Review article: Advances in the study of boundary value problems for nonlinear integrable PDEs

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Abstract

In this review I summarise some of the most signi cant advances of the last decade in the analysis and solution of boundary value problems posed for integrable partial di erential equations in two independent variables. These equations arise widely in mathematical physics, and in order to model realistic applications, it is essential to consider bounded domain and inhomogeneous boundary conditions.

I focus speci cally on a general and widely applicable approach, usually referred to as the *Uni ed Transform* or *Fokas Transform*, that provides a substantial generalisation of the classical Inverse Scattering Transform. This approach preserves the conceptual e ciency and aesthetic appeal of the more classical transform approaches, but presents a distinctive and important di erence. While the Inverse Scattering Transform follows the "separation of variables" philosophy, albeit in a nonlinear setting, the Uni ed Transform is a based on the idea of synthesis, rather than separation, of variables.

I will outline the main ideas in the case of linear evolution equations, and then illustrate their generalisation to certain nonlinear cases of particular signi cance.

1 Introduction

The Inverse Scattering Transform is one of the most celebrated advances in the study of nonlinear systems, pioneered at the end of the 1960's by Kruskal et al. [44] and consolidated throughout the 1970's by the work of many others [4, 5, 50, 57].
This transform is essentially a nonlinear version of the Fourier transform in one variable, and can be used to unravel the behaviour of many systems with the property that the nonlinearity is exactly balanced by other e ects, such as dispersive e ects. This implies that, in many important respects, the behaviour of the solutions of the system is highly regular. For example, when posed on an in nite spatial domain, these systems admit localised solutions (often referred to as solitons) that interact elastically - the interaction does not destroy the amplitude or speed of the solutions. More importantly still, localised initial conditions with su cient energy will eventually evolve into a train of solitons, followed by a dispersive tail. These properties are remarkable for a nonlinear system, and were rst described heuristically by Zabuski and Kruskal who observed this elastic interaction in numerical experiments modelling solutions of the Korteweg-deVries equation.

Systems with the particular properties described above are called *integrable*.

given PDE. Combining this idea with the two observations above, it is possible to construct algorithmically a formal integral representation of the solution of a given boundary value problem for an integrable PDE - this construction is the basis of the Uni ed Transform approach. It is important to note that this representation generally involves contours in the procedure yields the following integral representation of the solution of (1.1) on the half-line:

$$
\begin{aligned}
q(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^3t}\left(\int_0^{\infty} e^{-iky}\,q_0(y)\,dy\right)dk \; + \qquad (1.3)\\
+ {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^3t}\int_0^{t} e^{-ik^3s}\left(q_{xx}(0,s)+ik\,q_x(0,s)-k^2 f_0(s)\right)ds\,dk.
\end{aligned}
$$

The first term in this representation is the contribution of the initial condition. This would be the only term present when solving the Cauchy value problem for decaying data, with the integration in y extending over $\mathbb{R}$. In this term, the x and t dependence is explicit through the exponential term.

The second term involves the boundary values of the solution at $x = 0$, but two of the boundary values involved in the integrand are not directly available. This is the generic

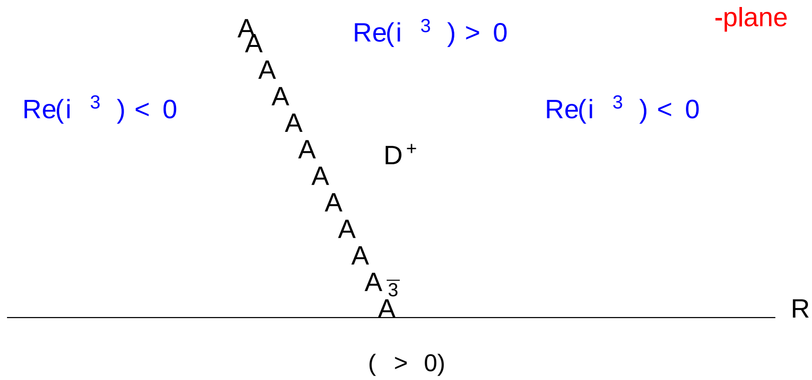

Figure 1: The domain D^+

with

$$\hat{q}_0(\lambda) = \int_0^\infty e^{-i\lambda y} q(y;0)dy, \quad \lambda \in \mathbb{C}; \quad \tilde{f}_0(\lambda) = \int_0^T e^{i\lambda^3 s} f(s)ds; \quad \omega = e^{2\pi i/3}. \quad (1.6)$$

