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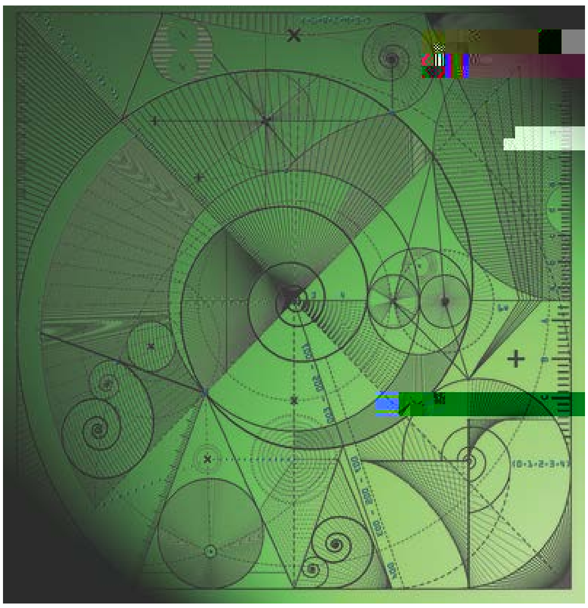

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GENERALISED SOLUTIONS FOR FULLY NONLINEAR PDE SYSTEMS AND EXISTENCE-UNIQUENESS THEOREMS

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Abstract.

existence can be proved given certain boundary conditions. Subsequent considerations typically include uniqueness, qualitative properties, regularity and numerics. This approach has been enormously successful but unfortunately only equations and systems with fairly special structure have been considered so far. A standing idea in this regard consists of using duality and integration-by-parts in order to interpret rigorously derivatives by \passing them to test functions". This method which dates back to the 1930s ([S1, S2, So]) is basically restricted to divergence structure equations and systems. A more recent approach discovered in the 1980s is that of viscosity solutions ([CL]) which builds on the maximum principle as a device to \pass derivatives to test functions". Although it applies mostly to single equations supporting the maximum principle, it has been hugely successful because it includes the fully nonlinear case.

In this paper we introduce a new theory of generalised solutions which applies to nonlinear PDE systems of any order. Our approach allows formerely measurable in approximating sequences due to the combination of phenomena of oscillations and/or concentrations ([

Our motivation to introduce and study generalised solutions for nonlinear PDE systems primarily comes from the need to study the recently discovered ∞-Laplace system rigorously, which is the fundamental equation of Vectorial Calculus of Variations in the space L^∞. Calculus of Variations in L^∞ has a long history which started in the 1960s by Aronsson ([A1]-[A5]) who was the first to consider variational problems for supremal functionals of the form

$$E_\infty(u,\Omega) := \big\| H(\cdot, u, Du)\big\|_{L^\infty(\Omega)}. \tag{1.6}$$

Aronsson introduced the appropriate notion of minimisers for such functionals and studied classical solutions of the respective equation which is the L^∞-analogue of the Euler-Lagrange equation. In the simplest case of $H(p) = |p|$ (the Euclidean norm on $\mathbb{R}^n$), the L^∞-equation is called the ∞-Laplacian and reads

$$\Delta_\infty u := Du \otimes Du : D^2u = 0. \tag{1.7}$$

Since then, the field has undergone huge development due to both the intrinsic mathematical interest and the important for applications: minimisation of the maximum provides more realistic models when compared to the classical case of integral functionals where the average is minimised instead. A basic difficulty in the study of (1.6) is that (1.7) possesses singular solutions. Aronsson himself exhibited this in [A6, A7] and the field had to wait until the development of viscosity solutions for 2nd order equations in the early 1990s in order to study general solutions (see [C, BEJ, E, E2] and for a pedagogical introduction see [K8]).

Until recently, the study of supremal functionals was restricted exclusively to the scalar case of $N = 1$ and to first order problems. The principal reason for this was the absence of an efficient theory of generalised solutions which would allow the rigorous study of non-divergence PDE systems or higher order equations, including those arising in L^∞. The foundations of the vector case of (1.6), including the discovery of the appropriate system version of (1.7), the correct vectorial minimality notion and the study of classical solutions have been laid in a series of recent papers of the author ([K1]-[K6]). In the simplest case of

$$E_\infty(u,\Omega) = \|Du\|_{L^\infty(\Omega)} \tag{1.8}$$

applied to Lipschitz maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ (where the L^∞ norm is interpreted as the essential supremum of the Euclidean norm $|Du|$ on $\mathbb{R}^{N\times n}$), the analogue of the Euler-Lagrange equation is the ∞-Laplace system:

(1.9) $\Delta_\infty u := \big(Du \otimes Du + |Du|^2[Du]$ndJ/F8(m:).9626 Tf 4.? 0 Td [(D)-28(u)]TJJ/F7 6.(m:).9626 Tf 1

Basics. Let $n, N \in \mathbb{N}$ be fixed, which in this paper will always be the dimensions of domain and range respectively of our candidate solutions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$. By Ω we will always mean an open subset of $\mathbb{R}^n$, even if it is not explicitly mentioned. Unless indicated otherwise, Greek indices $\alpha, \beta, \gamma, \ldots$ will run in $\{1, \ldots, N\}$ and latin indices $i, j, k, \ldots$ (perhaps indexed $i_1, i_2, \ldots$) will run in $\{1, \ldots, n\}$, even when the range is not given explicitly. The norms $|\cdot|$ appearing throughout will always be the Euclidean, while the Euclidean inner products will be denoted by either "$\cdot$" on $\mathbb{R}^n, \mathbb{R}^N$ or by ":" on tensor spaces, e.g. on $\mathbb{R}^{N\times n}$ and $\mathbb{R}^{N\times n^2}_s$ we have

$$|X|^2 = \sum_{\alpha,i} X_{\alpha i} X_{\alpha i} \equiv X : X, \qquad |\mathbf{X}|^2 = \sum_{\alpha,i,j} \mathbf{X}_{\alpha ij} \mathbf{X}_{\alpha ij} \equiv \mathbf{X} : \mathbf{X},$$

etc. The standard bases on $\mathbb{R}^n$, $\mathbb{R}^N$, $\mathbb{R}^{N\times n}$ will be denoted by $\{e^i\}$, $\{e_\alpha\}$ and $\{e_\alpha \otimes e^i\}$. By introducing the symmetrised tensor product

$$a \vee b := \frac{1}{2}\big(a \otimes b + b \otimes a\big), \quad a, b \in \mathbb{R}^n, \tag{2.1}$$

we will write $e_\alpha \otimes (e^{i_1} \vee \cdots \vee e^{i_p})$ for the standard basis of the $\mathbb{R}^{N\times n^p}_s$. We will follow the convention of denoting vector subspaces of Euclidean spaces as well as the orthogonal projections on them by the same symbol. For example, if $\Sigma \subseteq \mathbb{R}^N$ is a subspace, we denote the projection map $\mathrm{Proj}_\Sigma : \mathbb{R}^N \longrightarrow$

