While the literature on Tre tz nite elements for time-harmonic wave propagation problems is nowadays quite developed (see e.g. [4, 6, 11, 14, 23, 29, 32] for di erent approaches using Tre tz-type basis functions and e.g. [2, 15, 19, 20] for theoretical analyses 10 in Remark 2.1 (in one space dimension) and again in Remark 3.4 (in any space dimension).

Given a space domain $\Omega = (x_L; x_R)$ and a time domain $I = (0; T)$, we set $Q := \Omega \times I$. We denote by $n_Q = (n_Q^x; n_Q^t)$ the outward pointing unit normal vector on ∂Q

We assume the electric permittivity $\epsilon = \epsilon(x$

3.1 Mesh and DG notation

We introduce a mesh $\mathcal{T}_h$ on Q, such that its elements are rectangles with sides parallel to the space and time axes, and all the discontinuities of the parameters ε and μ lie on interelement boundaries (note that the method described in this paper can be generalised to allow discontinuities lying inside the elements as in [25]). The mesh may have hanging nodes.

We denote with $\mathcal{F}_h = \bigcup_{K \in \mathcal{T}_h} \partial K$ the mesh skeleton and its subsets:

$$
\begin{aligned}
\mathcal{F}_h^{hor} &:= \text{the union of the internal horizontal element sides } (t = \text{constant})\,;\\
\mathcal{F}_h^{ver} &:= \text{the union of the internal vertical element sides } (x = \text{constant})\,;\\
\mathcal{F}_h^{0} &:= [x_L, x_R] \times \{0\};\\
\mathcal{F}_h^{T} &:= [x_L, x_R] \times \{T\};\\
\mathcal{F}_h^{L} &:= \{x_L\} \times [0, T];\\
\mathcal{F}_h^{R} &:= \{x_R\} \times [0, T].
\end{aligned}
$$

We define the following broken Sobolev space:

$$H^1(\mathcal{T}_h) := \quad .12:$$

where

a

$$a^{\mathrm{wave}}_{\mathrm{TDG}}(v_{hp}, \sigma_{hp}; w, \tau) = \ell^{\mathrm{wave}}_{\mathrm{TDG}}(w, \tau) \qquad \forall (w, \tau) \in \mathbf{V}_p(\mathcal{T}_h); \tag{11}$$

with

$$\begin{aligned}
a^{\mathrm{wave}}_{\mathrm{TDG}}(v_{hp}, \sigma_{hp}; w, \tau) :=& \int_{\mathcal{F}_h^{\mathrm{hor}}} \left(c^{-2} v_{hp}^- [\![w]\!]_t + \sigma_{hp}^- \cdot [\![\tau]\!]_t \right) \mathrm{d}\mathbf{x} + \int_{\mathcal{F}_h^T} (c^{-2} v_{hp} w + \sigma_{hp} \cdot \tau) \,\mathrm{d}\mathbf{x} \\
&+ \int_{\mathcal{F}_h^{\mathrm{ver}}} \left(\{\!\{v_{hp}\}\!\} [\![\tau]\!]_N + \{\!\{\sigma_{hp}\}\!\} \cdot [\![w]\!]_N + \alpha [\![v_{hp}]\!]_N \cdot [\![w]\!]_N + \beta [\![\sigma_{hp}]\!]_N [\![\tau]\!]_N \right) \mathrm{d}S \\
&+ \int_{\mathcal{F}_h^{\partial}} \left(\sigma_{hp} \cdot \mathbf{n}_\Omega^x + \alpha v_{hp} \right) w \,\mathrm{d}S; \\
\ell^{\mathrm{wave}}_{\mathrm{TDG}}(w, \tau) :=& \int_{\mathcal{F}_h^0} (c^{-2} v_0 w + \sigma_0 \cdot \tau) \,\mathrm{d}\mathbf{x} + \int_{\mathcal{F}_h^{\partial}} g (\alpha w - \tau \cdot \mathbf{n}_\Omega^x) \,\mathrm{d}S;
\end{aligned}$$

