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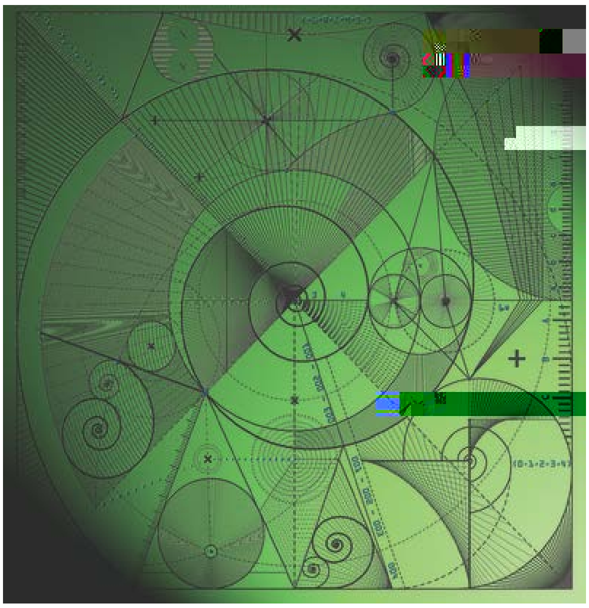

Coburn's Lemma and the Finite Section Method for Random Jacobi Operators

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Abstract. We study the spectra and pseudospectra of semi-in nite and bi-in nite tridiagonal random matrices and their nite principal submatrices, in the case where each of the three diagonals varies over a separate compact set, say $U;V;W$ C. Such matrices are sometimes termed stochastic Toeplitz matrices A_+ in the semi-in nite case and stochastic Laurent matrices A in the bi-in nite case. Their spectra, = spec A and $_+$ = spec A_+, are independent of A and A_+ as long as A and A_+ are pseudoergodic (in the sense of E.B. Davies, Commun. Math. Phys. 216 (2001), 687{704), which holds almost surely in the random case. This was shown in Davies (2001) for A; that the same holds for A_+ is one main result of this paper. Although the computation of and $_+$ in terms of U, V and W is intrinsically di cult, we give upper and lower spectral bounds, and we explicitly compute a set G that lls the gap between and $_+$

with entries $u_i \in U$, $v_i \in V$ and $w_i \in W$ for all i under consideration. The sets U, V and W are nonempty and compact subsets of the complex plane $\mathbb{C}$, and the box marks the matrix entry of A at $(0; 0)$. We will be especially interested in the case where the matrix entries are random (say i.i.d.) samples from U, V and W. Trefethen et al. [54] call the operator A a *stochastic Laurent matrix* in this case and A_+ a *stochastic Toeplitz matrix*. We will adopt this terminology which seems appropriate given that one of our aims is to highlight parallels between the analysis of standard and stochastic Laurent and Toeplitz matrices.

It is known that the spectrum of A depends only on the sets U, V, and W, as long as A is pseudoergodic in the sense of Davies [19], which holds almost surely if A is stochastic (see the discussion below). Via a version, which applies to stochastic Toeplitz matrices, of the famous Coburn lemma [17] for (standard) Toeplitz matrices, a main result of this paper is to show that, with the same assumption of pseudoergodicity implied by stochasticity, also the spectrum of A_+ depends only on U, V, and W. Moreoever, we tease out very explicitly what the difference is between the spectrum of a stochastic Laurent matrix and the spectrum of the corresponding stochastic Toeplitz matrix. (The difference will be that certain 'holes' in the spectrum of the stochastic Laurent case may be 'filled in' in the stochastic Toeplitz case, rather similar to the standard Laurent and Toeplitz cases.)

The second main result is to show that infinite linear systems, in which the matrix, taking one of the forms (1), is a stochastic Laurent or Toeplitz matrix, can be solved effectively by the standard finite section method, provided only that the respective infinite matrices are invertible. In particular, our results show that, if the stochastic Toeplitz matrix is invertible, then every finite $n \times n$ matrix formed by taking the first n rows and columns of A_+ is invertible, and moreover the inverses are uniformly bounded. Again, this result, which can be interpreted as showing that the finite section method for stochastic Toeplitz matrices does not suffer from spectral pollution (cf. [40]), is reminiscent of the standard Toeplitz case.