$\{E^{(\alpha)1},...,E^{(\alpha)n}\}$ of $\mathbb{R}^n$. Given such bases, we will always equip the spaces $\mathbb{R}^{N\otimes n}$ and $\mathbb{R}_s^{N\otimes n^p}$ with the following induced orthonormal bases:

$$(2.2)\qquad \begin{aligned} \mathbb{R}^{N\otimes n} &= \mathrm{span}\big[\{E^{\alpha i}\}\big]; \qquad E^{\alpha i} := E^{\alpha}\otimes E^{(\alpha)i};\\ \mathbb{R}_s^{N\otimes n^p} &= \mathrm{span}\big[\{E^{\alpha i_1\cdots i_p}\}\big]; \qquad E^{\alpha i_1\cdots i_p} := E^{\alpha}\otimes \big(E^{(\alpha)i_1}\vee\cdots\vee E^{(\alpha)i_p}\big). \end{aligned}$$

Given such frames, let $D_{E^{(\alpha)i}}$ and $D^p_{E^{(\alpha)i_p}\cdots E^{(\alpha)i_1}} = D_{E^{(\alpha)i_p}}\cdots D_{E^{(\alpha)i_1}}$ denote the usual directional derivatives of 1st and pth order along the respective directions. Then, the gradient Du of a map $u : \Omega\subseteq\mathbb{R}^n \to \mathbb{R}^N$ can be written as

$$(2.3)\qquad Du = \sum_{\alpha,i}\big(E^{\alpha i} : Du\big)E^{\alpha i} = \sum_{\alpha,i} D_{E^{(\alpha)i}}(E^{\alpha}\cdot u)\,E^{\alpha i}$$

and the pth order derivative $D^p u$ as

$$(2.4)\qquad \begin{aligned} D^p u &= \sum_{\alpha,i_1,...,i_p}\big(E^{\alpha i_1\cdots i_p} : D^p u\big)E^{\alpha i_1\cdots i_p}\\ &= \sum_{\alpha,i_1,...,i_p} D^p_{E^{(\alpha)i_1}\cdots E^{(\alpha)i_p}}(E^{\alpha}\cdot u)\,E^{\alpha i_1\cdots i_p}. \end{aligned}$$

We will also use the following notation for the pth order Jet of u:

$$D^{[p]}u := \big(Du, D^2u, ..., D^p u\big).$$

Given $a\in\mathbb{R}^n$ with $|a| = 1$ and $h\in\mathbb{R}\setminus\{0\}$, when $x, x+ah\in\Omega$ the 1st difference quotient of u along the direction a at x will be denoted by

$$(2.5)\qquad D_a^{1,h}u(x) := \frac{u(x+ha)-u(x)}{h}.$$

By iteration, if $h_1,...,h_p\neq 0$ the pth order difference quotient along $a_1,...,a_p$ is

$$(2.6)\qquad D^{p,h_p\cdots h_1}_{a_p\cdots a_1}u := D^{1,h_p}_{a_p}\big(\cdots\big(D^{1,h_1}_{a_1}u\big)\cdots\big).$$

Young Measures. Let $E\subseteq\mathbb{R}^n$ be a measurable set and $K\Subset\mathbb{R}^d$ a compact subset of some Euclidean space, which we will later take to be $\overline{\mathbb{R}}_s^{N\otimes n^p}$. Consider the L^1 space of strongly measurable maps valued in the (separable Banach) space $C^0(K)$ of real continuous functions over K, in the standard Bochner sense:

$$L^1\big(E; C^0(K)\big).$$

For details about these spaces we refer e.g. to [FL, F, V] (and references therein). The elements of $L^1\big(E; C^0(K)\big)$ can be identified with the Carathéodory functions

$$\Phi : E\times K\to\mathbb{R}, \quad (x,X)\mapsto\Phi(x,X)$$

for which

$$\|\Phi\|_{L^1(E;C^0(K))} := \int_E \max_{X\in K}\big|\Phi(x,X)\big|\,dx < \infty$$

and the identification is given by considering Φ as a map $E\ni x\mapsto\Phi(x,\cdot)\in C^0(K)$. The notion of Carathéodory functions is meant in the usual sense, that is for every $X\in K$ the function $x\mapsto\Phi(x,X)$ is measurable and for a.e. $x\in E$ the function $X\mapsto\Phi(x,X)$ is continuous. The space $L^1\big(E; C^0(K)\big)$ is separable and the simple functions of this space (which are norm-dense) have the form

$$E\ni x\mapsto\sum_{i=1}^{q}\chi_{E_i}(x)\,\Phi_i\in C^0(K);$$

where $E_1, \ldots, E_q$ are measurable disjoint subsets of E and $_i \in C^0(K$

Lemma 4. Suppose $E \subseteq \mathbb{R}^n$ is measurable and $v^m, v^\infty : E \to \mathbb{K}$ are measurable maps, $m \in \mathbb{N}$. Then, there exist subsequences $(m_k)_1^\infty$, $(m_l)_1^\infty$:

$$(1) \quad v^m \to v^\infty \text{ a.e. on } E \implies \delta_{v^{m_k}} \overset{*}{\rightharpoonup} \delta_{v^\infty} \text{ in } \mathscr{Y}(E;\mathbb{K});$$

$$(2) \quad \delta_{v^m} \overset{*}{\rightharpoonup} \delta_{v^\infty} \text{ in } \mathscr{Y}(E;\mathbb{K}) \implies v^{m_l} \to v^\infty \text{ a.e. on } E.$$

for any compactly supported "test" function $\Phi \in C^0_c\big(\Omega \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s\big)$. This gives the idea that we can embed the difference quotient maps

$$\big(D^{1,h_m}u, D^{2,h_m' h_m''}u\big) : \Omega \longrightarrow \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s$$

into the spaces of Young measures and consider instead

$$\delta_{D^{1,h_m}u} : \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn}\big), \qquad \delta_{D^{2,h_m' h_m''}u} : \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)$$

over the Alexandroff compactifications. The reason we need to attach the point at ∞

Remark 8. As a consequence of the separate convergence, the pth order Jet is always a (bre) product Young measure:

$$D^{[p]}u = Du \quad D \quad {}^{p}u:$$

The weak* compactness of the spaces of Young measures readily implies the existence of plenty of di use derivatives for measurable mappings.

Lemma 9 (Existence of di use derivatives). Every measurable mapping u : R^n ! R^N possesses di use derivatives of all orders, actually at least one for every choice of in nitesimal sequence.

Remark 10 (Nonexistence of distributional derivatives). Since we do not require our maps to be in $L^1_{loc}(\; ; R^N)$, they may not possess distributional derivatives.