where $\sigma_0 = -\nabla U(\cdot, 0)$ and $v_0 = \frac{\partial U}{\partial t}(\cdot, 0)$ are (given) initial data. Here, the jumps are defined as follows: $[\![w]\!]_t := (w^- - w^+)$ and $[\![\tau]\!]_t := (\tau^- - \tau^+)$ on horizontal faces, $[\![w]\!]_N := w_{|K_1} \mathbf{n}^x_{K_1} + w_{|K_2} \mathbf{n}^x_{K_2}$ and $[\![\tau]\!]_N := \tau_{|K_1} \cdot \mathbf{n}^x_{K_1} + \tau_{|K_2} \cdot \mathbf{n}^x_{K_2}$ on vertical faces.

In particular, the bilinear form $a_{TDG}(\cdot;\cdot)$ is coercive in the space $T(\mathcal{T}_h)$ with respect to the DG norm, with coercivity constant equal to 1.

Proof. Using the elementwise integration by parts in time and space

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h}\iint_K \frac{\partial F}{\partial t}\,dx\,dt &= \int_{\mathcal{F}_h^{hor}} [\![F]\!]_t\,dx + \int_{\mathcal{F}_h^{T}} F\,dx \quad \int_{\mathcal{F}_h^{0}} F\,dx\\
\sum_{K\in\mathcal{T}_h}\iint_K \frac{\partial F}{\partial x}\,dx\,dt &= \int_{\mathcal{F}_h^{ver}} [\![F]\!]_x\,dt + \int_{\mathcal{F}_h^{R}} F\,dt \quad \int_{\mathcal{F}_h^{L}} F\,dt \qquad \forall F\in W^{1,1}(\mathcal{T}_h);
\end{aligned}
\tag{14}
$$

and the jump identity

$$
v\,[\![v]\!]_t \quad \frac{1}{2}[\![v^2]\!]_t = \frac{1}{2}[\![v]\!]_t^2 \qquad \text{on } \mathcal{F}_h^{hor}; \quad \forall v\in H^1(\mathcal{T}_h);
\tag{15}
$$

we obtain the identity in the assertion:

$$
\begin{aligned}
a_{DG}(v_E;v_H;v_E;v_H) \overset{(7)}{=} &\sum_{K\in\mathcal{T}_h}\iint_K \left(\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon v_E^2 + \mu v_H^2\right) + \frac{\partial}{\partial x}(v_E v_H)\right) dx\,dt\\
&+ \int_{\mathcal{F}_h^{hor}} \left(\epsilon v_E\, [\![v_E]\!]_t + \mu v
\end{aligned}
$$

Proof. To prove uniqueness, assume that $E_L = E_R = E_0 = H_0 = 0$. Proposition 4.2 implies $E_{hp} = H_{hp} = 0$. Existence follows from uniqueness. For (16), the triangle inequality gives

$$|||(E,H) - (E_{hp},H_{hp})|||_{DG} \le |||(E,H) - (v_E,v_H)|||_{DG} + |||(E_{hp},H_{hp}) - (v_E,v_H)|||_{DG} \quad (17)$$

for all $(v_E,v_H) \in V_p(T_h)$. Since $(E_{hp},H_{hp}) - (v_E,v_H) \in V_p(T_h) \subset T(T_h)$, Proposition 4.2, consistency (which follows by construction and from the consistency of the numerical fluxes), and Proposition 4.3 give

$$|||(E_{hp},H_{hp}) - (v_E,v_H)|||^2_{DG} = a_{TDG}(E - v_E, H - v_H; E_{hp} - v_E, H_{hp} - v_H$$
$$\le 2|||(E,H) - (v_E,v_H)|||_{DG}$$

