1 s$$

for all $n \in \mathbb{N}$ and $s > 0$.

Theorem 2.5 will be essential when it comes to showing that a generic $\beta \in J$ satisfies a separation property. Importantly the C_1 appearing in Theorem 2.5 does not depend on n or s. In [B2] the first author studied the approximation properties of β-expansions. To understand these properties the following set was studied

$$T(s;\beta;n) := \left\{ a \in A_n(\beta) : \exists b \in A_n(\beta) \text{ satisfying } a \neq b \text{ and } |a-b| \leq \frac{s}{2^n} \right\}.$$

In [B2] it was shown that

$$\#T(s;\beta;n) \leq \#R_2(s;\beta;n). \tag{2.2}$$

If $T(s;\beta;n)$ is a small set then the elements of $A_n(\beta)$ are well spread out within $[0,1]$. As was seen in the proof of Theorem 2.1, if the elements of $A_n(\beta)$ are well spread out then F_C can be a counterexample to (1.1). We do not show that a separation condition as strong as Lemma 2.3 holds for a generic $\beta \in J$, but we can prove a weaker condition holds, a condition which turns out to be sufficient to prove Theorem 2.2.

Proposition 2.6. *For Lebesgue almost every $\beta \in J$, the following inequality holds for all but finitely many $n \in \mathbb{N}$:*

$$2^{n-1} \leq \#\left\{ a \in A_n(\beta) : |a-b| > \frac{1}{n^2 2^n} \text{ for all } b \in A_n(\beta) \setminus \{a\} \right\}.$$

To prove Proposition 2.6 we use the Borel-Cantelli lemma and the counting bounds provided by Theorem 2.5. The following lemma gives an upper bound on the Lebesgue measure of the set of β which exhibit contrary behaviour to that described in Proposition 2.6. The proof of this lemma is based upon an argument given in [B2]. We write $\mathcal{L}$ for Lebesgue measure.

Lemma 2.7. *We have*

$$\mathcal{L}\left(\beta \in J : 2^{n-1} \leq \#T(n^{-2};\beta;n)\right) \leq \frac{2C_1}{n^2}.$$

Proof. Observe that

$$\begin{aligned}
\mathcal{L}\left(\beta \in J : 2^{n-1} \leq \#T(n^{-2};\beta;n)\right) &\leq \mathcal{L}\left(\beta \in J : 2^{n-1} \leq \#R_2(n^{-2};\beta;n)\right) \qquad \text{(by (2.2))}\\
&= \frac{2^n \mathcal{L}\left(\beta \in J : 2^{n-1} \leq \#R_2(n^{-2};\beta;n)\right)}{2^n}\\
&\leq 2\int_{\beta \in J : 2^{n-1} \leq \#R_2(n^{-2};\beta;n)} \frac{\#R_2(n^{-2};\beta;n)}{2^n}\,d\beta\\
&\leq 2\int_J \frac{\#R_2(n^{-2};\beta;n)}{2^n}\,d\beta\\
&\leq \frac{2C_1}{n^2} \qquad \text{(by Theorem 2.5)}
\end{aligned}$$

as required. □

Applying Lemma 2.7 we see that

$$\sum_{n=1}^{\infty} \mathcal{L}\left(\beta \in J : 2^{n-1} \leq \#T(n^{-2};\beta;n)\right) \leq \sum_{n=1}^{\infty} \frac{2C_1}{n^2} < \infty.$$

Thus, the Borel-Cantelli lemma implies that for almost every $\beta \in J$ there exists finitely many n satisfying $2^{n-1} \leq \#T(n^{-2};\beta;n)$. If β does not satisfy a height one polynomial then $\#A_n(\beta) = 2^n$. Combining this statement with the above consequence of the Borel-Cantelli lemma we may conclude Proposition 2.6.