In this linear example, the expression (1.4) could be derived from (1.3) by simply considering analyticity properties with respect to the variable λ, extended from $\mathbb{R}$ to $\mathbb{C}$, and deforming the contour of integration. However, this deformation is not possible in the general nonlinear case. The general methodology to obtain the representation (1.4) is based instead on formulating and solving an associated Riemann-Hilbert problem, and this approach can indeed be extended to the case of nonlinear integrable evolution equations in one space variable, e.g. to the famous KdV or mKdV equation, posed on the domain $0 < x < \infty, \; 0 < t < T$:

$$(\text{KdV}) \qquad q_t + q_{xxx} + q_x + 6qq_x = 0; \qquad (1.7)$$

$$(\text{mKdV}) \qquad q_t + q_{xxx} + 6\lambda q^2 q_x = 0; \qquad \lambda = \pm 1. \qquad (1.8)$$

However, in this case, the elimination of the unknown boundary values is only as effective as in the linear case for special boundary conditions, called *linearisable* [26, 28]. For general boundary conditions, the characterisation can be obtained by a perturbation scheme effective to all orders [41, 42].

In this article, I present a summary of the main results obtained by this approach for boundary value problems, unifying the treatment of linear and integrable nonlinear PDEs

I will also leave out of my exposition the case of problems with periodicity in the variable x. The solution of this case, for integrable nonlinear evolution PDEs, was developed in the seventies, and it involves algebro-geometric techniques, through a formulation as a Riemann-Hilbert problem on a Riemann surface [13, 23, 48].

2 Integral transforms and Riemann-Hilbert problems

The Fourier transform provides the most effective way of solving initial value problems for linear evolution PDEs.
Consider for example the PDE $\partial_t q + \partial_x^n q = 0$. The solution "algorithm", assuming $q(x;0) = q_0(x) \in S(\mathbb{R})$, is given by (1.2). Schematically, this algorithm is given by

$$q_0(x) \xrightarrow{\text{Direct map}} \hat{q}_0(\lambda) \xrightarrow{\text{Inverse map}} q(x;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - (i\lambda)^n t}\,\hat{q}_0(\lambda)\,d\lambda\,;$$

with $\hat{q}_0(\lambda)$ is the Fourier transform of $q_0(x)$.
This solution algorithm is generally applicable, it yields an exact solution representation and, on account of the explicit x and t dependence appearing in the solution representation, it contains qualitative information (in particular, asymptotic information for large

For 2 R, where both functions are de ned, the di di

2.2 The Inverse Scattering Trasform: a nonlinear Fourier transform

The approach just described for formulating and inverting integral transforms has a nonlinear analogue. This nonlinear transform can be used to solve the initial value problem for integrable nonlinear evolution PDEs, in a way analogous to the use of Fourier transform in the linear case.
The starting point, rather than the scalar ODE (2.1), is a *matrix*-valued ODE. Namely, define the matrix Q in terms of the given arbitrary function $q(x) \in S(\mathbb{R})$ (although much less regularity is required, see [6]) by

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix}; \qquad (2.5)$$

where $\bar{\ }$ denotes complex conjugation, and consider the ODE

$$M_x + i\lambda[\sigma_3, M] = QM \quad x \in \mathbb{R}; \quad \lambda \in \mathbb{C}; \quad M(x;\lambda) \text{ a } 2\times 2 \text{ matrix}; \qquad (2.6)$$

where

$$[\sigma_3; M] = \sigma_3 M - M\sigma_3; \quad \sigma_3 = \mathrm{diag}(1; -1): \qquad (2.7)$$

One seeks a solution $M(x;\lambda)$ of this ODE well-defined for all $\lambda \in \mathbb{C}$. As for the linear case, one can define a solution $M^+(x;\lambda)$ well-defined and bounded for $\lambda \in \mathbb{C}^+$, indeed, such that $M^+ \to I$ as $x \to \infty$, and a solution $M^-(x;\lambda)$ well-defined and bounded for $\lambda \in \mathbb{C}^-$, with $M^- \to I$ as $x \to -\infty$. These matrix-valued solutions are not explicit, but characterised as the unique solution of a linear integral equation, namely

$$M^\pm(x;\lambda) = I - \int_x^{\pm\infty} e^{-i\lambda(x-y)\hat{\sigma}_3} Q(y) M^\pm(y;\lambda)\,dy; \quad \lambda \in \mathbb{C}^\pm:$$