for $\varphi, \psi \in L^2(Q)$. More precisely, we will need a bound on the L^2 norm of the traces of v_E and v_H on horizontal and vertical segments in terms of the $L^2(Q)$ norm of $(\varphi; \psi)$:

$$\begin{aligned}
&\left\|\varepsilon^{1/2} v_E\right\|^2_{L^2(F_h^{hor}\cup F_h^{T})} + \left\|\mu^{1/2} v_H\right\|^2_{L^2(F_h^{hor}\cup F_h^{T})} \\
&+ \left\|\alpha^{1/2} v_E\right\|^2_{L^2(F_h^{ver})} + \left\|\beta^{1/2} v_H\right\|^2_{L^2(F_h^{ver}\cup F_h^{L}\cup F_h^{R})} \\
&\leq C_{stab}^2 \left(\left\|\varepsilon^{1/2}\varphi\right\|^2_{L^2(Q)} + \left\|\mu^{1/2}\psi\right\|^2_{L^2(Q)} \right) \qquad \forall (\varphi; \psi) \in L^2(Q)^2; \qquad (19)
\end{aligned}$$

for some $C_{stab} > 0$. We have inserted the numerical flux parameters within the third and fourth term on the left-hand side of (19) because this is what we need in the proof of Proposition 4.7 below; then the constant C_{stab} will also depend on α and β.

Proposition 4.7. *Assume that the estimate (19) holds true for $(v_E; v_H)$ solution of problem (18). Then, for any Trefftz function $(w_E; w_H) \in T(\mathcal{T}_h)$, the $L^2(Q)$ norm is bounded by the DG norm:*

$$\left(\left\|\mu^{1/2} w_E\right\|^2_{L^2(Q)} + \left\|\varepsilon^{1/2} w_H\right\|^2_{L^2(Q)} \right)^{1/2} \leq \sqrt{2}\, C_{stab} \,|||(w_E; w_H)|||_{DG};$$

with C_{stab} as in (19).

Proof. Let $(v_E; v_H)$ be the solution of the auxiliary problem (18 we obtain the desired estimate. $\square$

Recalling that the error $((E - E_{hp}); (H - H_{hp})) \in T(T_h)$, and combining Proposition 4.7 and the quasi-optimality in DG norm proved in Theoremosition which in turns implies

$$E(t) = \underbrace{E(0)}_{=0} + \iint_{(0;t)} v_E + v_H \quad d$$

$$\overset{(18)}{=} \frac{2A}{h^x} \int\int_K \quad {}^{1}v_E^2 + \text{"} \; {}^{1}v_H^2 + 2(x \quad x_K) \quad \frac{@}{@t}(v_E v_H) + \quad {}^{1}v_E \quad + \text{"} \; {}^{1}v_H \qquad dx\,dt$$

$$\frac{2A}{h^x} \int\int_K \quad 2 \quad {}^{1}v_E^2 + 2\text{"} \; {}^{1}v_H^2 + (x \quad x_K)^2 \quad {}^{1} \; {}^{2} + \text{"} \; {}^{1} \; {}^{2} \qquad dx\,dt$$

$$+ \frac{2A}{h^x} \int_{@K^{SN}} 2j(x \quad x_K)v_E v_H j \; dx$$

Using $2v_E v_H \quad (\; v_E^2 + {}^{1}v_H^2)$ with weight $\quad = \text{"}c = (\;\; c)^{\;1}$, we have the bound

$$\sum_{K \, 2T_h} \quad {}^{1=2}v_E \; {}^{2}_{L^2(@K^{WE})} + \quad {}^{1=2}v_H \; {}^{2}_{L^2(@K^{WE})}$$

$$= \frac{4A}{h^x} \quad {}^{1=2}v_E \; {}^{2}_{L^2(Q)} + \frac{4A}{h^x} \; \text{"} \; {}^{1=2}v_H \; {}^{2}_{L^2(Q)} + \frac{Ah^x}{2} \quad {}^{1=2} \; {}^{2}_{L^2(Q)} + \frac{Ah^x}{2} \; \text{"} \; {}^{1=2} \; {}^{2}_{L^2(Q)}$$