In general di use derivatives may not be uniquefor nonsmooth maps. However, they are compatible with weak derivatives and a fortiori with classical derivatives:

Lemma 11 (Compatibility of weak and di use derivatives) . If u 2 $W^{1;1}_{loc}(\; ; R^N)$, then the di use gradient Du is unique and

$$_{Du} = Du; \quad \text{a.e. on} \quad :$$

More generally, if q 2 f 1; :::; p 1g and u 2 $W^{q;1}_{loc}(\; ; R^N)$, then $D^{[q]}u$ is unique and

$$D^{[p]}u = {}_{(Du;:::;D \; ^{q}u)} \; D \; {}^{q+1} \; ::: \; D \; {}^{p}u; \quad \text{a.e. on} \quad :$$

Proof of Lemma 11. It su ce to establish only the 1st order case. For any xed e 2 R^n we have $D^{1;h}_e u$! $D_e u$ in $L^1_{loc}(\; ; R^N)$ as h ! 0. We choose e := $E^{(\;)i}$ and h := h_m to get

$$D^{1;h_m}_{E(\;)i}(E \quad u) \; ! \quad D_{E(\;)i}(E \quad u); \quad \text{in } L^1_{loc}(\,) \text{ as } m \,!\, 1 \quad :m$$

(b) If there exists a measurable map $U : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}$ such that for any diffuse gradient $\mathcal{D}u \in \mathscr{Y}(\Omega;\overline{\mathbb{R}}^{Nn})$ we have

$$\mathcal{D}u = \delta_U, \quad \text{a.e. on } \Omega,$$

then it follows that u is differentiable in measure and $U = \mathcal{L}Du$ a.e. on Ω.

Proof of Lemma 13. (a) By choosing $y := hE^{(\alpha)i}$ in Definition 12 applied to the projection $E^{\alpha}\cdot u$ we get that $D^{1,h}_{E^{(\alpha)i}}(E^{\alpha}\cdot u) \longrightarrow E^{\alpha i} : (\mathcal{L}Du)$ as $h \rightarrow 0$ locally in measure on Ω. Thus, for any $h_m \rightarrow 0$, there is $h_{m_k} \rightarrow 0$ such that the convergence is a.e. on Ω, whence $\mathcal{D}u = \delta_{\mathcal{L}Du}$ by Lemma 4.

(b) We begin by observing a triviality: for any map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ we have $f(y) \longrightarrow \ell$ as $y \rightarrow 0$ if and only if for any $y_m \rightarrow 0$, there is $y_{m_k} \rightarrow 0$ such that $f(y_{m_k}) \longrightarrow \ell$ as $k \rightarrow \infty$. We continue by noting that by Lemma 4 and our assumption we have that for any $h_m \rightarrow 0$ there is $h_{m_k} \rightarrow 0$ such that $D^{1,h_{m_k}}u \longrightarrow U$ a.e. on Ω, as $k \rightarrow \infty$. Hence, we obtain that $D^{1,h}u \longrightarrow U$ as $h \rightarrow 0$ (full limit), a.e. on Ω. Since a.e. convergence implies convergence locally in measure, we deduce that $U = \mathcal{L}Du$ a.e. on Ω, as desired.

The next notion of solution will be central in this work. For pedagogical reasons, we give it first for $W^{1,1}_{loc}$ solutions of 2nd order systems and then in the general case.

Definition 14 (Weakly differentiable $\mathcal{D}$-solutions of 2nd order systems). Let $\Omega \subseteq \mathbb{R}^n$ be open,

$$F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s \longrightarrow \mathbb{R}^M$$

a Carathéodory map and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ a map in $W^{1,1}_{loc}(\Omega;\mathbb{R}^N)$. Suppose we have fixed some reference frames as in Definition 5 and consider the PDE system

$$F\big(\cdot, u, Du, D^2u\big) = 0, \quad \text{on } \Omega. \tag{2.10}$$

We say that u is a $\mathcal{D}$-solution of (2.10) when for any diffuse hessian of u arising from any infinitesimal sequence (Definition 7)

$$\delta_{D^{1,h_m}Du} \overset{*}{\rightharpoonup} \mathcal{D}^2u \quad \text{in } \mathscr{Y}\big(\Omega;\overline{\mathbb{R}}^{Nn^2}_s\big),$$

as $m \rightarrow \infty$, we have

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(X)\, F\big(\cdot, u, Du, X\big)\, d[\mathcal{D}^2u](X) = 0, \quad \text{a.e. on } \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$.

Now we consider the general pth order case. For brevity, we will write

$$\underline{X} \equiv (X_1, ..., X_p) \in \overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s.$$

Definition 15 ($\mathcal{D}$-solutions for pth order systems). Let $\Omega \subseteq \mathbb{R}^n$ be open,

$$F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s \longrightarrow \mathbb{R}^M$$

a Carathéodory map and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ a measurable map. Suppose also we have fixed some reference frames as in Definition 5 and consider the PDE system

$$F\big(x, u(x), D^{[p]}u(x)\big) = 0, \quad x \in \Omega. \tag{2.11}$$

Then, we say that u is a D-solution of (2.11) when for any di use pth order Jet of u arising from any in nitesimal sequence (De nition 7)

$$D^{[p];h_{\underline{m}}} u \ \ast\ D^{[p]}u \quad \text{in } Y \quad ; \overline{R}^{Nn} \qquad \overline{R}_s^{Nn^p} \ ;$$

as $\underline{m} \,! \,\underline{1}$, we have

$$\int_{\overline{R}^{Nn}} \int_{\overline{R}_s^{Nn^p}} (\underline{X})\, F\ \ x; u(x); \underline{X}\ \ d\, D^{[p]}u(x)\ (\underline{X}) = 0 ; \qquad \text{a.e. } x \ 2 \quad ;$$

for any $2\ C_c^0\ R^{Nn} \qquad R_s^{Nn^p}$.

Note that De nition 14 can be deduced from De nition 15 by using Lemmas 11 and 4 and that the convergence is separate. These imply when $p = 2$ that $D^{2;h_{(m^0;m)}} u\ ! \quad D^{1;h_m} Du$ a.e. on as $m^0 \,! \,1$.

The following result asserts the fairly obvious fact that D-solutions and strong solutions are compatible.

Proposition 16 (Compatibility of strong and D-solutions). Let F a Caratheodory map as in (1.1) and u : R^n

(2) All reduced pth order Jets of u satisfy the differential inclusion:

$$\text{For a.e. } x \in \Omega, \quad \mathrm{supp}\big(\mathcal{D}^{[p]}u(x)\big) \subseteq \Big\{ F\big(x, u(x), \cdot\big) = 0 \Big\}.$$

(3) For any pth order Jet of u

Since $U^m \overset{*}{\rightharpoonup}$ as $m \to \infty$ and ^{k} is an admissible Carathéodory function, we have

$$\int {}^k\big(x; U^m(x)\big)\, dx \ \to \ \int \int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} {}^k\big(x; \underline{X}\big)\, d[\#(x)](\underline{X})\, dx;$$

as $m \to \infty$. By assumption, we have that $F\big(\cdot; u; [U^m]^R\big) \to 0$ a.e. on as $m \to \infty$. By the properties of and of the truncations, we have the identity

$$[U^m]^R \; F\big(\cdot; u; [U^m]^R\big) = (U^m)\, F\big(\cdot; u; U^m\big)$$

valid a.e. on . Together these last facts give that $^k(\cdot; U^m) \to 0$ a.e. on . Moreover, by using the bound $|{}^k| \le k$ and that $|{}_k| < \infty$, the Dominated convergence theorem allows to infer that $^k(\cdot; U^m) \to 0$ in $L^1(\,)$ as $m \to \infty$. Hence, by the above convergence and the definition of , for a.e. $x \in {}_k$ we have that

$$\begin{aligned}
0 &= \int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} {}^k\big(x; \underline{X}\big)\, d[\#(x)](\underline{X}) \\
&= \int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} (\underline{X})\, F\big(x; u(x); \underline{X}\big)\, d[\#(x)](\underline{X}) \\
&\ge \int_{B_{R/2}(0)} F\big(x; u(x); \underline{X}\big)\, d[\#(x)](\underline{X}).
\end{aligned}$$

The conclusion follows by letting $k \to \infty$ and then $R \to \infty$.