They satisfy a jump condition across $\mathbb{R}$ of the form

$$M^-(x;\lambda) = M^+(x;\lambda)\tilde{J}(x;\lambda); \quad \lambda \in \mathbb{R}; \qquad (2.8)$$

as well as asymptotic conditions $M^\pm = I + O\left(\frac{1}{\lambda}\right)$ as $|\lambda| \to \infty$. The jump $\tilde{J}(x;\lambda)$, defined for $\lambda \in \mathbb{R}$, is now matrix-valued. The entries of the 2×2 matrix $\tilde{J}(x;\lambda)$ are defined in terms of certain -transforms of the given function $q(x)$, called the *spectral data*, multiplied by explicit exponentials encoding the dependence on x:

$$\tilde{J}(x;\lambda) = e^{i\lambda x\hat{\sigma}_3} J(\lambda): \qquad (2.9)$$

The notation expresses that the action of $\exp(x\hat{\sigma}_3)$ on a 2×2 matrix A is given by

$$e^{x\hat{\sigma}_3}A = \begin{pmatrix} a_{11} & e^{2x}a_{12} \\ e^{-2x}a_{21} & a_{22} \end{pmatrix}: \qquad (2.10)$$

Hence given the matrix $Q(x)$, the jump condition (2.8) defines the transform $J(\lambda)$.
Conversely, given $J(\lambda)$, the jump and decay data above determine a Riemann-Hilbert problem for $M(x;\lambda)$ on the real line. Note that the jump matrix $J(\lambda)$ and the function $M(x;\lambda)$ are well-defined only modulo the possible existence of isolated singularities. However, the

role of such isolated singularities is well understood and we will ignore them for the purpose of this review, see also remark 5.2 below.

The difference with the linear case is that the jump condition in this case is multiplicative. The multiplicative, non-commutative structure of the Riemann-Hilbert problem is a manifestation of the nonlinearity of the equation, and it implies that the solution does not have an explicit expression analogous to the one in (2.3) given by the Plemelj formula. However, it is a classical result that the solution $M(x;\ )$ of the Riemann-Hilbert determined by the data above can be characterised as the solution of a linear singular integral equation, and its unique solvability can be rigorously proved, appealing to the symmetries forced on the system by the choice of the form of the matrix $Q(x)$ [10, 11, 12, 22].

From the expression for $M(x;\ )$ one must derive an expression for $q(x)$ and thus formulate the inverse transform. Recall that, in the linear case, $\quad iq = \quad$ as $\quad \to \infty \quad$. Similarly, from the expression for $M(x;\ )$, one can determine the arbitrary function

the Lax pair may be altered by changes of variable to regularise the dependence, see for example the elliptic sine-Gordon case, equation (3.4) below).
A useful Lax pair for several important evolution PDEs takes the general form

$$
\begin{aligned}
M_x + i f_1(\ )[\ _3; M] &= Q(x;t;\ )M;\\
M_t + i f_2(\ )[\ _3; M] &= \tilde{Q}(x;t;\ )M;
\end{aligned}
\qquad (3.1)
$$

where $_3$ and the commutator [;] are defined in (2.5), (2.7).
The particular form of the functions $f_i(\ )$, $Q(x;t;\ )$, $\tilde{Q}(x;t;\ )$ depends on the specific PDE. Throughout this review, I will refer to two of the most common and important integrable evolution PDEs, arising as models in mathematical physics, namely the nonlinear Schrödinger (NLS) and modified Korteweg-deVries (mKdV) equations. The case of this second- and third-order evolution equation illustrate the general approach for integrable evolution PDEs in one space variable. For these equations, the Lax pair is given by the following choices:

<u>NLS</u>

$$
\text{equation}: \quad iq_t + q_{xx} \quad 2\ q_x |q|^2 = 0; \qquad = \quad 1; \qquad (3.2)
$$

$$
\text{Lax pair}: \quad f_1(\ ) = \ ; \quad f_2(\ ) = 2\ ^2;
$$

$$
Q = \begin{pmatrix} 0 & q(x) \\ q(x) & 0 \end{pmatrix}; \qquad \tilde{Q} = 2\ Q \quad iQ_x\ _3 \ \ |q|^2\ _3;
$$