$$+ 2A \int_{F_h^{hor} [F_h^T} c(\text{"}v_E^2 + \;\; v_H^2)\,dx$$

$$\overset{(27);(26)}{\quad} A \quad \frac{4T^2}{h^x}c_1^4 + \frac{h^x}{2}c_1^2 + 2c_1^3 \; N_{hor}\, e^T \qquad \text{"}^{1=2} \; {}^{2}_{L^2(Q)} + \quad {}^{1=2} \; {}^{2}_{L^2(Q)} \; ;$$

recalling that $c_1 = kck_{L^1(Q)}$.

This, together with (26), gives the bound (19) with constant C^2_{stab} as in (21). □

In case of a tensor product mesh with all elements having horizontal edges of length h^x and vertical edges of length $h^t = h^x = c$, the constant C_{stab} is proportional to $(h^x)^{1=2}$. We stress that we cannot expect a bound like (19) with C_{stab} independent of the meshwidth: indeed if the mesh is refined, say, uniformly, while the term in the brackets in the right-hand side of (19) is not modified, the left-hand side grows (consider e.g. the simple case $=$, $= 0$, $v_E = 0$, $v_H = t$).

One could attempt to derive the stability bound (19) by controlling with (;) either the $H^1(Q)$ or the Lx(singgth

$\frac{1}{2}Z$

for all $(E; H); (v_E; v_H)$ 2 $T(T$

and denote their length by

$$h_D := x_1 - x_0 + c(t_1 - t_0). \tag{32}$$

Their relevance is the following: the restriction to D of the solution of a Maxwell initial value problem posed in $\mathbb{R} \times \mathbb{R}^+$ will depend only on the initial conditions posed on $_D^- \cup {}_D^+$; see Figure 1.

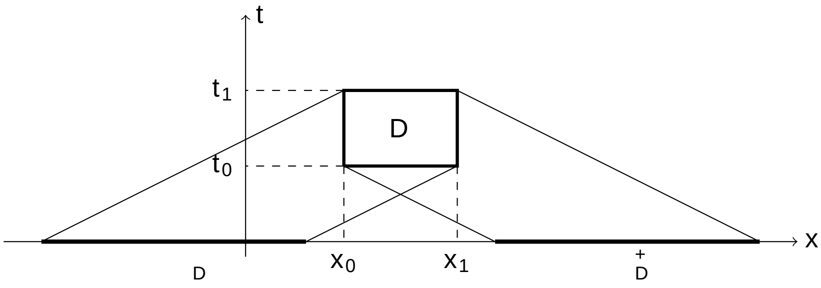

Figure 1: The intervals $_D^{\pm}$ in (31) corresponding to the space–time rectangle D.

Let $\alpha = (\alpha_x, \alpha_t) \in \mathbb{N}_0^2$ be a multi-index; for a sufficiently smooth function v, we define its anisotropic derivative $D_c^\alpha v$ as

$$D_c^\alpha v(x,t) := \frac{1}{c^{\alpha_t}} D^\alpha v(x,t) = \frac{1}{c^{\alpha_t}} \frac{\partial^{|\alpha|} v(x,t)}{\partial x^{\alpha_x} \partial t^{\alpha_t}}.$$

Note that, if u and w satisfy

$$u(x,t) = u_0(x - ct), \qquad w(x,t) = w_0(x + ct), \tag{33}$$

with u_0 and w_0 defined in $_D^-$ and $_D^+$, respectively, then

$$D_c^\alpha u(x,t) = (-1)^{\alpha_t} u_0^{(|\alpha|)}(x - ct),$$
$$D_c^\alpha w(x,t) = w_0^{(|\alpha|)}(x + ct).$$