(3) ⇒ (2): We argue as in the case "(1) ⇒ (2)". Suppose that

$$\int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s} F\big(x; u(x); \underline{X}\big)\, d[\#(x)](\underline{X}) = 0, \quad \text{a.e. } x \in ,$$

while the conclusion fails. Fix $x \in$ for which the above holds and assume that $\mathrm{supp}\,\#(x) \not\subseteq \big\{F\big(x; u(x); \cdot\big) = 0\big\}$. Then, there exists

$$\underline{X}_0 \in \big(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\big) \setminus \Big\{F\big(x; u(x); \cdot\big) = 0\Big\}$$

such that, for all $R > 0$ we have that $[\#(x)]\big(B_R(\underline{X}_0)\big) > 0$. Since $F\big(x; u(x); \cdot\big)$ is continuous and $F\big(x; u(x); \underline{X}_0\big) > 0$, there exist $c_0, R_0 > 0$ such that

$$F\big(x; u(x); \cdot\big) \ge c_0 > 0, \quad \text{on } B_{R_0}(\underline{X}_0).$$

Then, we have

$$0 = \int_{\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s} F\big(x; u(x); \underline{X}\big)\, d[\#(x)](\underline{X}) \ge c_0\, [\#(x)]\big(B_{R_0}(\underline{X}_0)\big) .$$

The above contradiction establishes the desired inclusion.

(2) ⇒ (5): We fix $R > 0$ and define the function

$$: \ \times \overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s \to [0, \infty)$$

given by

$$(x; \underline{X}) := \chi_{\overline{B_R(0)}}(\underline{X})\, \mathrm{dist}\Big(\underline{X}, B_R(0) \cap \Big\{F\big(x; u(x); \cdot\big) = 0\Big\}\Big) .$$

Then, is measurable in x for all $\underline{X}$ (this is a consequence of Aumann's theorem, see e.g. [FL]), upper semicontinuous in $\underline{X}$ for a.e. x and also bounded. Hence, since

Proof of Proposition 26. If suffices to establish b) and only for $p = 1$. By assumption, we have that $A^{1}(x) : LDv(x) = g(x)$ and also that for any $2\ C^0_c(\mathbb{R}^{Nn})$,

$$\int_{\overline{\mathbb{R}}^{Nn}} (X)\, \Big[A^{1}(x) : X \quad f(x) \Big]\, d[\mathcal{D}u(x)](X) = 0 ;$$

both being valid for a.e. on $x\ 2$. Here $\mathcal{D}u$ is any diffuse gradient. We fix any point x as above and replace by $+ LDv(x)$. Then, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}} \quad X + LDv(x) \quad \Big[A^{1}(x) : \ X + LDv(x) \quad f(x) \quad g(x) \Big]\, d[\mathcal{D}u(x)](X) = 0 :$$

By the definition of $\mathcal{D}u \quad T_{LDv}$, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}} (Y)\, \Big[A^{1}(x) : Y \quad (f + g)(x) \Big]\, d\ \mathcal{D}u(x) \quad T_{LDv(x)}\ (Y) = 0 :$$

By utilising part a), the conclusion ensues.

Example 28 (Nonlinearity of diffuse derivatives) . Let $K \quad \mathbb{R}$ be a compact nowhere dense set of positive measure (e.g. $K = [0;1] \setminus ([{}^{1}_{1} (r_j \quad 3^{\ j} ; r_j + 3^{\ j}))$ where $(r_j)^1_1$ is an enumeration of $\mathbb{Q} \cap [0;1]$). Then, for u

(see also Remark 17). However, $\mathcal{D}$-solutions completely avoid the impossibility to multiply distributions. For example, if $A \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$,

$$A\cdot D^{1,h_m}u \;\; \overset{\mathcal{D}}{\underset{}{\overset{*}{\rightharpoonup}}} \begin{cases} A\cdot Du, & \text{in } \mathcal{D}'\big(\mathbb{R}^n;\mathbb{R}^n\big), \quad \text{[not well defined!]} \\ A\cdot Du, & \text{in } \mathscr{Y}\big(\mathbb{R}^n;\overline{\mathbb{R}}^n\big), \quad \text{[well defined!]} \end{cases}$$

Hence, for the equation $A\cdot Du = 0$, solutions $u \in L^1_{loc}(\mathbb{R}^n)$ make perfect sense in our context by interpreting the equation as

$$A(x)\cdot\int_{\overline{\mathbb{R}}^N}\Phi(X)\,X\,d[\mathcal{D}u(x)](X) = 0, \quad \text{a.e. } x\in\mathbb{R}^n,$$

for all $\Phi \in C^0_c(\mathbb{R}^n)$, while in the sense of distributions it is not well defined:

$$\text{``}A(x)\cdot Du(x) = A\cdot \mathrm{bar}\big(\mathcal{D}u(x)\big) = A(x)\cdot\int_{\mathbb{R}^n} X\,d[\mathcal{D}u(x)](X)\text{''} = ?$$

We conclude this discussion by underlining the simplicity and handiness of our theory, as opposed to the more cumbersome algebraic theories of multiplication of distributions and the inconsistencies they present (e.g. [Co]).

3. $\mathcal{D}$-solutions of the ∞-Laplacian and tangent systems

In this section we establish our first main result concerning the existence of $\mathcal{D}$-solutions. We treat the Dirichlet problem for the ∞-Laplace system (1.9) which is the fundamental equation of vectorial Calculus of Variations in the space L^∞ and arises from the functional (1.8). A central ingredient in the proof of Theorem 29 below is a result of independent interest, Theorem 33 that follows, which provides a method of constructing nonsmooth $\mathcal{D}$-solutions to nonlinear systems by "differentiating an equation".

Theorem 29 (Existence of ∞-Harmonic maps). Let $\Omega \subseteq \mathbb{R}^n$ be an open set with $|\Omega| < \infty$ and $n \geq 1$. Then, for any $g \in W^{1,\infty}(\Omega;\mathbb{R}^n)$, the Dirichlet problem

$$\begin{cases} \Big(Du\otimes Du + |Du|^2[Du]^\perp\otimes I\Big) : D^2u = 0, & \text{on } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

has a $\mathcal{D}$-solution $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in $W^{1,\infty}_g(\Omega;\mathbb{R}^n)$. In particular, u satisfies Definition 14 (with respect to the standard frames): for any diffuse hessian, we have

$$\int_{\overline{\mathbb{R}}^{n\times n^2}_s}\Phi(X)\,\Big(Du\otimes Du + |Du|^2[Du]^\perp\otimes I\Big) : X\,d[\mathcal{D}^2u](X) = 0,$$

a.e. on Ω X

Fix an $M > 0$ as in statement and consider the Dirichlet problem:

$$\begin{cases} \sigma_i(Dv) = 1, & \text{a.e. in } \Omega, \quad i = 1, \dots, n, \\ v = g/M, & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Then, we have the estimate

$$\begin{aligned} \big\| \sigma_n(Dg) \big\|_{L^\infty(\Omega)} &= \Big\| \max_{|e|=1} \big(Dg^\top Dg\big)^{1/2} : e \otimes e \Big\|_{L^\infty(\Omega)} \\ &\leq \Big\| \big(Dg^\top Dg\big)^{1/2} \Big\|_{L^\infty(\Omega)}. \end{aligned} \tag{3.7}$$

In view of the results of [DM], the estimate (3.7) implies that the required compatibility condition is satisfied in regard to the problem (3.6). Hence there is a strong solution v to (3.6) such that $v - (g/M) \in$

We close this section we a discussion regarding the nonuniqueness problems related to the ∞-Laplace system.