Related work. The study of random Jacobi operators and their spectra has one of its main roots in the famous Anderson model [1, 2] from the late 1950's. In the 1990's the study of a non-selfadjoint (NSA) Anderson model, the Hatano-Nelson model [30, 42], led to a series of papers on NSA random operators and their spectra, see e.g. [24, 19, 18, 41]. Other examples of NSA models are discussed in [15, 16, 54, 35]: one example that has attracted significant recent attention (and which, arguably, has a particularly intriguing spectrum) is the randomly hopping particle model due to Feinberg and Zee [20, 21, 11, 13, 12, 26, 27, 28]. A comprehensive discussion of this history, its main contributors, and many more references can be found in Sections 36 and 37 of [55].

A theme of many of these studies [54, 55, 11, 13, 12], a theme that is central to this paper, is the relationships between the spectra, norms of inverses, and pseudospectra of random operators, and the corresponding properties of the random matrices that are their finite sections. Strongly related to this (see the discussion in the 'Main Results' paragraphs below) is work on the relation between norms of inverses and pseudospectra of finite and infinite *classical* Toeplitz and Laurent matrices [49, 3, 5]. In between the classical and stochastic Toeplitz cases, the same issues have also been studied for randomly perturbed Toeplitz and Laurent operators [8, 9, 10, 7]. This paper, while focussed on the specific features of the random case, draws strongly on results on the finite section method in much more general contexts: see [33] and the 'Finite sections' discussion below. With no assumption of randomness, the finite section method for a particular class of NSA perturbations of (selfadjoint) Jacobi matrices is analysed recently in [40].

We will build particularly on two recent studies of random Jacobi matrices A and A_+ and their finite sections. In [26] it is shown that the closure of the numerical range of these operators is the convex hull of the spectrum, this holding whether or not A and A_+ are normal operators. Further, an explicit expression for this numerical range is given: see (33) below. In [37] progress is made in bounding the spectrum and understanding the finite section method applied to solving infinite linear systems where the matrix is a tridiagonal stochastic Laurent or Toeplitz matrix. This last paper is the main starting point for this present work and we recall key notations and concepts that we will build on from [37] in the following paragraphs.

Matrix notations. We understand A and A_+ as linear operators, again denoted by A and A_+, acting boundedly, by matrix-vector multiplication, on the standard spaces $\ell^p(\mathbb{Z})$ and $\ell^p(\mathbb{N})$ of bi- and singly-infinite complex sequences with $p \in [1, \infty]$. The sets of all operators A and

$\mathrm{spec}^p B$ is independent of p (and so abbreviated as $\mathrm{spec}\, B$), the set $\mathrm{spec}^p_\varepsilon B$ depends on p in general. It is a standard result (see [55] for this and the other standard results we quote) that

$$\mathrm{spec}\, B + \varepsilon \mathbb{D} \subset \mathrm{spec}^p_\varepsilon B; \tag{3}$$

with $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk. If $p = 2$ and B is a normal matrix or operator then equality holds in (3). Clearly, for $0 < \varepsilon_1 < \varepsilon_2$, $\mathrm{spec}\, B \subset \mathrm{spec}^p_{\varepsilon_1} B \subset \mathrm{spec}^p_{\varepsilon_2} B$, and $\mathrm{spec}\, B = \bigcap_{\varepsilon > 0} \mathrm{spec}^p_\varepsilon B$ (for $1 \le p \le \infty$). Where $\overline{S}$ denotes the closure of $S \subset \mathbb{C}$, a deeper result, see the discussion in [52] (summarised in [12]), is that

$$\overline{\mathrm{spec}^p_\varepsilon B} = \mathrm{Spec}^p_\varepsilon B := \left\{\lambda \in \mathbb{C} : \|(B - \lambda I)^{-1}\|_p \ge \varepsilon^{-1}\right\}. \tag{4}$$

Interest in pseudospectra has many motivations [55]. One is that $\mathrm{spec}^p_\varepsilon B$ is the union of $\mathrm{spec}(B + T)$ over all perturbations T with $\|T\|_p < \varepsilon$. Another is that, unlike $\mathrm{spec}\, B$ in general, the pseudospectrum depends continuously on B with respect to the standard Hausdorff metric (see (63) below).