Note that Proposition 2.6 implies that for Lebesgue almost every 2 J , there exists a constant > 0 for which

$$2^{n \quad 1} 6 \ \# \quad {}^{n} \ a \ 2 \ A_n \ ($$

Proof. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$ endowed with the probability Lebesgue measure. Let $L^2(S^{d-1})$ denote the L^2 space of real valued functions $f : S^{d-1} \to \mathbb{R}$. By Drinfeld [D] (when $d = 3$) and by Margulis [M] and Sullivan [Su] (when $d \ge 4$), there exist rotations $g_1, \dots, g_k$ and $\varepsilon > 0$ for which the operator

$$A : L^2(S^{d-1}) \to L^2(S^{d-1}),$$

$$Af(x) = \frac{1}{k}\sum_{i=1}^{k} f(g_i^{-1}(x))$$

has a spectral gap, namely

$$\|Af\|_2 \le (1-\varepsilon)\|f\|_2 \quad \text{whenever} \quad \int f = 0.$$

Let f be the function which is 1 on the δ neighbourhood of x in S^{d-1}, and zero otherwise. Then $\int f \asymp \delta^{d-1}$. We have

$$\left\|A^n f - \int f\right\|_2 = \left\|A^n\left(f - \int f\right)\right\|_2 \le (1-\varepsilon)^n \left\|f - \int f\right\|_2 \le O(1)(1-\varepsilon)^n \delta^{(d-1)/2}. \tag{3.1}$$

Notice that $\int A^n f = \int f$. Let $E \subset S^{d-1}$ denote the support of $A^n f$ and μ denote the measure of E. Observe that E is contained by the δ-neighbourhood of $G^n(x)$, which implies

$$\mu = O(1)\delta^{d-1} N_\delta(G^n(x)). \tag{3.2}$$

Then

$$\int_{S^{d-1}\setminus E} \left(A^n f - \int f\right)^2 = (1-\mu)\left(\int f\right)^2$$

and by Cauchy–Schwarz,

$$\int_E \left(A^n f - \int f\right)^2 \int_E 1 \ge \left(\int_E A^n f - \int f\right)^2 = (1-\mu)^2\left(\int f\right)^2.$$

Combining these two, we obtain

$$\int_{S^{d-1}} \left(A^n f - \int f\right)^2 \ge \left(1 - \mu + (1-\mu)^2/\mu\right)\left(\int f\right)^2 = (1/\mu - 1)\left(\int f\right)^2.$$

Comparing this to (3.1) gives

$$1/\mu \le 1 + O(1)(1-\varepsilon)^{2n}\delta^{-(d-1)}.$$

Using (3.2) we obtain

$$1/N_\delta(G^n(x)) \le O(1)\delta^{d-1} + O(1)(1-\varepsilon)^{2n}.$$

This implies the theorem (with a different ε). □

Theorem 3.2. *Let $d \ge 3$. For every $\varepsilon > 0$ there is an IFS of similarities in $\mathbb{R}^d$ such that the attractor, $F_\emptyset$, consists of 1 point, but*

$$\underline{\dim}_B F_C \ge d - 1 - \varepsilon$$

whenever C is a singleton not equal to $F_\emptyset$.

Proof. Let $g_1, \dots, g_k$ and $\varepsilon > 0$ be as in Theorem 3.1. Fix $c < 1$ sufficiently close to 1. Consider the contractive similarities

$$S_i(x) = c\, g_i(x).$$

Then $F_\emptyset = \{0\}$ and let $C = \{x\}$ for some $x \ne 0$, which we may assume satisfies $\|x\| = 1$. Then

It can be shown that $(1+\varepsilon)^n \geqslant \varepsilon c^{(n-m)(d-1)}$ for $n = \delta(c)m + O(1)$ where

$$\delta(c) := \frac{(d-1)\log 1/c}{\log(1+\varepsilon) - (d-1)\log c}.$$

Note that $\delta(c) \to 0$ as $c \to 1$. This choice of n then yields

$$\underline{\dim}_B(F_C) \geqslant (1-\delta(c))(d-1).$$

Choosing c sufficiently close to 1 proves our result. □

Remark 3.1. When C and $F_\emptyset$ are singletons, the (lower and upper) box dimension of F_C is always less than $d-1$ (see Corollary 4.12).

4 New upper bounds

4.1 Upper bound for the box dimension of an inhomogeneous self-similar set

Recall that $S_i : \mathbb{R}^d \to \mathbb{R}^d$ $(i \in I)$ are contracting similarity maps with scaling ratio c_i defining the self-similar set $F_\emptyset \subseteq \mathbb{R}^d$. We assume that $C \subseteq \mathbb{R}^d$. To simplify notation in this section, we will write:

$$\begin{aligned} s &= \text{similarity dimension of } F_\emptyset \text{ (with the given similarities)};\\ \alpha &= \dim_H F_\emptyset = \dim_B F_\emptyset;\\ \beta &= \overline{\dim}_B C. \end{aligned}$$

Recall that the box dimension of a (homogeneous) self-similar set always exists and equals the Hausdorff dimension, see [F1, Corollary 3.3].