We define the Sobolev spaces $W_c^{j,\infty}(D)$ and $H_c^j(D)$ as the spaces of functions whose D_c^α derivatives, $0 \le |\alpha| \le j$, belong to $L^\infty(D)$ and $L^2(D)$, respectively. We define the following seminorms:

$$|v|_{W_c^{j,\infty}(D)} := \sup_{|\alpha|=j} \|D_c^\alpha v\|_{L^\infty(D)}, \qquad |v|^2_{H_c^j(D)} := \sum_{|\alpha|=j} \|D_c^\alpha v\|^2_{L^2(D)}.$$

Note that for j $W^{0,\infty}(D)$ and L $D)$ an

A similar result holds for $w(x;t) = w_0(x+ct)$, with ^+_D instead of $_D$.

Proof. For the $W_c^{j;1}(D)$-seminorms in (i), we have

$$|u|_{W_c^{j;1}(D)} = \sup_{|\ |=j} \|D_c u\|_{L^1(D)} = \sup_{|\ |=j} \left\| u_0^{(j\ j)}(x\ \ ct) \right\|_{L^1(D)} = |u_0|_{W^{j;1}(\ _D)}:$$

For the bound of $|u_0|^2_{H^j(\ _D)}$ in (ii), we have

$$|u_0|^2_{H^j(\ _D)} = \int_D \left| u_0^{(j)}(z) \right|^2 dz \quad _D \sup_{z2\ _D} \left| u_0^{(j)}(z) \right|^2 = \quad _D \sup_{|\ |=j} \sup_{(x;t)2D} |D_c u(x;t)|^2$$

$$= h_D |u|^2_{W_c^{j;1}(D)}:$$

Considd [(2)1.169416 0 Td [(1)996864 Tf 13.6801 0 Td [())2.56329].15739]TJ /R14880-1.4,75T618 (

This suggests a construction of discrete subspaces of $T(D)$: given $p \in \mathbb{N}_0$ and two sets of $p+1$ linearly independent functions $\Phi^- = \{\varphi_0^-, \ldots, \varphi_p^-\} \subset C^m(\overline{\Gamma_D^-})$ and $\Phi^+ = \{\varphi_0^+, \ldots, \varphi_p^+\} \subset C^m(\overline{\Gamma_D^+})$ we define the space

$$V_p(D) := \operatorname{span}\left\{ \left(\frac{\varphi_0^-(x-ct)}{2\epsilon^{1/2}}, \frac{\varphi_0^-(x-ct)}{2\mu^{1/2}}\right), \ldots, \left(\frac{\varphi_p^-(x-ct)}{2\epsilon^{1/2}}, \frac{\varphi_p^-(x-ct)}{2\mu^{1/2}}\right), \right.$$
$$\left. \left(\frac{\varphi_0^+(x+ct)}{2\epsilon^{1/2}}, -\frac{\varphi_0^+(x+ct)}{2\mu^{1/2}}\right), \ldots, \left(\frac{\varphi_p^+(x+ct)}{2\epsilon^{1/2}}, -\frac{\varphi_p^+(x+ct)}{2\mu^{1/2}}\right) \right\},$$

which is a subspace of $T(D) \cap C^m(\overline{D})^2$ with dimension $2(p+1)$.

By virtue of Proposition 5.1, the approximation properties of $V_p(D)$ in $T(D)$ only depend on the approximation properties of the one-dimensional functions $\{\varphi_0^\pm, \ldots, \varphi_p^\pm\}$: for all $(E, H) \in T(D) \cap W_c^{j,\infty}(D)^2$, defining u, w, u_0 and w_0 from (E, H) using (35) and (33),

