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Generalised golden ratios

by

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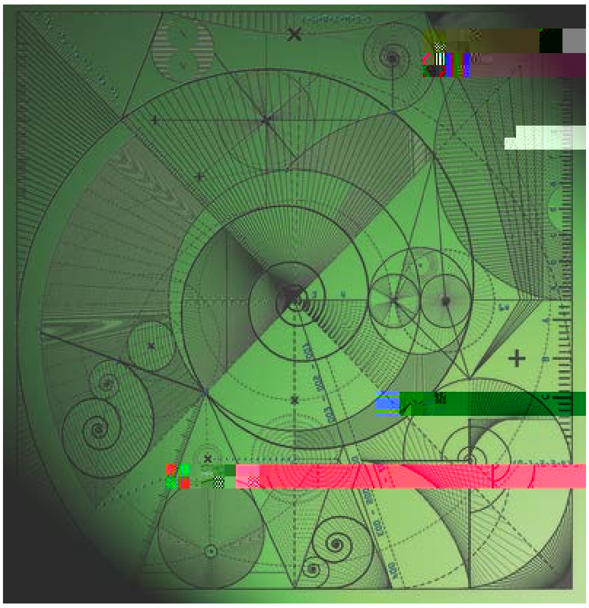

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Abstract

Given a finite set of real numbers A, the generalised golden ratio is the unique number $G(A) > 1$ for which we only have trivial unique expansions in smaller bases, and non-trivial unique expansions in larger bases. We show that $G(A)$ varies continuously with the alphabet A (of fixed size), and we calculate $G(A)$ for certain alphabets. As we vary a single parameter within A, the generalised golden ratio function may behave like a constant function, a linear function, and even a square root function.

We also build upon the work of Komornik, Lai, and Pedicini (2011) and study generalised golden ratios over ternary alphabets. We give a new proof of their main result, that is we explicitly calculate the function $G(\{0,1,m\})$. (For a ternary alphabet, it may be assumed without loss of generality that $A = \{0,1,m\}$.) We also study the set of $m \in (1,2]$ for which $G(\{0,1,m\}) = 1+\sqrt{m}$ and prove that it is an uncountable set of Hausdorff dimension 0. Last of all we show that the function mapping m to $G(\{0,1,m\})$ is of bounded variation yet has unbounded derivative.

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1 Introduction and statement of results

Let $A := \{a_0, a_1, \ldots, a$

that the golden ratio acts as a natural boundary between the possible cardinalities the set of expansions can take. It is natural to ask whether such a boundary exists for more general alphabets.

Before we state the definition of a generalised golden ratio it is necessary to define the univoque set. Given an alphabet A and $\beta > 1$ we set

$$U_\beta(A) := \left\{ (u_k)_{k=1}^{\infty} \in A^{\mathbb{N}} : \sum_{k=1}^{\infty} \frac{u_k}{\beta^k} \text{ has a unique expansion} \right\}.$$

We call $U_\beta(A)$ the univoque set. Note that for any alphabet A and $\beta > 1$ the points

$$\sum_{k=1}^{\infty} \frac{a_0}{\beta^k} \text{ and } \sum_{k=1}^{\infty} \frac{a_d}{\beta^k}$$

both have a unique expansion, so $\overline{a_0}$ and $\overline{a_d}$ are always contained in the univoque set. Here and throughout $\overline{w}$ denotes the infinite periodic word with period w. We are now in a position to define a generalised golden ratio for an arbitrary alphabet. Given an alphabet A, we call $G(A) \in (1, \infty)$ the generalised golden ratio for A if whenever $\beta \in (1, G(A))$ we have $U_\beta(A) = \{\overline{a_0}, \overline{a_d}\}$, and if $\beta > G(A)$ then $U_\beta(A$

The set M is uncountable, but its Hausdorff dimension is 0.

Theorem 3. We have $\dim_H(M) = 0$.

On certain intervals, the function G has the following simple form.

Theorem 4. Let h be a positive integer and $2^h \le m \le \left(1+\sqrt{\frac{m}{m-1}}\right)^h$. Then we have

$$G\left(\frac{m}{m-1}\right) = G(\{0,1,m\}) = m^{1/h}.$$

In [1] the first author studied $G(m)$ for integer m. In this case the following results hold.

Theorem B. Let $m \in \mathbb{Z}$ with $m \ge 2$. The following statements hold:

$$G(m) = \begin{cases} \frac{m}{2}+1 & \text{if } m \text{ is even,} \\ \frac{m+1+\sqrt{m^2+10m+9}}{4} & \text{if } m \text{ is odd.} \end{cases}$$

- If m is odd, then there exists $\varepsilon(m) > 0$ such that for all $\beta \in (G(m), G(m)+\varepsilon(m))$, the set $U_\beta(m)$ consists of $\overline{\frac{m-1}{2}\frac{m+1}{2}}$ and a subset of the sequences that end with $\overline{\frac{m-1}{2}\frac{m+1}{2}}$.
- If m is even, then there exists $\varepsilon(m) > 0$ such that for all $\beta \in (G(m), G(m)+\varepsilon(m))$, the set $U_\beta(m)$ consists of $\overline{\frac{m}{2}}$ and a subset of the sequences that end with $\overline{\frac{m}{2}}$.