<u>mKdV</u>

$$
\text{equation}: \quad q_t + q_{xxx} \quad 6\ q^2 q_x = 0; \qquad = \quad 1; \qquad (3.3)
$$

$$
\text{Lax pair}: \quad f_1(\ ) = \ ; \quad f_2(\ ) = 4\ ^3;
$$

$$
Q = \begin{pmatrix} 0 & q(x;t) \\ q(x;t) & 0 \end{pmatrix}; \qquad \tilde{Q} = 2Q^3 \quad Q_{xx} \quad 2i\ [Q^2 + Q_x]\ _3 + 4\ ^2 Q:
$$

There exist also integrable PDEs of elliptic type, with independent variables denoted x and y. The best-known such equation is the so-called elliptic sine-Gordon equation

$$
\text{equation}: \quad q_{xx} + q_{yy} \quad \sin q(x;y) = 0: \qquad (3.4)
$$

For this PDE, a convenient Lax pair is given by [33, 37, 54]

$$
\text{Lax pair}: \quad M_x + \frac{1}{4i} \quad \frac{1}{\ } \ [\ _3; M] = Q(x;y;
$$

3.1 The solution of the Cauchy problem for integrable PDEs in 2 independent variables using the Inverse Scattering Transform

I have introduced all the ingredients needed to extend the strategy for solving the Cauchy problem for linear evolution equations to nonlinear integrable evolution equations such as the NLS or mKdV equations. I focus on the particular example of the NLS equation (3.2) to discuss how this generalisation is obtained.

The Lax pair formulation of the NLS equation, given explicitly by

$$\begin{aligned} M_x + i\lambda[\sigma_3; M] &= QM \\ M_t + 2i\lambda^2[\sigma_3; M] &= (2\lambda Q - iQ_x\sigma_3 - i|q|^2\sigma_3)M \end{aligned} \qquad \lambda \in \mathbb{C}; \tag{3.6}$$

where $M = M(x;t;\lambda)$, implies that $q(x;t)$ solves the PDE (3.2) if and only if for all $\lambda \in \mathbb{C}$ there exists an invertible matrix-valued function $M(x;t;\lambda)$ solving (3.6), In practice, the PDE is obtained by imposing the compatibility condition $M_{xt} = M_{tx}$.

The first ODE in this pair is precisely the ODE associated with the nonlinear Fourier transform. The second part of the Lax pair is used to determine the time evolution of $M(x;t;\lambda)$, so that it is possible to write down an implicit expression for the solution $M(x;t;\lambda)$ of the Lax pair, in terms of a given initial condition $q(x;0) = q_0(x) \in S(\mathbb{R})$.

Conversely, given $J(\lambda)$ the function $q(x;t)$ can be represented in terms of the elements of the matrix $M(x;t;\lambda)$ as

$$q(x;t) = 2i \lim_{|\lambda|\to\infty} (\lambda M_{12}(x;t;\lambda)):$$

where the function $M(x;t;\lambda)$, sectionally analytic for $\lambda \in \mathbb{C}$, is the solution of the Riemann-Hilbert problem determined by

$$M^- = M^+ e^{(i\lambda x - i\lambda^2 t)\hat{\sigma}_3} J$$

4 An integral transform for linear boundary value problems

My aim is to extend the approach of the Inverse Scattering Transform to boundary value problems. I have already mentioned the di culty of solving a given boundary value problem using the Fourier transform, or the inverse scattering transform in the nonlinear case. This approach, that works well for the initial value problem when the time evolution is explicit and depends only on the initial condition, runs into the di cult problem of eliminating unknown boundary values. This problem is not purely nonlinear - it features also for linear PDEs. I will start therefore to consider how this di culty can be resolved in the linear case.

4.1 Lax pairs for linear PDEs in two variables and the global relation

with the condition that Re $w(\ )$ 0 for 2 R. This condition ensures that the pure initial value problem is well posed, excluding cases such as the "wrong" heat equation $q_t + q_{xx} = 0$, corresponding to $w(\ ) = {}^2$, whose one-parameter family of solutions exp$(i x + {}^2 t)$, with 2 R, grow exponentially in time. Taking the limit as ! 1 , this condition can be recognised as a condition on the leading coe cient $_n$. Namely, for n odd, it is enough to require the condition $_n =$ i, while for n even, the condition is that Re $_n$ 0.

To obtain a boundary value problem which is well posed in the sense of admitting a unique solution valid for all times, it is also necessary to prescribe an initial condition and an

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Once the PDE is cast in this form, involving the auxiliary complex parameter , the stage is set for deriving an appropriate transform. As stated earlier, instead of using only one ODE, this unified transform is determined by both ODEs in the Lax pair simultaneously This transform, valid only for the specific boundary value problem at hand, by construction involves both x and t as parameters.
In this way one derives the following formal representation for the solution of the PDE.