$$\inf_{(E_{hp}, H_{hp}) \in V_p(D)} \left\| \epsilon^{1/2}(E - E_{hp}) \right\|_{W_c^{j,\infty}(D)} + \left\| \mu^{1/2}(H - H_{hp}) \right\|_{W_c^{j,\infty}(D)} \tag{36}$$
$$\overset{(35)}{=} \inf_{\substack{u_{0,p} \in \operatorname{span}\{\varphi_0^-, \ldots, \varphi_p^-\}, \\ w_{0,p} \in \operatorname{span}\{\varphi_0^+, \ldots, \varphi_p^+\}}} \frac{1}{2} \left| u(x,t) - u_{0,p}(x-ct) + w(x,t) - w_{0,p}(x+ct) \right|_{W_c^{j,\infty}(D)}$$
$$+ \frac{1}{2} \left| u(x,t) - u_{0,p}(x-ct) - w(x,t) + w_{0,p}(x+ct) \right|_{W_c^{j,\infty}(D)}$$
$$\overset{\text{Prop. 5.1 (i)}}{\le} \inf_{u_{0,p} \in \operatorname{span}\{\varphi_0^-, \ldots, \varphi_p^-\}} \left| u_0 - u_{0,p} \right|_{W^{j,\infty}(\Gamma_D^-)} + \inf_{w_{0,p} \in \operatorname{span}\{\varphi_0^+, \ldots, \varphi_p^+\}} \left| w_0 - w_{0,p} \right|_{W^{j,\infty}(\Gamma_D^+)};$$

while for all $(E, H) \in T(D) \cap H_c^j(D)^2$

$$\inf_{(E_{hp}, H_{hp}) \in V_p(D)} \left\| \epsilon^{1/2}(E - E_{hp}) \right\|_{D)}^{D^2} (H - E_{hp}) \; w_{D)} \tag{36}$$

Following the second route, we prove simple approximation bounds in $H_c^1(Q)$, for a general rectangle

6 Convergence rates

We now derive the convergence rates of the Trefftz-DG method with polynomial approximating spaces

$$V_p(\mathcal{T}_h) = \left\{ (v_E; v_H) \in L^2(Q)^2 : (v_E; v_H)_{|K} \text{ are as in (38) with } p = p_K \right\}. \tag{43}$$

The two main ingredients are the quasi-optimality results investigated in section 4 and the best approximation bounds proved in section 5. To combine them, we need to control the DG$^+$ norm (12) of the approximation error with its $H^1_c(Q)$ norm, weighted with " and , to be able to use the bound (40). To this purpose, we define the following parameters:

$_K := \max$ "

If the bound (19) holds true for the solution of the auxiliary problem (18), we also have the following bound in $L^2(Q)$:

$$\left(\left\| \epsilon^{1/2}(E - E_{hp}) \right\|^2_{L^2(Q)} + \left\| \mu^{1/2}(H - H_{hp}) \right\|^2_{L^2(Q)} \right)^{1/2}$$
$$\leq \frac{12\sqrt{2}}{\sqrt{c}} C_{stab} \sum_{K \in \mathcal{T}_h} \left(6\left(c + \frac{h_K^x}{h_K^t}\right) + 8\sigma_K\left(1 + c\frac{h_K^t}{h_K^x}\right) \right)^{1/2} (e/2)^{\frac{s_K^2}{p_K}} \frac{\left(h_K^x + c h_K^t\right)^{s_K + \frac{3}{2}}}{p_K^{s_K}}$$
$$\left(\left\| \epsilon^{1/2} E \right\|_{W_c^{s_K+1,1}(K)} + \left\| \mu^{1/2} H \right\|_{W_c^{s_K+1,1}(K)} \right). \tag{47}$$

with C_{stab} from (19).

Proof. Given an element $K \in \mathcal{T}_h$, we denote by ∂K^N, ∂K^S, ∂K^W, and ∂K^E its North, South, West and East sides, respectively, North pointing in the positive time direction, and set $\partial K^{WE} := \partial K^W \cup \partial K^E$.

For all $(v_E; v_H) \in H^1(\mathcal{T}_h)^2$, expanding the DG+ norm(a)-5@H98 -3.6 Td [(:=)-[(+)-0.569-6.