For each positive integer k, we also calculate $G(m)$ on a small interval to the right of k. These calculations demonstrate that the function G can vary in different ways as we change a single parameter. For now we postpone the statement of these results.

2 Continuity of $G(A)$

Proof of Theorem 1. As $G(A) = G\left(\frac{A - a_0}{a_d - a_0}\right)$, we have $G(\{a_0; a_1; \ldots; a_d\}) = G(\rho \circ r(a_0; a_1; \ldots; a_d))$, with

$$r : \Delta_d \to \Delta'_d;\quad (a_0; a_1; \ldots; a_d) \mapsto \left(\frac{a_1 - a_0}{a_d - a_0}; \frac{a_2 - a_0}{a_d - a_0}; \ldots; \frac{a_{d-1} - a_0}{a_d - a_0}\right);$$
$$\rho : \Delta'_d \to \mathcal{P}(\mathbb{R});\quad (a_1; a_2; \ldots; a_{d-1}) \mapsto \{0; a_1; a_2; \ldots; a_{d-1}; 1\};$$

and $\Delta'_d = \{(a_1; a_2 \ldots; a_{d-1}) \in \mathbb{R}^{d-1} : 0 < a_1 < a_2 < \cdots < a_{d-1} < 1\}$. As r is continuous on Δ_d, it is sufficient to prove that $G \circ \rho$ is continuous on Δ'_d.

Let $a = (a_1; a_2 \ldots; a_{d-1}) \in \Delta'_d$ and $\varepsilon > 0$ arbitrary but fixed. We will show that $|G(\rho(b)) - G(\rho(a))| \le 3\varepsilon$ for all b in a neighbourhood of a. Let first $X \subset \Delta'_d$ be a closed neighbourhood of a such that $|q(\rho(b)) - q(\rho(a))| \le \varepsilon$ for all $b \in X$. (Note that $q \circ \rho$ is continuous on Δ'_d.) Set

$$\beta = \min_{b \in X} q(\rho(b)) - \varepsilon; \qquad Y = \{b \in X : G(\rho(b)) < \gamma\}.$$

If $Y = \emptyset$, then X is a neighbourhood of a with $|G(\rho(b)) - G(\rho(a))| \le 2\varepsilon$ for all $b \in X$. Otherwise, let $\ell \ge 2$ be such that $\sum_{k=1}^{\ell} \beta^{-k} \le (\beta + \varepsilon - 1)^{-1}$. Then

$$b_{j+1} - b_j \ge \frac{1}{q(\rho(b)) - 1} \ge \frac{1}{\beta + \varepsilon - 1} \ge \sum_{k=1}^{\ell} \frac{1}{\beta^k} \tag{2.3}$$

for all $(b_1; \ldots; b_{d-1}) \in Y$, $0 \le j < d$, with $b_0 = 0$, $b_d = 1$. Set

$$\delta(a; b) = \min_{0 \le j < d} (a_{j+1} - a_j) - (b_{j+1} - b_j)$$

(with $b_0 = a_0 = 0$, $b_d = a_d = 1$), and let $Z \subseteq X$ be a neighbourhood of a such that

$$\frac{a_j}{(\beta + \varepsilon)^k} - \frac{b_j}{\beta^k} \le \delta(a; b); \quad \frac{b_j}{(\beta + \varepsilon)} b \ldots$$

Similarly, we obtain from (2.2), (2.5) and (2.6) that

$$\sum_{k=1}^{\infty} \frac{1-u_{i+k}}{(\beta+\varepsilon)^k} - \sum_{k=1}^{\infty} \frac{1-u_{i+k}}{\beta^k} + \delta(a;b) < b_j - b_{j-1} + \delta(a;b) \le a_j - a_{j-1} \quad \text{when } u_i = b_j \neq 0.$$

Therefore, we have $u \in U_{\beta+\varepsilon}(\pi(a))$, thus $G(\pi(a)) \le G(\pi(b)) + \varepsilon$ for all $b \in Y \cap Z$.