A possible selection principle for Δ_∞. In view of Theorem 30 proved in [K2], among the many smooth solutions that (3.1) has, the boundary condition $g(x) = x$ is itself a solution. Moreover, it is the only solution which is a limit of p-Harmonic maps as $p \to \infty$: for each $p > 2$, the unique solution of the p-Laplacian $\Delta_p u = \mathrm{Div}\big(|Du|^{p-2}Du\big) = 0$ with data g on $\partial\Omega$ is g itself. On the other hand, in the scalar case all ∞-Harmonic functions arise as uniform limits of p-Harmonic functions (this is a consequence of Jensen's uniqueness theorem for the ∞-Laplacian and of the uniqueness for the p-Laplacian, see e.g. [C, K8] and references therein). Moreover, plenty of other examples seem to exhibit the same behaviour. Hence, we are led to the following conjecture regarding a selection ("entropy") principle of "good" solutions to the ∞-Laplace system:

Conjecture (Uniqueness for the Dirichlet problem for Δ_∞). *For any domain $\Omega \Subset \mathbb{R}^n$ with Lipschitz boundary and any $g \in W^{1,\infty}(\Omega;\mathbb{R}^N)$, the Dirichlet problem (3.1) has a unique $\mathcal{D}$-solution $u^\infty \in W^{1,\infty}_g(\Omega;\mathbb{R}^N)$ in the class of uniform subsequential limits of p-Harmonic mappings u^p as $p \to \infty$.*

Investigation of the validity of this conjecture is left for future work.

4. $\mathcal{D}$-solutions of fully nonlinear degenerate elliptic systems

Fix n, N

only certain partial regularity along rank-one lines. Our bre space counterparts which are adapted to the degenerate nature of the problem support feeble yet su cient versions of weak compactness, trace operators and Poincaré inequalities for D-solutions. The proof is completed by characterising the "bre" object we have obtained via xed point as the unique D-solution of the Dirichlet problem (4.1) inside the bre space.

4.2. Fibre spaces, degenerate ellipticity and the main result. Before stating our existence result we need some preparation. We will use the notation

$$A \in \mathbb{R}_s^{Nn \quad Nn}$$

to denote symmetric linear maps $A : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$, i.e. 4th order tensors satisfying $A_{i\ j} = A_{j\ i}$ for all indices $;\ = 1;\ldots;N$ and $i;j = 1;\ldots;n$. The notation

$$N\ A : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}\ ;\quad N\ A : \mathbb{R}_s^{Nn^2} \to \mathbb{R}^N$$

will be used to denote the nullspaces of A as linear map with domain and range those indicated in the brackets, i.e. when A acts respectively as

$$AQ := \sum_{;\ ;i;j} A_{i\ j}\ Q_j\ e\ e^i;\quad A : X := \sum_{;\ ;i;j} A_{i\ j}\ X_{ij}\ e :$$

We will also use similar notation for the respective ranges with "R" instead of "N". If A is rank-one positive, i.e. if the respective quadratic form is rank-one convex

$$A : \ a\ \ a = \sum_{;\ ;i;j} A_{i\ j}\ a_i\ a_j\ 0;\quad 2\ \mathbb{R}^N;\ a \in \mathbb{R}^n;$$

we de ne

$$
\begin{aligned}
&:= \ R\ A : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn} && \mathbb{R}^{Nn};\\
&:= \ \mathrm{span}\Big[\Big\{\ a \in \ \Big\}\Big] && \mathbb{R}^N;\\
&:= \ \mathrm{span}\Big[\Big\{\ (a \vee b)\ \ a;\ \ b \in \Big\}\Big] && \mathbb{R}_s^{Nn^2};\\
&:= \min_{|\ |=|a|=1;\ \ a \in} \Big\{ A : \ a\ \ a \Big\} > 0.
\end{aligned}
\tag{4.3}
$$

We will call the ellipticity constant of A, bearing in mind that strictly speaking A may not be elliptic and the respective in mum over $\mathbb{R}^{Nn}$ may vanish. We also recall that we will use the same letters $;\ ;$ to denote the subspaces as well as the orthogonal projections on them. Further, note that we may say "positive A" meaning "non-negative A", but "strictly positive" will always be used to clarify strictness.

The bre Sobolev spaces. Given $A \in \mathbb{R}_s^{Nn \quad Nn}$ rank-one positive, let $;\ ;$ be given by (4.3) and suppose that is spanned by rank-one directions A su cient condition regarding when this happens is when A is in a sense "decomposable", something we will require later in De nition 36 via the map $u \mapsto (u, Du, D^2u)$. We define the

Degenerate ellipticity and decomposability. Now we introduce our ellipticity hypothesis for (4.1) and a condition for tensors $\mathbf{A} \in \mathbb{R}_s^{Nn \otimes Nn}$ that will guarantee that their ranges are spanned by rank-one directions.

Definition 35 (Degenerate ellipticity)**.** We say that the Carathéodory map $F : \Omega \times \mathbb{R}_s^{Nn^2} \to \mathbb{R}^N$ (or the system $F(\cdot, D^2u) = f$) is degenerate elliptic when there exists $\mathbf{A} \in \mathbb{R}_s^{Nn \otimes Nn}$ rank-one positive, constants $B, C \geq 0$ with $B + C < 1$ and a positive measurable function α satisfying $\alpha, 1/\alpha \in L^\infty(\Omega)$ such that

$$\Big| \mathbf{A} : Z - \alpha(x)\big(F(x, X+Z) - F(x, X)\big) \Big| \leq B\,|\mathbf{A}:Z| + C$$

frames as in (2.2) depending only on F. In particular, u is well de ned and vanishes $H^{n\ 1}$-a.e. on @ and for any $2\ C^0_c\ R^{Nn^2}_s$, we have

$$\int_{\overline{R}^{Nn^2}_s} (\ X)\ \ F(x;X)\quad f(x)\quad d[D^2u(x)](X) = 0; \quad \text{a.e. } x\ 2 \quad ;$$

where D^2u is any di use hessian of u arising from any in nitesimal subsequences:

$$D^{2;h\,\underline{m}}u \quad * \quad D^2u \quad \text{in } Y \quad ;\overline{R}^{Nn^2}_s \quad ; \qquad \text{as } \underline{m} \ !\ 1 \quad :$$