S(5 Tf 5.4 -192(e).298(a)-5.8 0 3/9(h)v

where the last inequality follows noting that $f(x) = (1 - x)^{\frac{1}{x} - 1}(1 + x)^{-\frac{1}{x} - 1} 4^x e^{2 - 2x} \leq 1$ for all $0 < x < 1$ (which in turn can be verified by checking the convexity of $\log f$ and its limit values for $x \to 0$ and 1).

The estimate in

(What we actually need is only that u_0 and w_0 are analytic in a sufficiently large complex neighbourhood of the finite segments ${}_Q$ and ${}^+_Q$, respectively.)

For every mesh element K as above, we fix $h_K :=$ length(${}_K$). The complex ellipses with foci at the extrema of ${}_K$ and sum of the semiaxes equal to $\sigma + \sqrt{}$

Figure 3 shows convergence of the h-version for degree zero through three. Solid lines correspond to results obtained with the Trefftz basis whereas the dashed lines were obtained using the non-Trefftz basis. Uniform mesh step sizes are applied by reducing h_x and h_t simultaneously. The Trefftz method exhibits optimal algebraic convergence rates h^{p+1}. However, in the non-Trefftz case, the results seem to suggest an odd-even pattern of the convergence rates, with convergence being suboptimal for odd degrees (by one order). Numerical odd-even effects in the convergence rates of DG methods have also been reported, e.g. in [18, section 6.5], although it has been shown in [17] that in some situations this might

$$:= \max\left\{ (\varepsilon\mu)^{-1} + (\ldots)^{-1}\,\|\ldots\|_{L^1(\mathcal{F}_h^{ver})};\ (\ldots)^{-1}\,\|\ldots\|_{L^1(\mathcal{F}_h^{L}\cup\mathcal{F}_h^{R})} \right\}:$$

Proof. We assume that φ and ψ are continuous in Q; the general case will follow by a density argument.

First, we extend the initial problem to the entire space $\mathbb{R}$. Define $\tilde v_E; \tilde v_H; \tilde\varphi; \tilde\psi$ in $\mathbb{R}\times\mathbb{R}^+$ as the $2(x_R - x_L)$-periodic functions in x that satisfy $\tilde v_E|_Q = v_E; \tilde v_H|_Q = v_H; \tilde\varphi|_Q = \varphi; \tilde\psi|_Q = \psi$ and such that $\tilde v_E$ and $\tilde\psi$ are odd around x_L (and consequently also around x_R), and $\tilde v_H$ and $\tilde\varphi$ are even around the same points, i.e.

$$\tilde v_E(x_L + x; t) = -\tilde v_E(x_L - x; t); \qquad \tilde v_H(x_L + x; t) = \tilde v_H(x_L - x; t);$$
$$\tilde\psi(x_L + x; t) = -\tilde\psi(x_L - x; t); \qquad \tilde\varphi(x_L + x; t) = \tilde\varphi(x_L - x; t); \qquad \forall (x; t) \in \mathbb{R}\times\mathbb{R}^+:$$

(Note that the absolute values are $(x_R - x_L)$-periodic in x.) Since time derivatives preserve parities and space derivatives swap them, the extended functions $\tilde v_E$ and $\tilde v_H$ are continuous and satisfy the extended initial problem

$$\begin{aligned}
&\frac{\partial \tilde v_E}{\partial x} + \frac{\partial(\mu \tilde v_H)}{\partial t} = \tilde\varphi && \text{in } \mathbb{R}\times\mathbb{R}^+;\\
&\frac{\partial \tilde v_H}{\partial x} + \frac{\partial(\varepsilon \tilde v_E)}{\partial t} = \tilde\psi && \text{in } \mathbb{R}\times\mathbb{R}^+;\\
&\tilde v_E(\cdot; 0) = 0; \quad \tilde v_H(\cdot; 0) = 0 && \text{on } \mathbb{R}:
\end{aligned}$$

