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Equivalence between weak and -solutions for symmetric hyperbolic first order PDE systems

by

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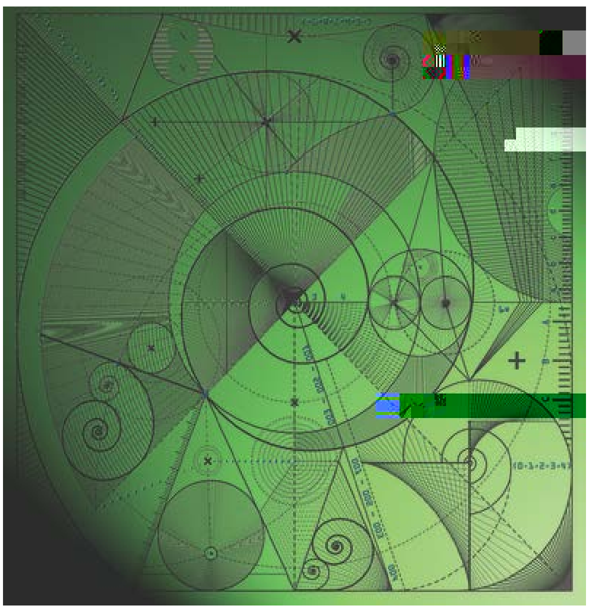

symmetric hyperbolic system (1.1) plays an important role in many contexts for both theory and applications.

In this paper we consider the question of equivalence between two completely di erent notions of generalised solutions to (1.1) and by assuming a commutativity

nonlinear degenerate elliptic systems

$$\begin{cases} F(\cdot, D^2 u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and we proved existence and uniqueness of a $\mathcal{D}$-solution to the problem (which in general may not be even once weakly differentiable).

2. Theory of $\mathcal{D}$-solutions for fully nonlinear systems

2.1. Preliminaries. We begin with some notation and some basic facts which will be used throughout the paper, perhaps without explicitly quoting this subsection.

Basics. Our measure theoretic and function space notation is either standard, e.g. as in [E2, EG] or self-explanatory. The norms $|\cdot|$ appearing will always be the Euclidean, while the Euclidean inner products will be denoted by either "$\cdot$" on $\mathbb{R}^m, \mathbb{R}^M$ or by ":" on matrix spaces, e.g. on $\mathbb{R}^{M \times m}$ we have

$$|X|^2 = \sum_{\alpha=1}^{M} \sum_{i=1}^{m} X_{\alpha i} X_{\alpha i}$$

Further, let $u : \quad \mathbb{R}^m \to \mathbb{R}^M$ be any measurable map which we understand to be extended by zero on $\mathbb{R}^m \setminus$. For any $a \in \mathbb{R}^n$ with $|a| = 1$ and $h \in \mathbb{R} \setminus \{0\}$, the difference quotients of u along the direction a at x will be denoted by

$$D_a^{1,h} u(x) := \frac{1}{} \tag{2.3}$$

De nition 1 (Young measures). The space of Young measures is the set of all probability-valued mappings $E \quad R^m \; ! \quad P \; \overline{R}^{Mm}$ which are weakly* measurable. Hence, the set of Young measures can be identi ed with a subset of the unit sphere of L^1_w

Definition 6 (D-solutions of 1st order systems). Let $\Omega \subseteq \mathbb{R}^m$ be open,

$$F : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M \times m} \longrightarrow \mathbb{R}^d$$

a Carathéodory map and $u : \Omega \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^M$ a map in $W^{1,1}_{loc}(\Omega; \mathbb{R}^M)$. Suppose we have fixed some reference frames as in (2.1) and consider the PDE system

$$F\big(x, u(x), Du(x)\big) = 0, \quad \text{on } \Omega. \tag{2.5}$$

We say that u is a D-solution of (2.5) when for any diffuse gradient of u arising from any infinitesimal sequence along subsequences (Definition 4)

$$\delta_{D^{1,h_k}u} \overset{*}{\rightharpoonup} \mathcal{D}u \quad \text{in } \mathscr{Y}\big(\Omega; \overline{\mathbb{R}}^{M \times m}\big), \quad \text{as } k \to \infty$$

and for any $\Phi \in C^0_c\big(\mathbb{R}^{M \times m}\big)$, we have

$$\int_{\overline{\mathbb{R}}^{M \times m}} \Phi(X)\, F\big(x, u(x), X\big)\, d[\mathcal{D}u(x)](X) = 0, \quad \text{a.e. } x \in \Omega.$$

The following result asserts the fairly obvious fact that D-solutions are compatible with strong solutions.