Remark 38. I) [Compatibility] f has to be valued in the subspace because this is a compatibility condition arising from the degeneracy of the problem. For example, the 2 2 system $u_1 = f_1$, $0 = f_2$ has no solution whatsoever in any weak sense unless f_2 0.
II) [Partial regularity] The solution we obtain in Theorem 37 possess di erentiable projections along certain rank-one lines, but in general this can not be improved further. For, choose any $f\ 2\ C^0(\overline{D})$ not weakly di erentiable with respect to x_1 for any x_2 over the unit disc of R^2. Then, the problem

$$D^2_{22}u = f \quad \text{on } D; \qquad u = 0 \quad \text{on } @D;$$

has the unique explicit D-solution (which is not in $W^{1;1}_{loc}(\ )$)

$$u(x_1;x_2) = \quad v(x_1;x_2) + \int$$

for some $0 < B < \infty$ and A positive such that $A, 1/A \in L^\infty(\Omega)$.
V) [Partial monotonicity] If F satisfies Definition 35 and Ξ (see (4.3)) satisfies

$$N\big(A : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\big) \subseteq \Xi^{\perp};$$

then the following "monotonicity" property holds true:

$$\text{(4.6)} \qquad \begin{cases} \text{For a.e. } x \in \Omega,\ F(x;\cdot) \text{ is constant along the subspace } \Xi^\perp: \\ F(x;X) = F(x;\Xi X); \quad X \in \mathbb{R}^{Nn^2}_s. \end{cases}$$

The above property of Ξ will turn out to be true when ΞA satisfies Definition 36. To see (4.6), note that since $\Xi^\perp \supseteq N\big(A : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\big)$, for any $Z \in \Xi^\perp$ we have $A : Z = 0$ and also $\Xi Z = 0$. Hence, Definition 35 gives

$$A(x)\,\big|F(x;X+Z) - F(x;X)\big| \le 0; \quad Z \in \Xi^\perp;\ X \in \mathbb{R}^{Nn^2}_s:$$

Obviously, we also have $A : X = A : (\Xi X)$. Observe that (4.6) is much weaker than the decoupling condition $F(X) = F(\Xi X)$ required for vector-valued viscosity solutions.

Next we gather some properties of the fibre spaces essentially proved in [K10] but without the formalism of the fibre spaces.

Remark 39 (Basic properties of the fibre Sobolev space counterparts, cf. [K10])
(I) [Poincaré inequality] For any $\Omega \Subset \mathbb{R}^n$, unit vectors a, and $u \in W^{1,2}_0(\Omega;\mathbb{R}^N)$, we have

$$\|\Xi u\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)\,\big\|D_a(\Xi u)\big\|_{L^2(\Omega)}:$$

(II) [Norm equivalence] The seminorm $\|G^2(\cdot)\|_{L^2(\Omega)}$ on the fibre space $(W^{1,2}_0 \cap W^{2,2})(\Omega;\Xi)$ (see (4.4), (4.5)) is equivalent to its natural norm

$$\|\cdot\|_{W^{2,2}(\Omega)} = \|\cdot\|_{L^2(\Omega)} + \|G(\cdot)\|_{L^2(\Omega)} + \|G^2(\cdot)\|_{L^2(\Omega)}:$$

(III) [Trace operator] If $\Omega \Subset \mathbb{R}^n$ is strictly convex and $a \in \mathbb{R}^n \setminus \{0\}$, then there is a closed set $E \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(E) = 0$ such that for any $\Gamma \Subset \partial\Omega \setminus E$, we have

$$\|v\|_{L^2(\Gamma;\mathcal{H}^{n-1})} \le C\Big(\|v\|_{L^2(\Omega)} + \big\|D_a v\big\|_{L^2(\Omega)}\Big);$$

for some universal $C = C(\Gamma) > 0$ and all $v \in C^1(\overline{\Omega})$. Hence, there is a well-defined trace operator $T : W^{1,2}(\Omega;\mathbb{R}^N) \to L^2_{loc}(\partial\Omega \setminus E;\mathcal{H}^{n-1};\mathbb{R}^N)$.

Before giving the proof of the main result, we need an important estimate. This is done in the next subsection.

4.3. A priori degenerate hessian estimates.

Herein we establish an a priori estimate for strong solutions in $(W^{2,2} \cap W^{1,2}_0)(\Omega;\mathbb{R}^N)$ of a regularisation of

$$A : D^2u = f; \quad \text{on } \Omega;$$

when A is decomposable. This is a generalisation of the elliptic estimate of [K11] (which extended the classical Miranda-Talenti identity) to the degenerate case.

Theorem 40 (Degenerate hessian estimate) Let $n, N \ge 1$ with $\Omega \subseteq \mathbb{R}^n$ a convex bounded C^2 domain. Suppose further that $A \in \mathbb{R}^{Nn \times Nn}_s$ satisfies Definition 36. If Ξ, ν are as in (4.3), then for any $u \in (W^{2,2} \cap W^{1,2}_0)(\Omega;\mathbb{R}^N)$ and any $\varepsilon \ge 0$ we have the estimate

$$\big\|D^2u\big\|_{L^2(\Omega)} \le \frac{1}{\nu}\big\|A^{(\varepsilon)} : D^2u\big\|_{L^2(\Omega)}$$

((A) is defined in the statement). We now define the subspaces of $\mathbb{R}^{n^2}_s$

$$
\begin{aligned}
H^0 &:= \left\{ X \in \mathbb{R}^{n^2}_s \ : \ X = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & (X_{ij})^{j=i_0,\dots,n}_{i=i_0,\dots,n} \end{array}\right] \right\}, \\
H &:= \left\{ X \in \mathbb{R}^{n^2}_s \ : \ O^\top X O \in H^0 \right\}.
\end{aligned} \tag{4.10}
$$

We may now apply the estimate (4.12) to v over $U \quad \mathbb{R}^n$ and by (4.12), (4.13) to obtain

$$\int_U D^2u(\ x) : (A + "I)^2 dx \qquad \int_U O^\top D^2u(\ x) O^{\ 2} dx \overset{(4.11)}{\quad} (A)^2 \int_U H^0 \ O^\top D^2u(\ x) O^{\ 2} dx:$$

By the change of variables $y :=\ x$ and by using that O is orthogonal, we obtain

$$D^2u : (A + "I)_{L^2(\,)} \qquad (A)\ \ O\ \ H^0\ \ O^\top D^2u\, O \quad O^\top_{\ L^2(\,)} : \tag{4.14}$$

Now we claim that the orthogonal projection on the subspace $H \quad \mathbb{R}^{n^2}_s$ is given by

$$H X = O\ \ H^0\ \ O^\top X O \quad O^\top : \tag{4.15}$$

Once (4.15) has been established, the desired estimate follows from (4.14), (4.15) and a standard density argument in the Sobolev norm. Indeed, if K denotes the linear operator de ned by the right hand side of (4.15), for any X 2 $\mathbb{R}^{n^2}_s$ we have

$$K\ \ K X \ = O\ \ H^0\ \ O^\top O\ \ H^0\ \ O^\top X O \quad O^\top O \quad O^\top =$$
$$= O\ \ H^0 H^0\ O^\top X O \quad O^\top = O\ \ H^0\ O^\top X O \quad O^\top = K X:$$