Limit operators. A main tool of our paper, and of [37], is the notion of limit operators. For $A = (a_{ij})_{i,j \in \mathbb{Z}} \in BDO(X)$ with $X = \ell^p(\mathbb{Z})$ and $h_1, h_2, \ldots$ in $\mathbb{Z}$ with $|h_n| \to \infty$ we say that $B = (b_{ij})_{i,j \in \mathbb{Z}}$ is a *limit operator* of A if, for all $i, j \in \mathbb{Z}$,

$$a_{i+h_n, j+h_n} \to b_{ij} \quad \text{as} \quad n \to \infty. \tag{5}$$

The boundedness of the diagonals of A ensures (by Bolzano-Weierstrass) the existence of such sequences (h_n) and the corresponding limit operators B. From $A \in BDO(X)$ it follows that $B \in BDO(X)$. The closedness of U, V and W implies that $B \in M(U;V;W)$ if $A \in M(U;V;W)$.

shall approximate the solution x of (6) (in the sense that, for every right hand side b, it holds as $n \to \infty$ that $\|x_n - x\| \to 0$, if $1 \le p < \infty$, that $\|x_n\| = O(1)$ and $x_n(j) \to x(j)$ for every $j \in \mathbb{Z}$, if $p = \infty$). If that is the case then the FSM is said to be *applicable* to A

where this supremum is taken over all limits B_+ and C_- in (9), and is attained as a maximum if the FSM is applicable to A.

Versions of Lemma 1.2, (10), and (11) hold for semi-infinite matrices $A_+ = (a_{ij})_{i,j\in\mathbb{N}} \in BDO(\ell^p(\mathbb{N}))$, with the modification that $l_n = -\infty$, which implies that every limit B_+ in (9) is nothing but the matrix A_+ again, so that in Lemma 1.2 iii), iv) and (11) it is the invertibility of only A_+ and C_- that is at issue.

Remark 1.3 – Reflections. Often we find it convenient to rearrange/reflect the matrices $C_- = (c_{ij})_{i,j=-\infty}^{0}$ from (9) as $B_+ = (c_{-j,-i})_{i,j=0}^{\infty}$. This rearrangement $C_- \mapsto B_+$ corresponds to a matrix reflection against the bi-infinite antidiagonal; it can be written as $B_+ = RC_-^\top R$, where R denotes the bi-infinite flip $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{-i})_{i\in\mathbb{Z}}$. As an operator on ℓ^p, one gets $\|B_+\|_p = \|RC_-^\top R\|_p = \|C_-^\top\|_p = \|C_-\|_q$

sets is. The key to describe the di erence between $\mathrm{spec}_{\mathrm{ess}} B_+$ and $\mathrm{spec} B_+$ is a new result that has a famous cousin in the theory of Toeplitz operators: Coburn's Lemma [17] says that, for every bounded and nonzero Toeplitz operator T_+, one has $(T_+) = 0$ or $(T_+) = 0$, so that T_+ is known to be invertible as soon as it is Fredholm and has index zero. We prove that the same statement holds with the Toeplitz operator T_+ replaced by any $B_+ \; 2 \; M_+(U; V; W)$ provided that 0 is not in (12). So $\mathrm{spec}_{\mathrm{ess}} B_+$ and $\mathrm{spec} B_+$ di er by the set of all $2 \; C$ for which $B_+ \quad I_+$ is Fredholm with a nonzero index. We give new upper and lower bounds on the sets $\mathrm{spec}_{\mathrm{ess}} B_+$ and $\mathrm{spec} B_+$, and we nd easily computable sets G that close the gap between the two, i.e., sets+

were the core of [37], our second main tool is a \glueing technique" { see (37) and (38) { that is used in the proofs of two of our main results, Theorems 2.2 and 3.3 as well asww

$|u| < |w|$, and collapses into a line segment if $|u| = |w|$. From (14) and spec $C = E(u,v,w)$ if C is Laurent, we get that the union of all ellipses $E(u,v,w)$ with $u \in U$, $v \in V$ and $w \in W$ is a simple lower bound on Σ:

$$\bigcup_{u\in U,\, v\in V,\, w\in W} E(u,v,w) \subset \Sigma(U,V,W). \tag{16}$$

