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and Physical Sciences

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Preprint MPS-2015-21

09 November 2015

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ON THE WELL-POSEDNESS OF GLOBAL FULLY NONLINEAR FIRST ORDER ELLIPTIC SYSTEMS

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Abstract. In the very recent paper [K1], the second author proved that for any $f \in L^2(\mathbb{R}^n;\mathbb{R}^N)$, the fully nonlinear first order system $F(\cdot, \mathrm{D}u) = f$ is well posed in the so-called J.L. Lions space and moreover the unique strong solution $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the problem satisfies a quantitative estimate. A central ingredient in the proof was the introduction of an appropriate notion of ellipticity for F inspired by Campanato's classical work in the 2nd order case. Herein we extend the results of [K1] by introducing a new strictly weaker ellipticity condition and by proving well posedness in the same "energy" space.

1. Introduction

In this paper we consider the problem of existence and uniqueness of global strong solutions $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the fully nonlinear first order PDE system

$$F(\cdot, \mathrm{D}u) = f, \quad \text{a.e. on } \mathbb{R}^n, \tag{1.1}$$

where $n, N \geq 2$ and $F : \mathbb{R}^n \times \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}^N$ is a Caratheodory map. The latter means that $F(\cdot, X)$ is a measurable map for all $X \in \mathbb{R}^{N \times n}$ and $F(x, \cdot)$ is a continuous map for almost every $x \in \mathbb{R}^n$. The gradient $\mathrm{D}u : \mathbb{R}^n \longrightarrow \mathbb{R}^{N \times n}$ of our solution $u = (u_1, ..., u_N)^\top$ is viewed as an $N \times n$ matrix-valued map $\mathrm{D}u = (\mathrm{D}_i u_\alpha)_{i=1...n}^{\alpha=1...N}$ and the right hand side f is assumed to be in $L^2(\mathbb{R}^n;\mathbb{R}^N)$.

The method we use in this paper to study (1.1) follows that of the recent paper [K1] of the second author. Therein the author introduced and employed a new perturbation method in order to solve (1.1) which is based on the solvability of the respective linearised system and a structural ellipticity hypothesis on first order differential operator. In the linear F assumes the form

$$F(x, X) = \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \mathrm{A}_{\alpha i \beta j} X_{\beta j}\, e_\alpha,$$

for some linear map $\mathrm{A} : \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}^N$. We will follow almost the same conventions as in [K1], for instance we will denote the standard bases of $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{N \times n}$ by

2010 *Mathematics Subject Classification.* Primary 35J46, 35J47, 35J60; Secondary 35D30, 32A50, 32W50.

Key words and phrases. Cauchy-Riemann equations, fully nonlinear systems, elliptic first order systems, Calculus of Variations, Campanato's near operators, Cordes' condition, Compensated compactness, Baire Category method, Convex Integration.

$fe^i g$, $fe\ \ g$ and $fe\ \ \ e^i g$ respectively. In the linear case, (1.1) can be written as

$$\sum_{=1}^{N}\sum_{i;j=1} A\ \ _j D_j u\ \ = f\ ;\qquad = 1;\ldots;N;$$

and compactly in vector notation as

$$A : Du = f: \tag{1.2}$$

The appropriate well-known notion of ellipticity in the linear case is that the nullspace of the linear map A contains no rank-one lines This requirement can be quanti ed as

$$jA : \tag{1.3}$$

in terms of the distance of the respective right hand sides of (1.1). The main advance in this paper which distinguishes it from the results obtained in [K1] is that we introduce a new notion of ellipticity for (1.1) which is strictly weaker than (1.5), allowing for more general nonlinearities F to be considered. Our new hypothesis of ellipticity is inspired by an other recent work of the second author [K2] on the second order case. We will refer to our condition as the \AK-Condition" (De nition 4). In Examples 5, 6 we demonstrate that the new condition is genuinely weaker and hence our results indeed generalise those of [K1]. Further, otiv systems. We leave the study of the present problem in the context of \D-solutions"

In (2.4), $(h_m)_1^\infty$ is any sequence of even functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$0 \leq h_m(x) \leq \frac{1}{|x|} \quad \text{and} \quad h_m(x) \longrightarrow \frac{1}{|x|}, \quad \text{for a.e. } x \in \mathbb{R}^n, \quad \text{as } m \to \infty.$$

The limit in (2.4) is meant in the weak L^2 sense as well as a.e. on $\mathbb{R}^n$, and u is independent of the choice of sequence $(h_m)_1^\infty$.