Second, we split the right- and the left-propagating components. Define

$$u := \varepsilon^{1/2}\tilde v_E + \mu^{1/2}\tilde v_H; \quad w := \varepsilon^{1/2}\tilde v_E - \mu^{1/2}\tilde v_H; \qquad \text{so that} \qquad \tilde v_E = \frac{u+w}{2\varepsilon^{1/2}}; \quad \tilde v_H = \frac{u-w}{2\mu^{1/2}}:$$

They satisfy the inhomogeneous transport equations in $\mathbb{R}\times\mathbb{R}^+$

$$\frac{\partial u}{\partial x} + \frac{\partial(c^{-1}u)}{\partial t} = \varepsilon^{1/2}\tilde\varphi + \mu^{1/2}\tilde\psi =: f; \qquad \frac{\partial w}{\partial x} - \frac{\partial(c^{-1}w)}{\partial t} = \varepsilon^{1/2}\tilde\varphi - \mu^{1/2}\tilde\psi =: g;$$

recalling that $(\varepsilon\mu)^{1/2} = c^{-1}$, so they can be written explicitly with the following representation formula (e.g. [9, section 2.1.2, equation (5)], recall that from the assumptions made in the proof, f and g are piecewise continuous)

$$u(x; t) = \int_0^t c f\big(x + c(s - t); s\big)\,ds; \qquad w(x; t) = \int_0^t c g\big(x - c(s - t); s\big)\,ds:$$

We first bound the L^2 norm of u and w on horizontal and vertical segments with the data $f; g$; from the triangle inequality $(\tilde v_E; \tilde v_H)$ will be bounded by $\tilde\varphi$ and $\tilde\psi$, and the bound for v_E and v_H will follow. For all $0 \le t \le T$

$$\begin{aligned}
\|u(\cdot; t)\|^2_{L^2(\Omega)} &= \int_{x_L}^{x_R}\left(\int_0^t c f\big(x + c(s - t); s\big)\,ds\right)^2 dx\\
&\le tc^2 \int_{x_L}^{x_R}\int_0^t \big(f(x + c(s - t); s)\big)^2\,ds\,dx\\
&= tc^2 \int_0^t \int_{x_L + c(s-t)}^{x_R + c(s-t)} \big(f(y; s)\big)^2\,dy\,ds\\
&\le 2tc^2 \int_0^t \int_{x_L + c(s-t)}^{x_R + c(s-t)} \left(\varepsilon|\tilde\varphi(y; s)|^2 + \mu|\tilde\psi(y; s)|^2\right) dy\,ds\\
&= 2tc^2\left(\left\|\varepsilon^{1/2}\varphi\right\|^2_{L^2(\Omega\times(0;t))} + \left\|\mu^{1/2}\psi\right\|^2_{L^2(\Omega\times(0;t))}\right)
\end{aligned}$$

(the last equality follows from the symmetries of $\tilde\varphi$ and $\tilde\psi$ which ensure the equality of their L^2 norms on the rectangle $(x_L; x_R)\times(0; t)$ and on the parallelogram with vertices $(x_L - ct; 0); (x_R - ct; 0); (x_R; t); (x_L; t)$). Similarly, for all $x \in \Omega$

$$\|u(x; \cdot)\|^2_{L^2(0;T)} = \int_0^T\left(\int_0^t c f\big(x + c(s - t); s\big)\,ds\right)^2 dt$$

$$c^2 \int_0^T \int_0^t t \quad f \quad x + c(s \quad t); s \quad ^2 \, ds \, dt$$

$$= c^2 \int_0^T \int_0^T \quad t \; f \quad x + c(s \quad t); s \quad ^2 \, dt \, ds$$

$$= c^2 \int_0^T \int_{x \quad c(T \quad s)}^{s \; x} \quad x \quad y$$

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