Because we will come back to Laurent and Toeplitz operators, let us from now on write

$$T(u,v,w) := uS + vI + wS^{-1} \quad \text{and} \quad T_+(u,v,w)$$

for the Laurent operator $T \in M(\{u\},\{v\},\{w\})$, acting on $\ell^p(\mathbb{Z})$, and for its compression T_+ to $\ell^p(\mathbb{N})$, which is a Toeplitz operator. Here we write S for the forward shift operator, $S : x \mapsto y$ with $y(j+1) = x(j)$ for all $j \in \mathbb{Z}$, and S^{-1} for the backward shift. From (13) and (15) (or [4, 6]), $\mathrm{spec}_{ess}\, T_+(u,v,w) = \mathrm{spec}\, T(u,v,w) = E(u,v,w)$. Further,

$$\mathrm{spec}\, T_+(u,v,w) = \mathrm{conv}\, E(u,v,w) \tag{17}$$

is the same ellipse but now filled [4, 6]. (Here conv S denotes the convex hull of a set $S \subset \mathbb{C}$.) Let $E_{in}(u,v,w)$ and $E_{out}(u,v,w)$ denote the interior and exterior, respectively, of the ellipse $E(u,v,w)$, with the understanding that $E_{in}(u,v,w) = \varnothing$ and $E_{out}(u,v,w) = \mathbb{C} \setminus E(u,v,w)$ when $|u| = |w|$ and the ellipse $E(u,v,w)$ degenerates to a straight line. The reason why the spectrum of a Toeplitz operator T_+ is obtained from the spectrum of the Laurent operator T (which is at the same time the essential spectrum of both T_+ and T) by filling in the hole $E_{in}(u,v,w)$ can be found in the classical Coburn lemma [17]. We will carry that fact over to stochastic Toeplitz and Laurent operators. A key role will also be played by the following index formula. Let wind(γ, z) denote the winding number (counter-clockwise) of an oriented closed curve γ with respect to a point $z \notin \gamma$. For $0 \notin E(u,v,w)$, so that T_+ is Fredholm, it holds that [4, 6]

$$\mathrm{ind}\, T_+(u,v,w) = -\,\mathrm{wind}(E(u,v,w), 0) = \begin{cases} 0, & 0 \in E_{out}(u,v,w), \\ \mathrm{sign}(|w| - |u|), & 0 \in E_{in}(u,v,w). \end{cases} \tag{18}$$

To get a simple upper bound on Σ, write $A \in E(U,V,W)$ as $A = D + T$ with diagonal part $D = \mathrm{diag}(v_i)$ and off-diagonal part T and think of A as a perturbation of D by T with $\|T\| \le \varepsilon$, where $\varepsilon := u^* + w^*$ and

$$u^* := \max_{u\in U} |u|, \qquad w^* := \max_{w\in W} |w|.$$

Since A is in the ε-neighbourhood of D, its spectrum spec $A = \Sigma$ is in the ε-neighbourhood of spec $D \subset V$. (Note that D is normal or look at Lemma 3.3 in [37].) In short,

$$\Sigma(U,V,W) \subset V + (u^* + w^*)\overline{\mathbb{D}} \tag{19}$$

(recall that $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk, and $\overline{\mathbb{D}}$ is its closure). Note that the same argument, and hence the same upper bound, applies to the spectra of all (singly or bi-)infinite and all finite Jacobi matrices over U, V and W.

Sometimes equality holds in (19) but often it does not. For $U = \{1\}$, $V = \{0\}$ and $W = \mathbb{T}$, the lower (16) and upper bound (19) on Σ coincide so that equality holds in (19) saying that $\Sigma = 2\overline{\mathbb{D}}$. If we change W from $\mathbb{T}$ to $\{\pm 1\}$ then the right-hand side of (19) remains at $2\overline{\mathbb{D}}$ while Σ is now smaller (it is properly contained in the square with corners ± 2 and $\pm 2i$, see [12, 13]). Taking W even down to just $\{1\}$, the spectrum Σ clearly shrinks to $[-2, 2]$ with the right-hand side of (19) still at $2\overline{\mathbb{D}}$. So the gap in (19) can be considerable, or nothing, or anything in between, really.