Proposition 4.2 (The formal solution representation) Define

$$\bar{F}(\;;t) = \int_0^t e^{w(\;)s} F(0;s;\;)\,ds = \sum_{k=0}^{n\;1}$$

Proposition 4.3 (The global relation) Consider the PDE (4.2), and let $\tilde{F}(\lambda;t)$ be given by (4.10).
Consider also the Fourier transform in x of the solution $q(x;t)$ at time t:

$$\hat{q}(\lambda;t) = \int_0^\infty e^{-i\lambda y} q(y;t)\,dy; \qquad 0 \le t \le T; \qquad \lambda \in \mathbb{C}^-. \tag{4.14}$$

Then for every $t \in (0;T)$, the functions $\hat{q}_0(\lambda)$ given by (1.6), $\tilde{F}(\lambda;t)$ and $\hat{q}(\lambda;t)$ satisfy the global relation

$$\tilde{F}(\lambda;t) = \hat{q}_0(\lambda) - e^{w(\lambda)t}\hat{q}(\lambda;t); \qquad \lambda \in \mathbb{C}^-. \tag{4.15}$$

The restriction $\lambda \in \mathbb{C}^-$ in (4.15) is needed for the integral defining the terms $\hat{q}_0(\lambda)$ and $\hat{q}(\lambda;t)$ to be well defined. In general, the functions involved in the global relation may have isolated singularities in λ, that can arise from specific boundary conditions or for problems posed on bounded intervals, and whose residues play an important role in the explicit characterisation of the solution. I will touch on this point later, for the case of problems posed on bounded intervals. See [51] for other examples.

The global relation is a necessary condition that the boundary and initial values must satisfy.

4.2 The determination of the spectral data - generalised Dirichlet to Neumann map

To formulate and solve a boundary value problem for a PDE of the form (4.2), it is necessary to

(a) determine how many boundary conditions should be prescribed at the boundary in order to guarantee the existence of a unique solution of the problem;

(b) derive a solution representation that involves only the prescribed boundary conditions, and not all boundary values as in (4.12).

The answer to the question posed in (a) yields the number of boundary values that must be determined as part of the e ective solution of the boundary value problem.
The determination of these unknown boundary values yields the answer to (b). I refer to the expression for the unknown boundary values in terms of the prescribed data as the generalised Dirichlet to Neumann map

Since the Lax pair yields naturally a formal representation of the solutionyi0i334(guagp)-31(3ndary)-41sary where S is given by (4.3), $q_0(x) \in S(\mathbb{R}^+)$ is a given function, and there are N prescribed boundary conditions $\partial_x^j q(x,t)|_{x=0} = f_j(t)$, $j = 0, \ldots, N$

as well as in the derivation of a solution representation involving only the known boundary conditions.
The representation (4.12) involves all functions $\tilde{f}_k(; t)$, $k = 0; \ldots; n \quad 1$, evaluated for $\quad 2 \; @D^+$. On the other hand, the global relation (4.15) is only valid in C . The main idea involved in the derivation of (4.19) from (4.12) is to make use only of the transformations that map the connected components of D^+ into D . Indeed, it can be shown that while there are $n \quad 1$ transformations $\;_j(\;)$ leaving $w(\;)$ invariant (excluding the trivial one $\;(\;) = \;$), for each connected component of D^+ only $n \quad N$ of these map the given component into C ; namely (upon relabelling)

$$2 \; D^+ \; =) \quad {}_j(\;) \; 2 \; C \;; \quad j = 1; \ldots; n \quad N:$$

Recalling that $\tilde{f}_k(\;_j(\;)) = \tilde{f}_j(\;)$, by evaluating the global relation at the values $\;_j(\;)$ one nds a system of $n \quad N$ equations for the $n \quad N$ unknown boundary values:

$$\sum_{k=N}^{n\;1} c_k(\;_1(\;))\tilde{f}_k(\;; t) \;=\; \hat{q}_0(\;_1(\;)) \quad K(\;_1(\;); t) \quad e^{w(\;)t}\hat{q}(\;_1(\;); t)$$
$$\vdots$$
$$\vdots$$
$$\sum_{k=N}^{n\;1} c_k(\;_{n\;N}(\;))\tilde{f}_k(\;; t) \;=\; \hat{q}_0(\;_{n\;N}(\;)) \quad K(\;_{n\;N}(\;); t) \quad e^{w(\;)t}\hat{q}(\;_{n\;N}(\;); t)$$