Definition 4.1. Assuming $S_i(x) = c_i M_i(x) + b_i$ for an orthogonal matrix M_i and $b_i \in \mathbb{R}^d$, let $T_i(x) = c_i M_i(x)$. Let G_C be the inhomogeneous self-similar set defined by the maps T_i $(i \in I)$ and C.

Definition 4.2. For $k \geqslant 0$ let I_k denote the set of those multi-indices I for which $2^{-k-1} < c_I \leqslant 2^{-k}$.

The similarity dimension s gives the bound

$$|I_k| \leqslant 2^{(s+o_k(1))k} \tag{4.1}$$

where we use the standard o_k notation; i.e. $o_k(f(k))/f(k)$ tends to zero as $k \to \infty$ (this sequence may depend on the IFS). To see this, observe that

$$|I_k| \leqslant \sum_{I \in I_k} \left(2^{k+1} c_I\right)^s \leqslant 2^{sk+s} \sum_{I \in I_k} c_I^s \leqslant 2^{sk+s} O(1) = 2^{(s+o_k(1))k}.$$

From the definition of upper box dimension, we immediately have

$$N_{2^{-n}}(C) \leqslant 2^{\beta n + o_n(n)}. \tag{4.2}$$

Definition 4.3. Let $\gamma \geqslant 0$ be the unique real number for which

$$\limsup_{k \to \infty} |\{T_I : I \in I_k\}|^{1/k} = 2^{\gamma}.$$

(In fact, this sequence is essentially sub-multiplicative and therefore the limit exists.)

Note that $\gamma \leqslant s$ by (4.1).

Lemma 4.4. If $d = 1$ or $d = 2$ (that is, F_C is a subset of $\mathbb{R}$ or $\mathbb{R}^2$) or the matrices M_i are commuting, then $\gamma = 0$.

Proof. If the matrices are commuting then the maps T_i are commuting, hence every T_I $(I \in I\,)$ is of the form

$$\prod_{i \in I} T_i^{n_i}$$

with $c_i^{n_i} > 2^{\;k\;\;1}$. Therefore $|\{T_I : I \in I_k\}|$ is at most polynomial in k, implying $\;= 0$.

If $d = 1$ then the maps are commuting. For $d = 2$, notice that it is enough to show that $|\{M_I : I \in I_k\}|^{1=k} \to 0$. This then follows by noticing that rotations of R^2 commute and that $RM = M^{\;\;1}R$ for any reection R and rotation M. □

For sets $X; Y \quad R^d$, let $X + Y = \{x + y : x \in X; y \in Y\}$ and when $X \quad R$ let $XY = \{xy : x \in X; y \in Y\}$.

Lemma 4.5. *Let* $X; Y \quad R^d$. *Then* $N\;(X + Y) \le 2^d N\;(X)N\;(Y)$:

Proof. Consider the product set $X \quad Y \quad R^d \quad R^d$ and the map $p : R^d \quad R^d \to R^d$ dened by $p(x; y) = x + y$. Let $\{U_i\}_i$ and $\{V_j\}_j$ be the half open -cubes intersecting X and Y respectively. Then $\{p(U_i \quad V_j)\}_{i;j}$ is the set of half open 2-cubes intersecting $X + Y$. Each of these cubes intersects 2^d half open -cubes, which proves the result. □

Denition 4.6. Let

$$F_C^k = \bigcup_{I \in I_k} S_I(C) \qquad \text{and} \qquad G_C^k = \bigcup_{I \in I_k} T_I(C):$$

Proposition 4.7. *Let* $0 \le k \le n$. *Then*

$$N_{2\;\;n}(F_C^k) \le 2^{o_n(n)} 2^{ks+(n\;\;k)} \tag{4.3}$$

$$N_{2\;\;n}(F_C^k) \le 2^{o_n(n)} 2^{k\;\;+(n\;\;k)d} \tag{4.4}$$

$$N_{2\;\;n}(F_C^k) \le 2^{o_n(n)} 2^{n\;\;+(n\;\;k)(\;\;+d\;\;1)} \tag{4.5}$$

$$N_{2\;\;n}(F_C^k) \le 2^{o_n(n)} 2^{n\;\;+k\;\;+(n\;\;k)}\;: \tag{4.6}$$

Proof of (4.3). For $I \in I_k$ we have

$$N_{2\;\;n}(S_I(C)) = N_{2\;\;(n\;\;k)}(2^k\;\;S_I(C)) \le N_{2\;\;(n\;\;k)}(C):$$