Hence, $K^2 = K$. Moreover, K is symmetric as a map $\mathbb{R}^{n^2}_s$! $\mathbb{R}^{n^2}_s$: by using that H^0 is symmetric, we have

$$(K X) : Y = \ \ O\ \ H^0\ \ O^\top X O \quad O^\top \ : Y = H^0\ \ O^\top X O \ : \ O^\top Y O =$$
$$= \ \ O^\top X O \ : H^0\ \ O^\top Y O \ = X : \ O\ \ H^0\ \ O^\top Y O \quad O^\top \ =$$
$$= X : (K Y);$$

for any $X; Y$ 2 $\mathbb{R}^{n^2}_s$. Hence, (4.15) follows. It remains to exhibit the claimed property of H. To this end, $x\ \ X$? H. Then, we have that the projection of X on H vanishes and as a result of

O

we obtain that H has a basis consisting of matrices of the form $Oe^i \vee Oe^j$, $i,j = i_0,\dots,n$. We recall now that $A = O\Lambda O^\top$ where Λ is a diagonal matrix with entries the eigenvalues $\{0,\dots,0,\lambda_{i_0},\dots,\lambda_n\}$ of A. We define the vectors

$$a^i := Oe^i = \big(O_{1i},\dots,O_{ni}\big)^\top, \quad i = 1,\dots,n.$$

Then, $\{a^1,\dots,a^n\}$ is an orthonormal frame of $\mathbb{R}^n$ corresponding to the columns of the matrix A and is a set of eigenvectors of A. Since $\{a^{i_0},\dots,a^n\}$ correspond to the nonzero eigenvalues $\{\lambda_{i_0},\dots,\lambda_n\}$, the nullspace $N\big(A:\mathbb{R}^n\to\mathbb{R}^n\big)$ is spanned by $\{a^1,\dots,a^{i_0-1}\}$ and hence

$$R\big(A:\mathbb{R}^n\to\mathbb{R}^n\big) = \mathrm{span}\big[\{a^{i_0},\dots,a^n\}\big].$$

Since H has a basis of the form $\{a^i\vee a^j : i,j = i_0,\dots,n\}$, the claim follows.

Now we begin working towards the vector case $N\geq 2$. Let us first verify that $A^{(\varepsilon)}$ is strictly rank-one positive. Indeed, if $0<\varepsilon<1$, $\eta\in\mathbb{R}^N$

because ? if 6= . We now de ne Q := P ; a and we claim that A : Q = R. Indeed, we have

$$\sum_{;j} A_{ij} Q_j = \sum_{;\,;j} B\ A_{ij} \left(\sum_{;} a_j \right) = \sum_{;\,;j} B\ A_{ij} \left(\sum a_j \right) =$$

$$= \sum_{;\,;\,;j} B\ A_{ij}\ a_j = R_i\ :$$

This establishes that T , therefore completing the proof.

The next step is to nd an upper bound of the ellipticity constant of A in terms of the matrices B , A .

Claim 44. Let be given (4.3) and , T by (4.16). Then, we have the estimate

$$\min_{2} \min_{;j\ j=1} B\ :\ \ \min_{a2T\ ;jaj=1} A\ :a\ a\ :$$

Proof of Claim 44. We begin by noting that on top of the decomposability we may further assume that all the matrices A have the same smallest positive eigenvalue $_{i_0}$ equal to 1 for all = 1; :::; N which is realised at a common eigenvector a 2 R^n. Indeed, existence of a follows from De nition 36 since the eigenspaces N A $_{i_0}$I intersect for all at least along a common line in R^n. Further, by replacing f B^1; :::; B^N g, f A^1; :::; A^N g by the rescaled families f $\bar B^1$; :::; $\bar B^N$ g, f $\bar A^1$; :::; $\bar A^N$ g where $\bar B$:= $_{i_0}$B , $\bar A$:= (1= $_{i_0}$)A , we have that the new families have the same properties as the original and in addition all the new A matrices have the same minimum positive eigenvalue normalised to 1. Hence, we may assume that A is decomposable and moreover

$$(4.17)\quad 9\ a\ 2\ @B_1^n \bigcap_{=1}^{N} T\ :\ \ _{i_0} = \min_{a2T\ ;jaj=1} A\ :a\ a\ = A\ :a\ a = 1;$$

for all = 1; :::; N . By using (4.17), Claim 43 and that [T T , we calculate

$$= \min_{j\ j=jaj=1;} \sum_{a2} B\ :\ \ A\ :a\ a$$

$$\min_{j\ j=jaj=1;} \sum_{a2[\ (\ \ T\ )} B\ :\ \ A\ :a\ a$$

$$\min \left(\min_{j\ j=jaj=1;} \sum_{a2\ \ T} B\ :\ \ A\ :a\ a \right)$$

$$\min \left(\min_{j\ j=1;\ 2} \sum B\ :\ \ A\ :a\ a \right)$$

$$= \min \min_{j\ j=1;\ 2} \sum B\ :\ \ :$$

Since B : = 0 if 2 for 6= , by using (4.17) again we conclude that

$$\min_{2} \min_{;j\ j=1} B\ :$$

$$= \min_{2} \min_{;j\ j=1} B\ :\ \ \min_{a2T\ ;jaj=1} A\ :a\ a\ ;$$

as desired.

Now we complete the proof of the theorem by using the previous claims. We de ne

$$(4.18) \qquad := \qquad T \;\_\; T \qquad \mathbb{R}^{Nn^2}_s ;$$

and for brevity we set

$$:= \; T \;\_\; T \qquad \mathbb{R}^n$$

Hence, (4.21) gives

$$A^{(")} : D^2 u^{\ 2} \qquad \min_{=1\,;:::;N} \Big\{ B \ : \ \mathrm{sgn}\, C^{(")} \qquad \mathrm{sgn}\, C^{(")} \Big\}^2 \sum_{=1}^{N} C^{(")\ \ 2}$$

and as a result we obtain

$$A^{(")} \ \ \mathrm{sgn}$$

of the approximation $A^{(")}$ and De nition 36, we have

$$\sum_{=1}^{N} B^{(")} \quad D^2u^" : A^{(")} \quad = f \quad B^{(")0} \; D^2u^" : A^{(")0} \;;$$

a.e. on . By using that $B^{(")} = B$ for $= 1; :::; N$ and that $B^{(")0} ? B^1 + \quad + B^N$, we may project the system above on the range of $B^1 + \quad + B^N$ which we denote by . Then, since $f = f$ and $A^{(")} = A + "I$, we obtain

$$\sum_{=1}^{N} B \quad " \quad u^" + D^2u^" : A \quad = f;$$

a.e. on . Moreover, by (4.7) (and in view of Remark 38), we deduce

$$A : \quad D^2u^" \quad f = \quad " \sum_{=1}^{N} B \;(\quad u^");$$

a.e. on . Then, for any $2\; C^1_c\;(\;;\mathbb{R}^N)$, integration by parts gives

$$\int A : \quad D^2u^" \quad f \quad = \quad " \int \sum_{=1}^{N} B \;(\; u^") \quad :$$

By letting $"_k \,!\, 0$, we obtain $A : G^2(u) = f$, a.e. on . We nally show uniqueness. Let $v; w\; 2\; (W^{2;2} \setminus W^{1;2}_0)(\;;)$ be two solutions of the system. Then, there are sequences $(v^m)^1_1 ; (w^m)^1_1 \quad (W^{2;2} \setminus W^{1;2}_0)(\;; \mathbb{R}^N)$ such that $v^m \quad w^m \;! \quad v \quad w$ with respect to $k \;\; k_{W^{2;2}( )}$ as $m \,!\, 1$. By assumption we have $A : G^2(v \quad w) = 0$ a.e. on , and hence

$$A : D^2(v^m \quad w^m) =: f^m; \quad \text{a.e. on } ;$$

and $f^m \;! \;\; 0$ in $L^2(\;;\mathbb{R}^N)$ as $m \,!\, 1$. Hence, by Theorem 40 and Remark 39, we have

$$kf^m k_{L^2( )} \qquad : \; D^2(v^m \quad w^m) \;\; {}_{L^2( )} \qquad C \;\; (\; v^m \quad w^m) \;\; {}_{L^2( )}$$

and by letting $m \,!\, 1$ we see that $v \quad w$, hence uniqueness ensues.