For $b \in X \setminus Y$, recall that $G(\pi(a)) \le q(\pi(a)) \le \beta + 2\varepsilon \le G(\pi(b)) + 2\varepsilon$. Similarly, we obtain for all $b \in Z$ that $G(\pi(b)) \le G(\pi(a)) + \varepsilon$ when $a \in Y$, $G(\pi(b)) \le G(\pi(a)) + 3\varepsilon$ when $a \notin Y$. This gives that $|G(\pi(b)) - G(\pi(a))| \le 3\varepsilon$ for all $b \in Z$, thus $G \circ \pi$ is continuous at a. □

3 Generalised golden ratios over ternary alphabets

3.1 Statements

Komornik, Lai and Pedicini [4] described the function $m \mapsto G(m)$ on the interval $(1, 2]$. We provide more

Note that all the numbers and sequences do not change if we replace by $_0$ since $_0(0\bar{1}) = 0^{-}$

$_{[0;n]}(0)$ for all n 0, i.e., $u_i u_{i+1}$ = u

for all i 0 such that $u_i = 1$. By Lemma 3.6 and since u is aperiodic, $u_i = 1$ implies that $u < u_{i+1}u_{i+2} < u_1u_2$. These bounds cannot be improved because, for all n 0, $1\ _{[0;n]}(0)$ and $1\ _0(1)\ _{[0;n\ 1]}(1)$ (which is a su x of $_{[0;n]}(0)$) are factors of u. Therefore, we have u 2 $U_{1+\sqrt{m}}(m)$ if and only if

$$\sqrt{m} \quad 1 + \sum_{k=1}^{\infty} \frac{u_k}{(1+\sqrt{m})^{k+1}} \quad \text{and} \quad 1 + \sum_{k=1}^{\infty} \frac{u_k}{(1+\sqrt{m})^{k}} \quad m:$$

This means that $1 + \sum_{k=1}^{1} u_k (1+\sqrt{m})\ ^{k} = m$, i.e., $m = m_u$. □

Lemma 3.10. Let 2 S and $m > 1$. There is a unique number $f\ (m) > 1$ such that

$$m = 1 + \sum_{k=1}^{\infty} \frac{u_k^{(\ )}}{f\ (m)^k}: \tag{3.4}$$

We have $f^{\,0}(m) < 0$, $f\ (m_{(0\bar{1})}) = 1 + \sqrt{m_{(0\bar{1})}}$, f

Proof. The number is well de ned since $f^{0}(m) < 0$, $g^{0}(m) > 0$ on I ,

$$f\ (m_{(0\bar{1})}) = 1 + \sqrt{m_{(0\bar{1})}} > g\ (m_{(0\bar{1})}) \text{ and } f\ (m_{(\bar{1})}) < 1 + \sqrt{m_{(\bar{1})}} = g\ (m_{(\bar{1})}):$$

If $(\bar{1}) =$

Proof. Let 2, $u \in U(m) \cap \{0; 1\}^N$ and $\tilde{u}$ as in Lemma 3.15. Then $\tilde{u} \in U_{\tilde{}}(m)$ for all $\tilde{} >$. If $\tilde{u} \in S_1$, then Lemma 3.9 gives that $1 + \sqrt{m}$. If $\tilde{u} = (\bar{1})$,

4 Behaviour at the generalised golden ratio

In this section we discuss the behaviour of the univoque set at the generalised golden ratio. It was observed in [1] that when $\beta = G(L)$ for some $L \in \mathbb{N}$, then every $x \in (0, \frac{L}{\beta-1})$ either has a countable infinite of expansions or a continuum of expansions. In other words $U_{G(L)}(L)$ is still trivial. However, Lemma 3.9 demonstrates that this is not always the case. Indeed the following result is an immediate consequence of this lemma.

Proposition 4.1. *There exists A for which $U_{G(A)}(A)$ is non-trivial.*

In [9] it was shown that the smallest $\beta \in (1,2)$ for which an x has precisely two expansions over the alphabet $\{0,1\}$ was $\beta_2 \approx 1.71064$. In other words, there is a small gap between the golden ratio for the alphabet $\{0,1\}$, and the smallest β for which an x has precisely two expansion. As we show below, for certain alphabets it is possible that an x has precisely two expansions at the golden ratio.

Proposition 4.2. *For every $m \in M$, the number $m/G(m)$ has precisely two expansions in base $G(m)$ over the alphabet $\{0,1,m\}$.*

Proof. Let $u \in S$ be such that $m = m_u$, let $\beta = G(m) = 1 + \sqrt{m}$ and let $m/\beta = \sum_{k=1}^{\infty} v_k \beta^{-k}$ be an expansion of m/β over the alphabet $\{0,1,m\}$. Since $m > \frac{m}{\beta-1}$, we have $v_1 \in \{1,m\}$, thus $\sum_{k=1}^{\infty} v_{k+1}\beta^{-k}$ equals $m-1$ and 0 respectively. Clearly, 0 has a unique expansion, and $m-1$ has the expansion $u_1u_2\cdots$ by (3.1), which is also unique. □

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