Equality (13) contains the formula

$$\mathrm{spec}_{ess}\, B_+ = \bigcup_{C_+ \in M_+(U,V,W)} \mathrm{spec}_{ess}\, C_+ =$$

for all $B_+ \in E_+(U;V;W)$. One of our new results, Corollary 2.5 below, is that

$$\mathrm{spec}\, B_+ = \bigcup_{C_+ \in M_+(U;V;W)} \mathrm{spec}\, C_+ =: \ _+ \qquad (20)$$

holds independently of $B_+ \in E_+(U;V;W)$.

Upper and lower bounds on $_+ = \ _+(U;V;W)$ can be derived in the same way as above for . This time, because of (17), the ellipses in the lower bound (16) have to be filled in, while the upper bound from (19) remains the same, so that

$$\bigcup_{u \in U, v \in V, w \in W} \mathrm{conv}\, E(u;v;w) \quad _+(U;V;W) \quad V + (u + w)\overline{D}. \qquad (21)$$

The results in this section will also make precise the difference between and $_+$.

For nonzero Toeplitz operators T_+ (semi-infinite matrices with constant diagonals), acting boundedly on $\ell^p(\mathbb{N})$, the following classical result fills the gap between essential spectrum and spectrum: at least one of the two integers, (T_+) and (T_+), is always zero. So if their difference is zero (i.e. T_+ is Fredholm with index zero) then both numbers are zero (i.e. T_+ is injective and surjective, hence invertible). This is Coburn's Lemma [17], which was also found, some years earlier, by Gohberg [22] (but for the special case of Toeplitz operators with continuous symbol). Here is a new cousin of that more than 50 year old lemma:

Theorem 2.2 If $(U;V;W)$ is compatible (i.e. (i)–(vi) hold in Proposition 2.1) then every $B_+ \in M_+(U;V;W)$ is Fredholm and at least one of the non-negative integers (B_+) and (B_+) is zero.

Proof. Let $(U;V;W)$ be compatible and take $B_+ \in M_+(U;V;W)$ arbitrarily. Then B_+, with matrix representation $(b_{ij})_{i,j \in \mathbb{N}}$, is Fredholm since (i)–(vi) of Proposition 2.1 hold. Suppose that $(B_+) > 0$ and $(B_+) > 0$. Then there exist $x \in \ell^p(\mathbb{N})$ and $y \in \ell^p(\mathbb{N})$, with $x \neq 0$ and $y \neq 0$, such that $B_+ x = 0$ and $B_+^{>} y = 0$. Let $a, b \in \mathbb{C}$, set

$z = (\ ; ay_2; ay_1; \boxed{0}; bx_1; bx_2;$ 2>9626 Tf 6.614 1.494 Te 7.0 .9626 Tf 3.874 0 Td [(N)]TJ/

Similarly to the situation for Toeplitz operators, one can now derive invertibility of operators in $M_+(U;V;W)$ from their Fredholmness and index. The additional result here that every $B_+ \in M_+(U;V;W)$ is Fredholm with the same index was first pointed out in [37 complex plane $\mathbb{C}$ into four pairwise disjoint parts. To this end, x an arbitrary $B_+ \in E_+(U;V;W)$. The rst part of the plane is our set $= (\ U;V;W)$ from (13),

$$
\begin{aligned}
&= \{\ \in \mathbb{C} : B_+ \ \ I_+ \text{ is not Fredholm}\} \\
&= \operatorname{spec}_{ess} B_+ = \bigcup_{C_+ \in M_+(U;V;W)} \operatorname{spec}_{ess} C_+ \\
&= \{\ \in \mathbb{C} : (U;V\ \ ;W) \text{ is not compatible}\}.
\end{aligned}
$$

The rest of the complex plane now splits into the following three parts: for $k = \ 1;0;1$, let

$$
{}_k := \{\ \in \mathbb{C} : \operatorname{ind}(B_+ \ \ I_+) = k\} = \{\ \in \mathbb{C} : \ (U;V\ \ ;W) = k\}.
$$

Theorem 2.7 It holds that

$$_{+} = \quad [\quad _{1} = \quad [\; E \; _{1} = \quad [\; E_{\backslash} = \quad [\; E_{[} \tag{32}$$

with $_{1}$ and E $_{1}$ from (30) and $E_{\backslash}$ and E

b) There is a similar coincidence between the FSM for pseudoergodic bi-in nite matrices (called

Precisely, with $B = (b_{ij})_{i,j\in\mathbb{Z}}$, we put $B_+ := (b_{ij})_{i,j\in\mathbb{N}} \in M_+(U;V;W)$ and $B_- := (b_{ij})_{i,j\in-\mathbb{N}}$.