Proposition 7 (Compatibility of D-solutions with strong solutions). Let F be a Carathéodory map as above, $u : \Omega \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^M$ a mapping in $W^{1,1}_{loc}(\Omega; \mathbb{R}^M)$ and consider the PDE system (2.5). Then, u is a D-solution on Ω if and only if u is a strong a.e. solution on Ω.

The proof of Proposition 7 is an immediate consequence of Lemma 5 and of the motivation of the notions.

Remark 8 (Nonlinearity of diffuse derivatives and relation to distributions). We summarise here some of the discussions of [K8]. In the context of the usual notions of solution (smooth, strong, weak, distributional), it is standard that the generalised derivative is a linear operation. However, without extra assumptions this may be false for diffuse derivatives; D-solutions are a genuinely nonlinear approach even when we apply them to linear PDE. More precisely, let $T_a : \mathbb{R}^{M \times m} \longrightarrow \mathbb{R}^{M \times m}$ denote the translation by a. Given a Young measure $\vartheta \in \mathscr{Y}\big(\Omega; \overline{\mathbb{R}}^{M \times m}\big)$, we define $\vartheta \circ T_a \in$ $\mathscr{Y}\big(\Omega;$

3. Equivalence between weak and D-solutions for hyperbolic systems

3.1. Fibre spaces and the main result.

and for any $\in C^0_c(\mathbb{R}^{N+Nn})$, we have

$$\int_{\overline{\mathbb{R}}^{N+Nn}} (\underline{X}) \quad X_0 + A:X \quad f(t,x) \quad d\,\underline{D}u(t,x)\,(\underline{X}) = 0,$$

for a.e. $(t,x) \in (0,T) \quad \mathbb{R}^n$.

Moreover, any D-solution u to (3.7) (and hence any weak solution) has the following regularity: the projection of the space-time gradient $(D_t u, Du)$ on the subspace $\quad \mathbb{R}^{N+Nn}$ associated to A exists in L^2, $\quad$ is spanned by rank-one matrices and for any such direction $\quad \underline{a} \in \quad$, we have $D_{\underline{a}}(\quad u) \in L^2((0,T) \quad \mathbb{R}^n)$.

The commutativity hypothesis is always satisfied if either $n = 1$ (one spatial dimension) or $N = 1$ (scalar case). By standard results on hyperbolic systems, we readily have the following consequence of Theorem 9:

Corollary 10 (Well-posedness of the Cauchy problem for D-solutions). In the setting of Theorem 9 and under the same assumptions, the Cauchy problem

$$\begin{cases} D_t u + A:Du = f, & \text{in } (0,T) \quad \mathbb{R}^n, \\ u(0,\ ) = u_0, & \text{on } \{0\} \quad \mathbb{R}^n, \end{cases}$$

has a unique D-solution in the fibre space $W^{1,2}\big((0,T) \quad \mathbb{R}^n, \mathbb{R}^N\big)$, for any given $f \in L^2\big((0,T) \quad \mathbb{R}^n, \mathbb{R}^N\big)$, $u_0 \in L^2(\mathbb{R}^n, \mathbb{R}^N)$ and $T > 0$.

Proof of Theorem 9. The proof consists of three lemmas. In the first one below we show that the commutativity hypothesis on the matrices $A_1, \ldots, A_n$ implies that the vector space $\quad$ of (3.3) has an orthonormal basis of rank-one directions which can be completed to an orthonormal basis of rank-one directions spanning $\mathbb{R}^{N+Nn}$.