The solution of this system can be given using Cramer's rule to write each of the functions $\tilde{f}_k(\;; t)$, $k = N; \ldots n \quad 1$, in terms of the known function on the right hand side and the determinant of the system, which can be shown to have no zeros. This solution de nes the mapping F appearing in (4.19).
It would seem that this solution involves also the terms $\hat{q}(\;_j(\;); t)$ that are unknown. However, it can be proved that the combination of these terms appearing in the solution is always bounded and analytic in D^+, in fact that these terms have decay of order $O \; \frac{1}{\;}$ as $\; ! \; 1$, so that their integral along $@D^+$ vanishes. Hence these are ghost terms that do not contribute to the integral in (4.19), and can be ignored for the purpose of representing the solution in the form (4.19).
Finally, analyticity arguments can be used to substitute t with the nal time T in the term $K(\;; t)$ appearing in the nal integral representation.
QED

Remark 4.3 There is considerable exibility in deforming the contour of integration in (4.12) and (4.19). In .629 Td99.91nH322(nal)-321(t69I1nH322(nal)14nal)-3216 Tf 8.6 T294nal t-348(theour

The mapping F of theorem 4.1 (assuming for convenience that the homogeneous boundary condition $q(0;t) = 0$ is prescribed, hence that $\tilde{K}(\ ) = \hat{q}_0(\ )$) is given for this example by

$$F[\tilde{K}](\ ) = !\ \hat{q}_0(!\ \ ) + !$$

the domain D. In this case, the evaluation of the residue of these functions at the poles gives rise to the usual series representation.

The global relation is now the following equation, valid for all $\lambda \in \mathbb{C}$:

$$\tilde{F}(\lambda;t) - e^{-i\lambda L}\tilde{G}(\lambda;t) = \hat{q}_0(\lambda) - e^{w(\lambda)t}\hat{q}(\lambda;t); \qquad \lambda \in \mathbb{C};\ 0 < t \leq T. \tag{4.29}$$

Using the analysis of the global relation just as for the half-line case, it is possible to determine the value of $\tilde{F}(\lambda;t)$ and $\tilde{G}(\lambda;t)$ in terms only of the given initial and boundary

where $a_k, b_k \in \mathbb{C}$, $k = 0,\dots,n-1$.

The above discussion makes it clear that it is of crucial importance to determine the location of the zeros of the function $\Delta(\lambda)$. General results in complex analysis [49] allow one to determine the asymptotic location of the zeros of $\Delta(\lambda)$ for large λ and this is sufficient to give a complete characterisation of the solution of these boundary value problems.

The approach discussed here provides a constructive criterion to distinguish between boundary conditions that yield a self-adjoint spectral problem, and those that do not. In the latter case, the integral representation of the solution is not equivalent to a series representation. The first results on this phenomenon were presented in [35, 36, 53], but recently there has been a rigorous treatment of this issue from the point of view of classical spectral theory [56].

Series versus integral representation - an example

For the illustrative case of the PDE (1.1), two cases of boundary conditions exemplify the situation. Suppose that one has

$$q_t + q_{xxx} = 0, \quad 0 < x < 1, \quad 0 < t < T; \qquad q(x,0) = q_0(x), \quad 0 < x < 1, \tag{4.36}$$

and either of the following two sets of boundary conditions:

$$\text{(A)} \qquad q(0,t),\ q(1,t),\ q_x(0,t) \ \text{ given} \tag{4.37}$$

$$\text{(B)} \qquad q(0,t)\ q(1,t) \ \text{ given}; \quad q_x(0,t) = \alpha q_x(1,t), \quad \alpha \in \mathbb{R}\setminus\{0\}. \tag{4.38}$$

In both cases, since $w(\lambda) = -\lambda^3$ for this PDE, the solution is obtained as an integral along the boundary of the region

$$D = \{\lambda \in \mathbb{C} : \mathrm{Im}(i\lambda^3) < 0\}. \tag{4.39}$$

The global relation is now (dropping the dependence on t)

$$\tilde f_2(\lambda) + i\lambda \tilde f_1(\lambda) - \lambda^2 \tilde f_0(\lambda) - e^{-i\lambda}\left[\tilde g_2(\lambda) + i\lambda \tilde g_1(\lambda) - \lambda^2 \tilde g_0(\lambda)\right] = \hat q_0(\lambda) - e^{i\lambda^3 t}\hat q(\lambda; t), \quad \lambda \in \mathbb{C} \tag{4.40}$$

where $\tilde f_k$ represent transforms of the solution evaluated at $x = 0$ and $\tilde g_k$ represent transforms of the solution evaluated at $x = 1$, $k = 0, 1, 2$. Note that now the global relation is valid for all $\lambda \in \mathbb{C}$, as all functions involved are entire functions of λ.