Now, for a vector $x \in \mathbb{C}^n$ and a complex sequence $(r_k)_{k\in\mathbb{Z}}$, put

$$\tilde{x} := \begin{pmatrix} \vdots \\ 0 \\ r_{-1}x \\ \boxed{0} \\ r_0 x \\ 0 \\ r_1 x \\ 0 \\ \vdots \end{pmatrix}; \qquad \text{leading to} \qquad B\tilde{x} = \begin{pmatrix} \vdots \\ z_{-1} \\ r_{-1}Fx \\ \boxed{z_0} \\ r_0 Fx \\ z_1 \\ r_1 Fx \\ z_2 \\ \vdots \end{pmatrix}; \tag{39}$$

where $\boxed{0}$ and $\boxed{z_0}$ mark the respective 0 positions and

$$z_k = r_{k-1}ux_n + r_k w x_1$$

and the FSM cannot apply { no matter how the cut-o s are placed (e.g. [36, Prop 5.2]). In the semi-in nite case the operator A_+

Case 2: =

If moreover $(U;V;W)$ is compatible then

$$\|A^{-1}\| = \max_{B \in M(U;V;W)} \|B^{-1}\| =: N \tag{50}$$

If we have a particular $p \in [1;\infty]$ in mind, or want to emphasise the dependence on p, we will write M_p and N_p for the expressions M and N defined in Corollary 4.2 (cf. (35)).

The following proposition is a simple consequence of the observations that, if $A; B \in BDO(\ell^p(\mathbb{Z}))$ for all $1 \le p \le \infty$, and $A = RB^{\top}R$, where R is the reflection operator defined in Remark 1.3, then: (i) $\|A\|_p = \|B^{\top}\|_p = \|B\|_q$, for $1 \le p \le \infty$, if $p^{-1} + q^{-1} = 1$; (ii) $A \in M(U;V;W)$ iff $B \in M(U;V;W)$; (iii) A is invertible iff B is invertible, and if they are both invertible then $\|A^{-1}\|_p = \|(B^{\top})^{-1}\|_p = \|B^{-1}\|_q$.

Proposition 4.3 For $p; q \in [1;\infty]$, with $p^{-1} + q^{-1} = 1$, we have

$$M_p = M_q \quad \text{and} \quad N_p = N_q.$$

Proof. It is clear from the above observations that $M_p = \sup_{B \in M(U;V;W)} \|B$

Proof. a) If $(U; V; W)$ is compatible then all operators in $M(U; V; W)$ are invertible and $N_p = N_q$ holds by Proposition or (53) holds with p replaced by q. If (53) holds we say that p is favourable (for the triple $(U;V;W)$).

b) p and q are both favourable iff $N_{+;p} = N_{+;q}$. If p and q are both favourable, then

$$\|A_+^{-1}\|_p = N_{+;p} = N_p = N_q = N_{+;q} = \|A_+^{-1}\|_q; \quad \text{for all } A_+ \in E_+(U;V;W). \tag{54}$$

In particular this holds for $p = q = 2$.

Proof. a) Either $\|A_+^{-1}\|_p \leq N_p$ for all $A_+ \in M_+(U;V;W)$, or $\|A_+^{-1}\|_p > N_p$ for some $A_+ \in M_+(U;V;W)$. In the first case (53) follows immediately from Proposition 4.4 b). In the second case, by Proposition 4.5 b), $\|B_+^{-1}\|_q \leq N_q$ for all $B_+ \in M_+(U;V;W)$, and then (53), with p replaced by q, follows from Proposition 4.4 b).

b) is an immediate corollary of a) and Proposition 4.4 b). ∎

It is unclear to us whether every $p \in [1;\infty]$ is favourable for every triple $(U;V;W)$. Indeed, while, for every triple $(U;V;W)$, $p \in [1;\infty]$, and $A_+ \in E_+(U;V;W)$, it follows from Propositions for every $B \in M_n(U;V;W)$, $A = R_n B^{\top} R_n \in M(U;V;W)$, where $R_n = (r_{ij})_{i,j=1,\dots,n}$ is the $n \times n$ matrix with $r_{ij} = \delta_{i,n+1-j}$, where δ_{ij} is the Kronecker delta.