Lemma 11. GivenLemmconsequence of2

Proof of Lemma 11. We begin by observing that directly from the definitions of and of $^{?} = N(\underline{A})$, we have

(3.13)
$$\begin{cases} {}^{?} = \Big\{ \underline{X} = A : X \ X \quad X \in \mathbb{R}^{Nn} \Big\} ; \\ \quad = \Big\{ \underline{Y} = [Y_0|Y] \in \mathbb{R}^{N+Nn} \quad Y_0 \ (A : X) + Y : X = 0 ; \ X \in \mathbb{R}^{Nn} \Big\} : \end{cases}$$

Next, by standard linear algebra results ([L]) we obtain that the commutativity hypothesis of the (symmetric) matrices $\{A_1; \ldots; A_n\} \subseteq \mathbb{R}^{N \ N}_s$ is equivalent to the requirement that there exists an orthonormal basis $\{ ^1; \ldots; ^N\} \subseteq \mathbb{R}^N$ which diagonalises all the matrices $\{A_1; \ldots; A_n\}$ simultaneously, namely there is a common set of eigenvectors for perhaps different eigenvalues $\{c^{(i)1}; \ldots; c^{(i)N}\}$ of A_i. Thus, for any $i = 1 \ldots n$, we have

$$A_i \quad = c^{(i)} \quad ; \qquad = 1; \ldots; N;$$

or, in index form (see (1.4))

$$\sum_{=1}^{N} A \ _i \quad = c^{(i)} \quad ; \quad ; \quad = 1; \ldots; N;$$

whereas $(A_i) = \{c^{(i)1}; \ldots; c^{(i)N}\}$. We rewrite the above as

$$\sum_{=1}^{N} \sum_{j=1}^{n} A \ _j \quad e^i_j \ + \quad c^{(i)} \qquad = 0; \quad ; \quad = 1; \ldots; N; \ i = 1; \ldots; n;$$

and in view of (1.4) we may write it as

$$(3.14) \qquad A : \quad e^i \ + \quad c^{(i)} \qquad = 0 \qquad = 1; \ldots; N; \ i = 1; \ldots; n.$$

We now define

$$(3.15) \qquad N^{\ i} := \frac{c^{(i)}}{e^i} = \qquad c^{(i)} \ ; 0; \ldots; 0; \underbrace{1}_{(1+i)\text{-position}}; 0; \ldots 0 \ ^{\top} \ ;$$

for $= 1; \ldots; N$, $i = 1; \ldots; n$, and also

$$(3.16) \qquad N^{\ 0} := \frac{1}{c} = \qquad 1; c^{(1)} \ ; \ldots; c^{(n)} \ ^{\top} \ ;$$

where

$$c := c^{(1)} \ ; \ldots; c^{(n)} \ ^{\top}$$

is the -th eigenvalue vector of the matrices $\{A_1; \ldots; A_n\}$. The definition of $N^{\ i}$ and (3.14) with (3.1) immediately give that

$$\underline{A} : N^{\ i} = 0; \qquad = 1; \ldots; N; \ i = 1; \ldots; n;$$

and hence $N^{\ i} \in N(\underline{A}) = {}^{?}$. Moreover, by (3.13) and the fact that the (Nn)-many matrices $\{ \ e^i \ | \ ; i \}$ are an orthonormal basis of $\mathbb{R}^{Nn}$, we have that

$$\begin{aligned} \underline{Y} \in \quad & \Longleftrightarrow \quad Y_0 \ (A : X) + Y : X = 0; \qquad X \in \mathbb{R}^{Nn}; \\ & \Longleftrightarrow \quad Y_0 \quad A : (\quad e^i) \ + Y : (\quad e^i) = 0; \quad = 1; \ldots; N; \ i = 1; \ldots; n; \\ & \overset{(3.14)}{\Longleftrightarrow} \quad Y_0 \quad c^{(i)} \quad + Y : (\quad e^i) = 0; \qquad = 1; \ldots; N; \ i = 1; \ldots; n; \\ & \overset{(3.15)}{\Longleftrightarrow} \quad [Y_0|Y] : N^{\ i} = 0; \qquad = 1; \ldots; N; \ i = 1; \ldots; n. \end{aligned}$$

Hence, $\underline{Y}$? $N(\underline{A})$ if and only if $\underline{Y}$? $N^{\ i}$ for all $= 1; :::; N$ and $i = 1; :::; n$. Since $N(\underline{A}) = \ ^{?}$, this proves that

$$(3.17) \qquad N(\underline{A}) = \text{span}[\ \overset{n}{\ } N^{\ i} \quad = 1; :::; N;\ i = 1; :::; n \overset{o}{\ }\ ]:$$