In the above statement, "sgn", "cof" and "det" symbolise the sign function on $\mathbb{R}^n$, the cofactor and the determinant on $\mathbb{R}^{N\times N}$ respectively. Although the formula (2.4) involves complex quantities, u above is a real vectorial solution. Moreover, the symbol "$\widehat{\ }$" stands for Fourier transform (with the conventions of [F]) and "$\vee$" stands for its inverse.

Next, we recall the strict ellipticity condition of the second author taken from [K1] in an alternative form which is more convenient for our analysis. We will relax it in the next section.

Definition 2 (K-Condition of ellipticity, cf. [K1]). Let $F : \mathbb{R}^n \times \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$ be a Caratheodory map. We say that F is elliptic when there exists a linear map

$$A : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^N$$

satisfying (2.1) and $0 < \beta < 1$ such that for all X, Y

satisfying (1.3), a positive function with $\ ;\ 1/\ \in L^1(\mathbb{R}^n)$ and $\ ;\ > 0$ with $\ +\ < 1$ such that

$$(3.1)\qquad (x)\left[F(x;X+Y) - F(x;Y)\right] - A:X \quad (A)|X| + \ |A:X|:$$

for all $X,Y \in \mathbb{R}^{Nn}$ and a.e. $x \in \mathbb{R}^n$. Here (A) is the ellipticity constant of A given by (1.6).

Nontrivial fully nonlinear examples of maps F which are elliptic in the sense of the Definition 4 above are easy to find. Consider any fixed map $A : \mathbb{R}^{Nn} \to \mathbb{R}^N$ for which $(A) > 0$ and any Carathéodory map

$$L : \mathbb{R}^n \ \mathbb{R}^{Nn} \to \mathbb{R}^N$$

which is Lipschitz with respect to the second variable and

$$L(x;\ )_{C^{0,1}(\mathbb{R}^{Nn})} \quad (A); \quad \text{for a.e. } x \in \mathbb{R}^n$$

for some $0 < \ < 1$. Let also be a positive essentially bounded function with $1/$ essentially bounded as well. Then, the map $F : \mathbb{R}^n \ \mathbb{R}^{Nn} \to \mathbb{R}^N$ given by

$$F(x;X) := \frac{1}{(x)}A:X + L(x;X)$$

satisfies Definition 4, since

$$\begin{aligned} (x)\left[F(x;X+Y) - F(x;Y)\right] - A:X \quad & L(x;X+Y) - L(x;Y) \\ & (A)|X| \\ & (A)|X| + \frac{1}{2}|A:X|: \end{aligned}$$

As a consequence, F satisfies the AK-Condition for the same function $(\ )$ and for the constants and $= (1\ )/2$. 3X we have $|X_0| = 2$ and $|A : X_0| = 2$. Hence, for any $Y \in \mathbb{R}^{Nn}$ we have

$$
\begin{aligned}
\Big[F(\cdot;X_0+Y) \quad F(\cdot;Y)\Big] \quad A : X_0 &= \tfrac{1}{}A : (X_0+Y) \quad \tfrac{1}{}A : Y \quad A : X_0 \\
&= \tfrac{1}{}A : X_0 \quad A : X_0 \\
&= A : X_0 \quad \tfrac{1}{} \quad 1 \\
&= 2 \quad \tfrac{1}{} \quad 1 \\
&= \quad \tfrac{}{2}(A)\,|X_0|;
\end{aligned}
$$

where we have used that $(1= \ ) \quad 1 \quad 1$. Our claim ensues.

The essential point in the above example that makes Definition 4 more general than Definition 2 was the introduction of the rescaling function $(\ )$. Now we give a more elaborate example which shows that even if we ignore the rescaling function and normalise it to $(\ ) \quad 1$, Definition 4 is still more general that Definition 2.

Example 6. Fix $c, b > 0$ such that $c + b < 1$ and $\sqrt{2}c + b > 1$ and a unit vector $\in \mathbb{R}^N$. Consider the Lipschitz function $F \in C^0(\mathbb{R}^{2\times 2})$, given by:

$$F(x; X) := A : X + \quad b\ X \quad + c\ A : X \quad ; \tag{3.2}$$

where A is again the Cauchy-Riemann tensor (1.4). Then, this F satisfies

$$\Big[F(\cdot;X+Y) \quad F(\cdot;X)\Big] \quad A : Y \qquad (A)|Y| + \quad A : Y \ ; \tag{3.3}$$

for some $\ ; \ > 0$ with $\ + \ < 1$, but does not satisfy (3.3) with $\ = 0$ for any $0 < \ < 1$. Hence, F satisfies Definition 4 (even if we $x \quad (\ ) \quad 1$) but it does not satisfy Definition 2. Indeed we have:

$$
\begin{aligned}
A : Y \quad \Big[F(\cdot;X+Y) \quad F(\cdot;X)\Big]
&= A : Y \quad A : Y \quad b\ \ |X+Y| \quad |X| \quad c\ \ A:(X+Y) \quad |A : X| \\
&\quad b|\ |\,|X+Y| \quad |X| + c|\ |\,|A : X + A : Y| \quad |A : X| \\
&\quad b|Y| + c|A : Y|
\end{aligned}
$$

and hence (3.3) holds for $\ = b$ and $\ = c$. On the other hand, we choose

$$X_0 := 0; \quad Y_0 := \quad \begin{matrix}1 \\ 1\end{matrix} \quad ; \qquad := \frac{1 \quad b}{\sqrt{2c^2 \quad (1 \quad b)^2}}.$$

This choice of $\ $ is admissible because our assumption $\sqrt{2}c + b > 1$ implies $2c^2 \quad (1 \quad b)^2 > 0$. For these choices of X and Y, we calculate:

$$A : Y_0 \quad \Big[F(\cdot;X_0+Y_0) \quad F(\cdot;X_0)\Big] = \quad =$$

We now show that

$$b|Y_0| + c|\mathrm{A}:Y_0| = |Y_0|$$

and this will allow us to conclude that(3.3) can not hold for any < 1 if we impose $= 0$. Indeed, since $|Y_0|^2 = 2 + 2\ ^2$ and $|\mathrm{A}:Y_0|^2 = 4\ ^2$, we have

$$\begin{aligned} 1 \quad b\ ^2|Y_0|^2 \quad c^2|\mathrm{A}:Y_0|^2 &= 1 \quad b\ ^2 2\ 1 + \ ^2 \quad c^2 4\ ^2 \\ &= 2\ 1 \quad b\ ^2 + 2 \quad 1 \quad b\ ^2 \quad 2c^2 \quad ^2 \\ &= 2\ 1 \quad b\ ^2 + 2 \quad 1 \quad b\ ^2 \quad 2c^2 \frac{(1 \quad b)^2}{2c^2 \quad (1 \quad b)^2} \\ &= 0. \end{aligned}$$

We now show that our ellipticity assumption can be seen an a notion of pseudo-monotonicity coupled by a global Lipschitz continuity property. The statement and the proof are modelled after a similar result appearing in [K2] which however was in the second order case.

Lemma 7 (AK-Condition of ellipticity vs Pseudo-Monotonicity)**.** Definition 4 is equivalent to the following statements:

There exist $> \quad > 0$, a linear map $\mathrm{A} : \mathbb{R}^{Nn} \to \mathbb{R}^N$ satisfying (1.3) a positive function such that $, 1/ \in L^1(\mathbb{R}^n)$ with respect to which F satisfies

$$(3.4) \qquad (\mathrm{A}:Y)^\top \Big[F(x,X+Y) \quad F(x,X) \Big] \qquad \frac{}{(x)}|\mathrm{A}:Y|^2 \qquad \frac{}{(x)} (\mathrm{A})^2|Y|^2,$$

for all $X, Y \in \mathbb{R}^{Nn}$ and a.e. $x \in \mathbb{R}^n$. In addition, $F(x,\ )$ is Lipschitz continuous on $\mathbb{R}^{Nn}$, essentially uniformly in $x \in \mathbb{R}^n$; namely, there exists $M > 0$ such that

$$(3.5) \qquad F(x,X) \quad F(x,Y) \qquad M|X \quad Y|$$

for a.e. $x \in \mathbb{R}^n$ and all $X, Y \in \mathbb{R}^{Nn}$.

Proof of Lemma 7. Suppose that Definition 4 holds for some constant $, > 0$ with $+ \ < 1$, some positive function with $, 1/ \in L^1(\mathbb{R}^n)$ and some linear map $\mathrm{A} : \mathbb{R}^{Nn} \to \mathbb{R}^N$ satisfying (1.3). Fix $\varepsilon > 0$. Then, for a.e. $x \in \mathbb{R}^N$ and all $X, Y \in \mathbb{R}^{Nn}$ we have:

$$\begin{aligned} &|\mathrm{A}:Y|^2 + \ (x)^2 \ F(x,X+Y) \quad F(x,X)\ ^2 \\ &\quad 2\ (x)\,(\mathrm{A}:Y)^\top \Big[F(x,X+Y) \quad F(x,X) \Big] \\ &\qquad\qquad ^2\ (\mathrm{A})^2|Y|^2 + \ ^2|\mathrm{A}:Y|^2 + 2 \quad (\mathrm{A})|Y|\ |\mathrm{A}:Y| \end{aligned}$$