Lemma 4.8 For $p,q \in [1,\infty]$ with $p^{-1} + q^{-1} = 1$ and $n \in \mathbb{N}$, we have $M_{n,p} = M_{n,q}$ and $N_{n,p} = N_{n,q}$, so that $M_{\ast n,p} = M_{\ast n,q}$ and $N_{\ast n,p} = N_{\ast n,q}$, where

$$M_{n,p} := \sup_{F \in M_n(U;V;W)} \|F\|_p; \qquad N_{n,p} := \sup_{F \in M_n(U;V;W)} \|F^{-1}\|_p;$$
$$M_{\ast n,p} := \sup_{F \in M_{fin}(U;V;W)} \|F\|_p \quad \text{and} \quad N_{\ast n,p} := \sup_{F \in M_{fin}(U;V;W)} \|F^{-1}\|_p.$$

The following simple lemma relates $M_{\ast n,p}$ to M_p, defined in Corollary 4.2.

Lemma 4.9 For $p \in [1,\infty]$, $M_{\ast n,p} = \lim_{n\to\infty} M_{n,p} = M_p$.

Proof. Let $A \in E(U;V;W)$ so that $\|A\|_p = M_p$ by Corollary 4.2. For $F \in M_n(U;V;W)$, $\|F\|_p \le \|A\|_p$, since every F is an arbitrarily small perturbation of a finite section of A. On the other hand, if A_n is the finite section of A given by (8), then we have noted in (10) that $\liminf_{n\to\infty} \|A_n\|_p \ge \|A\|_p$. ∎

The following is a more quantitative version of Theorem 3.3:

Proposition 4.10 a) Properties (a)–(k) of Theorem 3.1 are equivalent to:

(l) all $F \in M_n(U;V;W)$ are invertible and their inverses are uniformly bounded.

If (a)–(l) are satisfied then

$$N_{\ast n,p} = \max(N_{+,p}\,;\, N_{+,q})\,; \tag{56}$$

for every $p,q \in [1,\infty]$ with $p^{-1} + q^{-1} = 1$.

b) In the case that p and q in a) are both favourable, (56) simplifies to

$$N_{\ast n,p} = N_{\ast n,q} = N_{+,p} = N_{+,q} = N_p = N_q. \tag{57}$$

Proof. a) If (l) holds then, by the equivalence of ii) and iii) in Lemma 1.2 and the definition of stability, (e) holds. But this implies invertibility of all $F \in M_n(U;V;W)$ by Theorem 3.3. The uniform boundedness of the inverses F^{-1} (and hence (l)) will follow if we can prove “$\le$” in (56).

To see that (56) holds, fix $p \in [1,\infty]$, $n \in \mathbb{N}$, and an $F \in M_n(U;V;W)$. To estimate $\|F^{-1}\|_p =: f$, fix $x \in \mathbb{C}^n$ with $\|Fx\|_p = 1$ and $\|x\|_p = f$. As in the proof of Theorem 3.3, define B by (37) and B_+ and B_- as in (38), and define $\tilde{x}$ by 38), and).

02.875 0 Td [(U)2(Lemma)]s362(in)rg 0.00 0.00 0.50 RG [-362(1.2)]TJ 0 g 0 G [-362(and)-362(the)82 2(de nition)-362(8 0 T6350 rg 0.00

Case 3: $p < \infty$ and $\tilde{x}_+ \notin \ell^p(\mathbb{N})$, i.e. $(r_k)_{k=0}^{+\infty} \notin \ell^p(\mathbb{N})$. Then $s_m := \sum_{k=0}^{m} |r_k|^p \to \infty$ as $m \to \infty$. Let $m \in \mathbb{N}$ and put $\tilde{x}_m := (\tilde{x}(1), \tilde{x}(2), \ldots, \tilde{x}((m+1)(n+1)), 0, 0, \ldots) \in \ell^p(\mathbb{N})$. Then

$$\|B_+^{-1}\|_p^p \geq \frac{\|\tilde{x}_m\|_p^p}{\|B_+\tilde{x}_m\|_p^p} = \frac{\sum_{k=0}^{m} |r_k|^p \|x\|_p^p}{\sum_{k=0}^{m} |r_k|^p \|Fx\|_p^p + |u r_m x_n|^p} = \frac{s_m f^p}{s_m + |u r_m x_n|^p} \xrightarrow{m\to\infty} f^p = \|F^{-1}\|_p^p$$

since $s_m \to \infty$ as $m \to \infty$ and r_m is bounded.