Moreover, the matrices $N^{\ i}$ spanning $N(\underline{A})$ are linearly independent and hence exactly Nn-many. Indeed, for $;\ = 1; :::; N$ and $i; j = 1; :::; n$, by (3.15) we have

$$N^{\ i} : N^{\ j} = \frac{c^{(i)}}{e^{i}} : \frac{c^{(i)}}{e^{j}}$$

$$= c^{(i)}\ c^{(j)} + \ _{ij} :$$

It follows that for any $\ \not=\ $, $N^{\ i}$ is orthogonal to $N^{\ j}$. Moreover, for any for $= 1; :::; N$ and $i \not= j$ in $f1; :::; ng$, by (3.15) we have

$$\frac{N^{\ i}}{jN^{\ i}j} : \frac{N^{\ j}}{jN^{\ j}j} = \frac{c^{(i)}}{p\overline{1 + (c^{(i)}\ )^2}}\ \frac{c^{(j)}}{p\overline{1 + (c^{(j)}\ )^2}}\ 2\ (\ 1; +1)$$

and hence for each the set of matrices $f N^{\ i}\ j\,i\,g$ is linearly independent. Further, by (3.15), (3.16) we have that

$$N^{\ 0} : N^{\ i} = \frac{1}{c} : \frac{c^{(i)}}{e^{i}}$$

$$= \overset{n}{\ }\ 1; c^{(1)}\ ; :::; c^{(n)} \qquad c^{(i)}\ ; 0; :::; 0; 1; 0; :::0 \overset{o}{\ }$$

(1+i)-position

$$= c^{(i)} + c^{(i)}$$

$$= 0;$$

for all $;\ = 1; :::; N$ and $i = 1; :::; n$. Moreover, by (3.16) we have

$$N^{\ 0} : N^{\ 0} = \frac{1}{c} : \frac{1}{c}$$

$$= 1 + c\ \ c$$

and as a consequence the matrices $fN^{\ 0}j\ \ g$ form an orthogonal set of N-many elements which is orthogonal to $N(\underline{A})$. Since the dimension of the space is $N + Nn$, all the above together with (3.13), (3.15), (3.16) prove that

$$(3.18) \qquad = \text{span}[\ \overset{n}{\ } N^{\ 0} \quad = 1; :::; N \overset{o}{\ }\ ]:$$

We now show that the frame $f N^{\ i}\ j\ ; i\ g$ can be modi ed in order to be made an orthonormal basis and still consisting of rank-one matrices. First note that the matrices spanning are orthogonal and we only need to x their length. Further, note that $\ ^{?}$ can be decomposed as the following direct sum of mutually orthogonal subspaces

$$^{?} = \bigoplus_{=1}^{N} \text{span}[\ \overset{n}{\ } N^{\ i} \quad i = 1; :::; n \overset{o}{\ }\ ] =: \bigoplus_{=1}^{N} W :$$

Since

$$W = \text{span}[\ \frac{c^{(i)}}{e^{i}} : i = 1; :::; n\ ];$$

by the Gram-Schmidt method, we can nd an orthonormal basis of W consisting of matrices of the form

(3.19) $$N^{i} = \underline{a}^{( )i}; \quad \underline{a}^{( )i} \quad \underline{a}^{( )j} = {}_{ij} :$$

Finally, we de ne

$$E^{0} := \frac{N^{0}}{|N^{0}|} = \frac{1}{\sqrt{1+|c|^2}} \quad \frac{1}{c} \quad ! \quad 2\ \mathbb{R}^{N+Nn};$$

$$E^{i} := N^{i} = \underline{a}^{( )i} \quad 2\ \mathbb{R}^{N+Nn};$$

and also

$$E := \quad 2\ \mathbb{R}^{N};$$

$$E^{( )0} := \frac{1}{\sqrt{1+|c|^2}} \quad \frac{1}{c} \ ; \ E^{( )i} := \underline{a}^{( )i} \quad 2\ \mathbb{R}^{1+n};$$

where $= 1;:::;N$ and $i = 1;:::;n$. By the previous it follows that $f E^{i} j = 1;:::;N; i = 0;1;:::;ng$ is an orthonormal basis of $\mathbb{R}^{N+Nn}$ consisting or rank-one directions such that $f E^{0} j = 1;:::;N g$ span the subspace and $f E^{i} j = 1;:::;N; i = 1;:::;ng$ span its complement $^{?}$. Moreover, $E^{i} = E \quad E^{( )i}$. We conclude the proof of the lemma by noting that (3.11), (3.12) follow by the de nition of and standard linear algebra results.