An essential ingredient in order to pass from the linear to the non-linear problem is the next result of Campanato taken from [C3] (see also [K7]) which we recall for the convenience of the reader.

Lemma 46 (Campanato's bijectivity of near operators). Let $X \,6\, ;$ be a set and $(X; k \;\; k)$ a Banach space. Let also $F \,; A \;: X \,! \quad X$ be two mappings and suppose there is a $K\; 2\; (0; 1)$ such that

F (u) F (v) A626 Tf 9.962 0 Td [(F)]TJ/F8 9.9626 Tf 10.28 0 Td [(()]TJ/F11 9.( )]TJ/F1(.07

where $G^2(u)$ is the bre hessian of u.

Proof of Claim 47. For any xed u 2 $(W^{2;2} \backslash W_0^{1;2})(\;;\;)$, we have that $A : G^2(u)$ is in $L^2(\;;\;)$ because $G^2(u)$ 2 $L^2(\;;\;)$ and also $A : X$ lies is in $\mathbb{R}^N$ for any X

Proof of Claim 48. Step 1 (The frames). By (4.3) and (4.16) we have that there is an orthonormal frame f E j g of R^N and for each there is a frame f $E^{( )i}$ j i g of R^n such that each of the mutually orthogonal subspaces R^N is spanned by a subset of vectors E and for the same index , T is spanned by f $E^{( )i_0}$; :::; $E^{( )n}$ g which is a set of eigenvectors of A . By (4.3) and (4.18) there are also induced orthonormal frames of R^{Nn} and $R_s^{Nn^2}$ consisting of matrices as in (2.2). These frames are such that a subset of the E^{ij} 's spans the subspace $R_s^{Nn^2}$ and the rest are orthogonal to .

Step 2 (Su ciency). Let now u 2 $(W^{2;2} \cap W_0^{1;2})$(;) be the map of Claim 47 which satis es F (; G^2(u)) = f a.e. on . Let also us x any in nitesimal sequence $(h_{\underline{m}})_{\underline{m} 2 N^2}$ with respect to the frames of Step 1 (see De nition 5) and let $D^2 u$ be any di use hessian of u arising from this sequence

$$D^{2;h_{\underline{m}}} u \ \ * \ \ D^2 u \quad \text{in } Y \quad ; \overline{R}_s^{Nn^2} \ ; \quad \text{as } \underline{m} ! 1$$

perhaps along subsequences. By the characterisation of the bre hessian $G^2(u)$ 2 L^2(;) in terms of directional derivatives of projections (Subsection 4.2), we have

$$(4.24) \qquad G^2(u) = \sum_{;i;j \ : E^{ij} \ 2} G^2(u) : E^{ij} \quad E^{ij} \ ; \quad \text{a.e. on } \ ;$$

because the projection of G^2(u) along E^{ij} is non-zero only for those E^{ij} spanning . Since F is a Carathéodory map and F x; G^2(u)(x) = f (x) for a.e. x 2 , by (4.24) and in view of (2.4) we get

$$F \left(x; \sum_{;i;j \ : E^{ij} \ 2} D^{2;h_{\underline{m}^2_1} h_{\underline{m}^2_2}}_{E^{( )i} E^{( )j}} \ E \quad u \quad (x) E^{ij} \right) ! \quad f (x);$$

for a.e. x 2 as $\underline{m}$! 1 . By Remark 38V), the above is equivalent to

$$F \quad x; D^{2;h_{\underline{m}}} u(x) \quad = F \left(x; \sum_{;i;j} D^{2;h_{\underline{m}^2_1} h_{\underline{m}^2_2}}_{E^{( )i} E^{( )j}} \ E \quad u \quad (x) E^{ij} \right) ! \quad f (x);$$

for a.e. x 2 , as $\underline{m}$! 1 . We set

$$f^{\underline{m}}(x) := F \quad x; D^{2;h_{\underline{m}}} u(x) \qquad f (x)$$

and note that we have $f^{\underline{m}}$! 0, a.e. on as $\underline{m}$! 1 . By the above, for any 2 C_c^0 $R_s^{Nn^2}$ we have

$$\int_{\overline{R}_s^{Nn^2}} (X) \ \Big[F (x; X) \quad f (x) + f^{\underline{m}}(x) \Big] \ d \ _{D^{2;s_{\underline{m}}} u(x)} (X) = 0 ; \quad \text{a.e. } x \ 2 \quad :$$

Since $f^{\underline{m}}$! 0 a.e. on as $\underline{m}$! 1 , we apply the Convergence Lemma 18 to obtain

$$\int_{\overline{R}_s^{Nn^2}} (X) \ F (x; X) \quad f (x) \ d \ D^2 u(x) \ (X) = 0 ; \quad \text{a.e. } x \ 2 \quad ;$$

for any 2 C_c^0 $R_s^{Nn^2}$. Hence, the map u of Claim 47 is a D-solution of (4.1).

Step 3 (Necessity). We now nish the proof by showing that any D-solution w of (4.1) with respect to the frames of Step 1 which lies in the bre space $(W^{2;2} \cap W_0^{1;2})$(;) actually coincides with the map u of Claim 47. By Theorem 22, we have that the D-solution w can be characterised by the property that for any $R > 0$, the cut off associated to F (see Definition 21) satisfies

$$F\big(x, D^{2,h_m} w(x)\big)^R \longrightarrow f(x), \quad \text{a.e. } x \in \Omega,$$

as $m \to \infty$. By using Remark 38V), we have for any $R > 0$ that

$$F\big(x, D^{2,h_m} w(x)\big)^R \longrightarrow f(x), \quad \text{a.e. } x \in \Omega,$$

as $m \to \infty$. Since w is in $(W^{2,2} \cap W_0^{1,2})(\Omega; \mathbb{R}^N)$, by using the properties of the fibre space we get that $D^{2,h_m} w \rightharpoonup G^2(w)$ in L^2 and hence a.e. on Ω along perhaps further subsequences. By passing to the limit as $m \to \infty$ and then as $R \to \infty$, we obtain that $F(\cdot, G^2(w)) = f$, a.e. on Ω. Hence, $w \equiv u$ and the claim ensues.

By recalling Remark 39 regarding the boundary trace values of maps in the fibre space, we conclude that the proof of Theorem 37 is now complete.

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