So in either case we get $\|F^{-1}\|_p \leq \|B_+^{-1}\|_p$ if (41) holds. The other case, (42), is analogous and leads to $\|F^{-1}\|_p \leq \|B_-^{-1}\|_p = \|C_+^{-1}\|_q$, where $C_+ := RB_-^{\top}R$ is the reflection of B_- as discussed in Remark 1.3. Since we only know that (41) or (42) applies, but not which one of them, we conclude $\|F^{-1}\|_p \leq \max(\|B_+^{-1}\|_p, \|C_+^{-1}\|_q)$. Since $F \in M_n(U, V, W)$ (U;V;W Tf 4.67 -1.4944 Td [(+)]TJ/F8/F14 9.9626 Tf 11.64

4.4 Pseudospectra

We can rephrase our results on the norms of inverses, $\|J^{-1}\|$, of Jacobi matrices J over (U,V,W) in terms of resolvent norms $\|(J-\lambda I)^{-1}\|$ and pseudospectra, noting that $J-\lambda I$ is a Jacobi matrix over $(U,V-\lambda,W)$. In particular, J and $J-\lambda I$ are both pseudoergodic at the same time. In the language of pseudospectra, Corollary 4.2 and Proposition 4.3 can be rewritten as follows:

Corollary 4.12 – bi-infinite matrices. For all $A \in \Psi E(U,V,W)$, $\varepsilon > 0$ and $p \in [1,\infty]$, it holds that

$$\mathrm{spec}_\varepsilon^p A = \Sigma_\varepsilon^p := \bigcup_{B \in M(U,V,W)} \mathrm{spec}_\varepsilon^p B \quad \text{and} \quad \Sigma_\varepsilon^p = \Sigma_\varepsilon^q,$$

where $p^{-1} + q^{-1} = 1$.

Summarizing Corollary 2.5, Theorem 2.7 and the results in Section 4.2, and recalling the notations $E_\cap$ and $E_\cup$ from (26), we obtain:

Proposition 4.13 – semi- vs. bi-infinite matrices.

a) For every $A_+ \in \Psi E_+(U,V,W)$, $\varepsilon > 0$ and all $p \in [1,\infty]$, it holds that

$$\Sigma_\varepsilon^p \cup \Sigma_+ \subset \mathrm{spec}_\varepsilon^p A_+ \subset \Sigma_{+,\varepsilon}^p := \bigcup_{C_+ \in M_+(U,V,W)} \mathrm{spec}_\varepsilon^p C_+ . \tag{59}$$

b) For all $A_+, B_+ \in M_+(U,V,W)$, $\varepsilon > 0$ and $p, q \in [1,\infty]$ with $p^{-1} + q^{-1} = 1$, it holds that

$$\mathrm{spec}_\varepsilon^p A_+ \cap \mathrm{spec}_\varepsilon^q B_+ \subset \Sigma_{+,\varepsilon}^p \cap \Sigma_{+,\varepsilon}^q = \Sigma_\varepsilon^p \cup G, \tag{60}$$

for each $G \in \{E_\infty, E_\cap, E_\cup, \Sigma_+\}$. Equality holds in (60) if $A_+, B_+ \in \Psi E_+(U,V,W)$. If $w \subset u$ and $u \subset w$, then $E_\infty = \varnothing$, so that (60) holds with $G = \varnothing$.

c) If $\varepsilon > 0$ and $p \in [1,\infty]$ is favourable, in particular if p

Now suppose that $A_+ ; B_+$ 2 $E_+(U; V; W)$ and that 2 ^{p} [$_+$. If 2 $_+$, then 2 specA_+ = spec B_+ by Corollary

$$\limsup S_n = \limsup$$

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