Next, we employ the orthonormal frames constructed in Lemma 11 and the properties (3.11), (3.12) of A in order to characterise weak solutions to (3.7) as mappings in the bre space (3.6) which solve the equation in a pointwise \strong bre-wise" sense: the equation is satis ed a.e. on $(0;T) \quad \mathbb{R}^n$ if we substitute the distributional gradient $(D_t u; Du$

that u 2 $W^{1;2}$ $(0;T)$ $\mathrm{R}^n;\mathrm{R}^N$ and in addition $\underline{D}u^{"}$! $\underline{G}(u)$ in L^2. Thus, by passing to the limit in (3.20) as " ! 0 and as ! 0, we obtain that

$$\underline{A}:\underline{G}(u) = f; \quad \text{a.e. on } (0;T) \quad \mathrm{R}^n; \tag{3.21}$$

as desired. Conversely, suppose that (3.21) holds. Then, by the de nition (3.6) of the bre space there are approximating sequences u ! u and $\underline{D}u$! $\underline{G}(u)$, both in L^2 as ! 1 . Hence, we have

$$\begin{aligned} \underline{A}: \quad \underline{D}u \quad f &= A: \quad \underline{D}u \quad \underline{G}(u) \\ &= o(1); \end{aligned}$$

as ! 1 , in L^2. By the above, (3.12) and (3.1)-(3.5), we have

$$\begin{aligned} D_t u + A:Du \quad f &= \underline{A}:\underline{D}u \quad f \\ &= \underline{A}: \quad \underline{D}u \quad f \\ &= o(1); \end{aligned}$$

as ! 1 , in L^2. Hence, for any 2 C^1_c $(0;T)$ R^n , we have

$$\int_{(0;T} Du$$

as desi 33427 6.9059 -4.114 Td [( )]TJ/F11 9.9626 Tf 11.623 0 Td [(f)]T00u

Further, for any fixed measurable set $E \subseteq (0,T) \times \mathbb{R}^n$ with finite measure and any $\phi \in C^0_c(\mathbb{R}^{N\times Nn})$, by using our hypothesis $\underline{A} : \underline{G}(u) = f$, we have the estimate

$$\left\| \phi\big(\underline{D}^{1,h}u\big)\big(\underline{A} : \underline{D}^{1,h}u - f\big) \right\|_{L^1(E)} \leq \sqrt{|E|}\, \|\phi\|_{C^0(\mathbb{R}^{N\times Nn})} \left\| \underline{A} : \underline{D}^{1,h}u - \underline{A} : \underline{G}(u) \right\|_{L^2((0,T)\times\mathbb{R}^n)}. \tag{3.24}$$

Hence, (3.23) and (3.24) imply

$$\phi\big(\underline{D}^{1,h}u\big)\big(\underline{A} : \underline{D}^{1,h}u - f\big) \longrightarrow 0, \quad \text{in } L^1(E;\mathbb{R}^N) \text{ as } h \to 0. \tag{3.25}$$

Moreover, the Carathéodory function

$$\Phi(\underline{x}, \underline{X}) := \phi(\underline{X})\big(\underline{A} : \underline{X} - f(\underline{x})\big)\chi_E(\underline{x}) \tag{3.26}$$

is an element of the space

$$L^1\Big((0,T)\times\mathbb{R}^n; C^0\big(\overline{\mathbb{R}}^{N\times Nn}\big)\Big)$$

because

$$\|\Phi\|_{L^1((0,T)\times\mathbb{R}^n;C^0(\overline{\mathbb{R}}^{N\times Nn}))} \leq |E| \left(\max_{\underline{X}\in\mathrm{supp}(\phi)} \big|\phi(\underline{X})\,\underline{A}:\underline{X}\big|\right) + \sqrt{|E|}\left(\max_{\underline{X}\in\mathrm{supp}(\phi)} \big|\phi(\underline{X})\big|\right)\|f\|_{L^2((0,T)\times\mathbb{R}^n)}.$$

Let now $(h_\nu)_1^\infty \subseteq \mathbb{R}\setminus\{0\}$ be any infinitesimal sequence. Then, there is a subsequence $h_{\nu_k} \to 0$ such that

$$\ldots^{1,h_{\nu_k}}u} \overset{*}{\rightharpoonup} \mathcal{D}u \quad \text{in } \mathscr{Y}\big((0,T)\times\mathbb{R}^n; \overline{\mathbb{R}}^{N\times Nn}\big), \quad \text{as } k \to \infty.$$

... strong continuity of the duality pairing between

... $\mathbb{R}^n; C^0$ ...