The unknown boundary values can be obtained as in the half-line case by solving a system, for each fixed $\lambda \in \mathbb{C}$, of three equations for three unknowns obtained by evaluating the

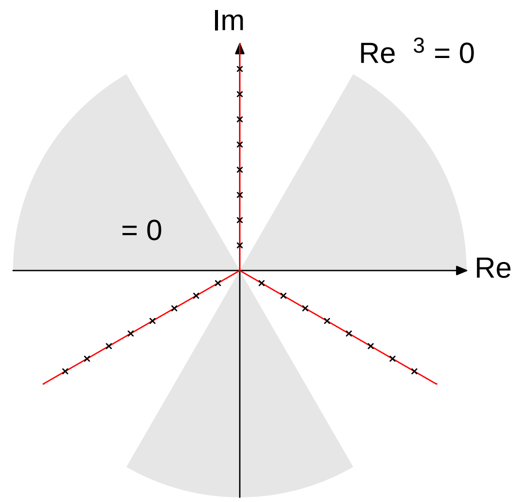

Figure 4: The location of the zeros of $\Delta(\lambda)$ for case (A)

global relation at the three roots given by (4.21):

(A) :

$$\begin{cases} \tilde f_2(\lambda) + i\lambda \tilde f_1(\lambda) - e^{-i\lambda}\tilde g_2(\lambda) = \hat q_0(\lambda) + K(\lambda) - e^{i\lambda^3 t}\hat q(\lambda;t);\\ \tilde f_2(\lambda) + i\omega\lambda \tilde f_1(\lambda) - e^{-i\omega\lambda}\tilde g_2(\lambda) = \hat q_0(\omega\lambda) + K(\omega\lambda) - e^{i\lambda^3 t}\hat q(\omega\lambda;t);\\ \tilde f_2(\lambda) + i\omega^2\lambda \tilde f_1(\lambda) - e^{-i\omega^2\lambda}\tilde g_2(\lambda) = \hat q_0(\omega^2\lambda) + K(\omega^2\lambda) - e^{i\lambda^3 t}\hat q(\omega^2\lambda;t); \end{cases}$$

$$\text{with } K(\lambda) = -\lambda^2\tilde f_0(\lambda) + e^{-i\lambda}[i\lambda\tilde g_1(\lambda) - \lambda^2\tilde g_0(\lambda)]$$

(B) :

$$\begin{cases} \tilde f_2(\lambda) + e^{-i\lambda}[i\lambda(\alpha-1)\tilde g_1(\lambda) - \tilde g_2(\lambda)] = \hat q_0(\lambda) + K(\lambda) - e^{i\lambda^3 t}\hat q(\lambda;t);\\ \tilde f_2(\lambda) + e^{-i\omega\lambda}[i\omega\lambda(\alpha-1)\tilde g_1(\lambda) - \tilde g_2(\lambda)] = \hat q_0(\omega\lambda) + K(\omega\lambda) - e^{i\lambda^3 T}\hat q(\omega\lambda;t);\\ \tilde f_2(\lambda) + e^{-i\omega^2\lambda}\, i\omega^2\lambda(\alpha-1)\tilde g_1(\lambda) - \tilde g_2(\lambda) = \hat q_0(\omega^2\lambda) + K(\omega^2\lambda) - e^{i\lambda^3 t}\hat q(\omega^2\lambda;t); \end{cases}$$

$$\text{with } K(\lambda) = -\lambda^2\tilde f_0(\lambda) + e^{-i\lambda}\lambda^2\tilde g_0(\lambda)$$

The determination of the spectral functions $\tilde F$ and $\tilde G$ in terms of given initial and boundary data only is obtained by solving the system - in this example, the explicit system in (A) or (B).

Case (A) The determinant of this linear system is

$$\Delta(\lambda) = (\omega - \omega^2)[e^{-i\lambda} + \omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda}]$$

The zeros lie asymptotically on the rays bisecting the three connected components of D^c, see Figure 4.