Using (4.1),

$$\begin{aligned} N_{2\;\;n}(F_C^k) &\le |I_k| N_{2\;\;(n\;\;k)}(C) \le 2^{sk+o_k(k)} N_{2\;\;(n\;\;k)}(C) \\ &\le 2^{sk+o_k(k)} 2^{(n\;\;k)\;\;+o_{n\;\;k}(n\;\;k)} \le 2^{o_n(n)} 2^{ks+(n\;\;k)}\;: \end{aligned}$$

□

Proof of (4.4). Let $B(X; r)$ stand for the r-neighbourhood of a set X. Assuming $C \quad B(F_{;}\,; r)$, we clearly have $F_C^k \quad B(F_{;}\,; r2^{\;\;k})$. As $F_{;}$ intersects at most $2^{k\;\;+o_k(k)}$ grid cubes of side-length 2 where in the last step we used that $2^k c_l \leqslant 1$ for $l \in I_k$. Since multiplication of scalars is commutative, $|\{2^k c_l : l \in I_k\}|$ is polynomial in k, bounded by k^D for some $D \in \mathbb{N}$ say. The upper box dimension of $|C|S^{d-1}$ is at most $+ d - 1$. Therefore (4.7) implies

$$\begin{aligned} N_{2^{-n}}(F_C^k) &\leqslant 2^d 2^{n + o_n(n)} k^D 2^{(n-k)(+ d - 1) + o_{n-k}(n-k)} \\ &\leqslant 2^{o_n(n)} 2^{n + (n-k)(+ d - 1)}. \end{aligned}$$

□

Proof of (4.6). For $l \in I_k$,

$$S_l(C) = S_l(0) + T_l(C) \quad F_\emptyset + T_l(C);$$

so

$$F_C^k \quad F_\emptyset + \bigcup_{l \in I_k} T_l(C).$$

We again have $N_{2^{-n}}(F_\emptyset) \leqslant 2^{n + o_n(n)}$, and

$$N_{2^{-n}}(\textstyle\bigcup_{l \in I_k} T_l(C)) \leqslant |\{T_l : l \in I_k\}| N_{2^{-n}}(T_l(C)) \leqslant 2^{k + o_k(k)} 2^{(n-k) + o_{n-k}(n-k)}.$$

So by Lemma 4.5,

$$N_{2^{-n}}(F_C^k) \leqslant 2^{d +}$$

Proof of Corollary 4.9. We will use the rst and the fourth estimate of Theorem 4.1. Let $f(x) = xs +$ $(1 \quad x)$ and $g(x) = \quad + (1 \quad x)$. Both function are monotone. We have $f(x) = g(x)$ if $x = \quad = s$ and at this point, the common value is $\quad + (1 \quad = s) \quad = \quad + \quad = s$. We also have $f(0) =$ and $g(1) =$.

Theorem 5.1. If F_C is an inhomogeneous self-similar set, then

$$\max\{\overline{\dim}_B F_{\emptyset}, \overline{\dim}_B C\} \leqslant \overline{\dim}_B F_C \leqslant \max\{s, \overline{\dim}_B C\}.$$

Proof. It suffices to show that $\overline{\dim}_B F_C \leqslant \max\{ (r), \overline{\dim}_B C\}$ for all $r \in (0,1)$, recalling that the lower bound is trivial. Fix $r \in (0,1)$ and let

$$J(r) = \{I \in I : I \text{ is a subword of } I' \text{ for some } I' \in I(r)\}.$$

Let

$$C(r) = \bigcup_{I \in J(r)} S_I(C) \cup C$$

and observe that this is a finite union of compact sets and so is itself compact and, moreover, has upper box dimension equal to that of C. This latter fact is due to upper box dimension being stable under taking finite unions and bi-Lipschitz images, see [F2, Chapter 3]. Let $F_{C(r)}$ denote the inhomogeneous attractor of the reduced IFS corresponding to $I(r) =$ along with the compact condensation set $C(r)$. It follows from (1.2) thatfollows

w 0 0 m B6.604 0 l S Q BT /F8 9.9626 Tf 199.229 720.357 Td [ws ws

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