which implies

$$\begin{aligned} &|\mathrm{A}:Y|^2 \quad 2\ (x)\,(\mathrm{A}:Y)^\top \Big[F(x,X+Y) \quad F(x,X) \Big] \\ &\qquad\qquad ^2\ (\mathrm{A})^2|Y|^2 + \ ^2|\mathrm{A}:Y|^2 + \frac{^2\ (\mathrm{A})^2|Y|^2}{\varepsilon} + \varepsilon\ ^2|\mathrm{A}:Y|^2. \end{aligned}$$

Hence,

$$\begin{aligned} &(\mathrm{A}:Y)^\top \Big[F(x,X+Y) \quad F(x,X) \Big] \\ &\qquad\qquad \frac{1}{(x)} \ \frac{1 \quad ^2 \quad \varepsilon\ ^2}{2} \ |\mathrm{A}:Y|^2 \qquad \frac{1}{(x)} \ \frac{\varepsilon\ ^2 + \ ^2}{2\varepsilon} \quad (\mathrm{A})^2|Y|^2. \end{aligned}$$

By choosing $'' := \; = \;$, from the above inequality we obtain (3.4) for the values

$$:= \frac{1 \quad (\; + \;)}{2}; \qquad := \frac{(\; + \;)}{2}:$$

These are admissible because $\; > 0$ and $\; > \;$ since

$$= \frac{1 \quad (\; + \;)^2}{2} > 0:$$

In addition, again by (3.1) we have:

$$(x)\, F(x; X) \quad F(x; Y) \qquad (\mathrm{A})|X \quad Y| + \quad \mathrm{A} : (X \quad Y) \; + \; \mathrm{A} : (X \quad Y) \; ;$$

and hence,

$$F(x; X) \quad F(x; Y) \qquad \frac{1}{(x)} \Big((1 + \;)\, \mathrm{A} : (X \quad Y) \; + \quad (\mathrm{A}) \; X \quad Y \Big)$$
$$\frac{1}{(\;)}_{L^1(\mathbb{R}^n)} \quad (1 + \;)|\mathrm{A}| + \quad (\mathrm{A}) \quad |X \quad Y|$$

for a.e. $x \in \mathbb{R}^N$ and all $X; Y \in \mathbb{R}^{Nn}$, which immediately leads to (3.5). Conversely, suppose that (3.4) and (3.5) hold and x a constant $\; > 2$. Then, by (3.5) we have the inequality

$$\frac{M^2 \;\; (x)^2}{{}^2 \; {}^2 \;\; (\mathrm{A})^2} \; (\mathrm{A})^2 |Y|^2 \qquad \frac{(x)^2 \; F(x; X + Y) \quad F(x; X)^{\;\;2}}{(\quad)^2}: \tag{3.6}$$

Further, by (3.4) we have

$$\begin{aligned} |\mathrm{A} : Y|^2 \qquad & \frac{2 \;\; (x)}{} \; (\mathrm{A} : Y)^{>} \Big[F(x; X + Y) \quad F(x; X) \Big] \\ & 1 \quad \frac{2}{} \; |\mathrm{A} : Y|^2 \; + \; \frac{2}{} \; (\mathrm{A})^2 |Y|^2 : \end{aligned} \tag{3.7}$$

By adding the inequalities (3.6) and (3.7), we obtain

$$\mathrm{A} : Y \quad \frac{(x)}{} \Big[F(x; X + Y) \quad F(x; X) \Big]^2$$
$$1 \quad \frac{2}{} \; |\mathrm{A} : Y|^2 \; + \; \left[\frac{2}{} \; + \; \frac{1}{{}^2} \; \frac{M \;\; (x)}{(\mathrm{A})} \quad {}^2 \right] \; (\mathrm{A})^2 |Y|^2$$

Hence,

$$\mathrm{A} : Y \quad \frac{(x)}{} \Big[F(x; X + Y) \quad F(x; X) \Big] \qquad {}_2\#$$
$$\sqrt{1 \quad \frac{2}{}} \; |\mathrm{A} : Y| \; + \; (\mathrm{A}) \; \sqrt{\frac{2}{}} \; \sqrt{\; \quad + \; \frac{1}{2} \quad M k \; k_{L^1} \;\; \mathrm{A} : Y} \qquad)$$

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