Conversely, suppose that

Such selections exist for large enough $R > 0$ by Aumann's measurable selection theorem (see e.g. [FL]), but in this specific case they can also be constructed explicitly because of the simple structure of the multi-valued mapping. By using (3.34), (3.33) implies that

$$\lim_{k\to\infty} \int_{E^R} \underline{A} : T^R \left(\underline{x}; \underline{D}^{1,h_k} u(\underline{x})\right) f(\underline{x})\, d\underline{x} = 0$$

and by recalling (3.12), we rewrite this as

$$\lim_{k\to\infty} \int_{E^R} \underline{A} : T^R \left(\underline{x};\ \underline{D}^{1,h_k} u(\underline{x})\right) f(\underline{x})\, d\underline{x} = 0: \tag{3.35}$$

Hence, (3.35) implies that

$$\begin{aligned}\int_{E^R} \underline{A} : \underline{G}(u)\ f \ &\ \int_{E^R} \underline{A} : T^R\ ;\ \underline{D}^{1,h_k} u\ \ f \\ &+ \int_{E^R} \underline{A} : T^R\ ;\ \underline{D}^{1,h_k} u\ \ \underline{A} : \underline{G}(u) \\ &\ o(1) + |\underline{A}| \int_{E^R} T^R\ ;\ \underline{D}^{1,h_k} u\ \ \underline{G}(u)\end{aligned}$$

as $k \to \infty$, and as a consequence we have

$$\begin{aligned}\int_{E^R} \underline{A} : \underline{G}(u)\ f \ &\ |\underline{A}| \int_{E^R} T^R\ ;\ \underline{D}^{1,h_k} u\ \ T^R\ ;\underline{G}(u) \\ &+ |\underline{A}| \int_{E^R} T^R\ ;\underline{G}(u)\ \ \underline{G}(u) + o(1);\end{aligned} \tag{3.36}$$

as $k \to \infty$, for any $R > 0$. Moreover, by assumption u is in the fibre space (3.6). Hence by invoking (3.23), the Dominated convergence theorem, the fact that $|E| < \infty$ and (3.34), we may pass to the limit in (3.36) as $k \to \infty$ to obtain

$$\int_{E^R} \underline{A} : \underline{G}(u)\ f \ \ |\underline{A}| \int_{E^R} T^R\ ;\underline{G}(u)\ \ \underline{G}(u)\ ;$$

for any $R > 0$. Finally, we let $R \to \infty$ and recall the arbitrariness of the set E $(0,T)$ $\mathbb{R}^n$ and (3.34) to infer that $\underline{A} : \underline{G}(u) = f$, a.e. on $(0,T)$ $\mathbb{R}^n$. The lemma has been established.

The proof of Theorem 9 is now complete.

Remark 14 (Functional representation of the diffuse gradients). In a sense, Lemma 13 says that all the diffuse gradients of the D-solution u when restricted on the subspace of non-degeneracies have a certain "functional" representation inside the coefficients , given by $\underline{G}(u)$. Namely, if we decompose $\mathbb{R}^{N+Nn}$ = $^?$, the restriction of any diffuse space-time gradient $\underline{D}u \in \mathscr{Y}$ $;\overline{\mathbb{R}}^{N+Nn}$ on is given by the fibre space-time gradient:

$$\underline{D}u(t,x)\, \mathsf{X} = \ _{\underline{G}(u)(t,x)}; \quad \text{a.e. } (t,x) \in (0,T)\ \mathbb{R}^n:$$

This is a statement of "partial regularity type" for D-solutions: although not all of the diffuse gradient is a Dirac mass, certain restrictions of it on subspaces are concentration measures.

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