These zeros are outside D and in fact it is not possible to deform the integration contour to include them - the integral representation on this case is not equivalent to a series representation.
Case (B) The determinant of this linear system is

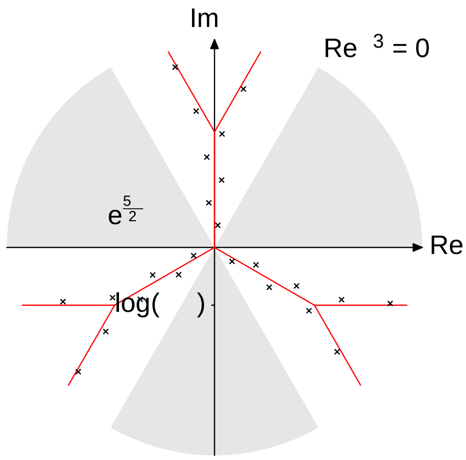

Figure 5: The location of the zeros of ( ) for case (B) - $e^{5=2}$

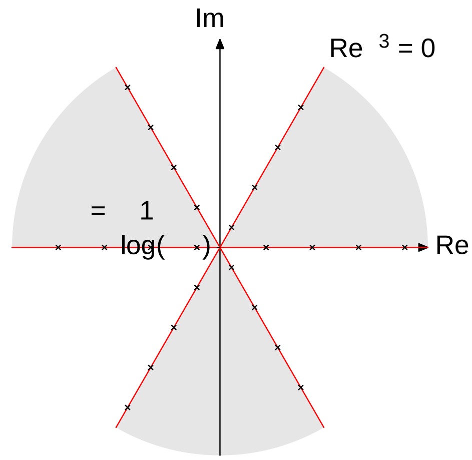

Figure 6: The location of the zeros of ( ) for case (B) - = 1

$$(\ ) = (!\ \ !^2)[e^{\ i} + !\, e^{\ i!} + !^2 e^{\ i!^2} + \ (e^{i} + !\, e^{i!} + !^2 e^{i!^2})]$$

The zeros lie asymptotically on $@D$, see Figures 5 and 6 for the example of two particular values of . Therefore the residues at these zeros must be computed and yield the series representation.

5 An integral transform for nonlinear boundary value problems

The Uni ed Transform to solve boundary value problems for linear equations, outlined in the previous section, is based on deriving an integral transform through a Riemann-Hilbert problem associated with both ODE in the Lax pair. This transform yields (1) an integral representation for the solution and (2) a global relation among certain transforms of the boundary values.
Since the starting point for the approach above is the Lax pair formulation of the PDE, it seems natural to expect that this construction can be generalised to the case of nonlinear integrable equations, that are precisely those characterised by a Lax pair formulation.
I will consider the case of evolution equations posed on a half-line, as this case gives the full avour of the techniques and results.
Consider an integrable nonlinear PDE in the variables x 2 R and t 2 R

Example of such equations are the NLS, KdV and mKdV equations given by (3.2), (1.7), (1.8) respectively, as well as many other important equations of mathematical physics. As for the linear case, it is indeed possible to derive, starting from the Lax pair, a global relation and a formal integral representation, that is now implicitly characterised through a singular linear integral equation.

I have discussed how the idea of the simultaneous spectral analysis of the Lax pair is implemented for the case of linear evolution PDEs, namely by formulating and solving a Riemann-Hilbert problem. The approach can be generalised and remains conceptually the same for the nonlinear case. However, there is an important technical di erence: the associated Riemann-Hilbert problem in the nonlinear case is matrix-valued, hence non commutative, rather than scalar as for the linear case. The lack of commutativity of the Riemann-Hilbert problem implies that it is not possible to write down explicit formulas.

To illustrate the di erence, I sketch the case of the defocusing NLS equation, namely equation (3.2) with $\ \ = 1$ (more details on the construction of the solution representation are given below in section 5.1). Then, in analogy with the case of the full line in section 3.1, the role of the Fourier transform of the initial condition $\hat{q}_0(\ \ )$, where $q_0(x) = q(x; 0)$ is played by the spectral data de ned in terms of the initial information. These spectral data are the pair of functions $a(\ \ )$, $b(\ \ )$ such that

$$\begin{matrix} b(\ \ ) \\ a(\ \ ) \end{matrix} = \begin{matrix} M_{21}(x; 0;\ \ ) \\ M_{22}(x; 0;\ \ ) \end{matrix} :$$

where $M(x; 0;\ \ )$ is the solution of the x part of the Lax pair, i.e. the rst ODE in (3.6), evaluated at $t = 0$. It is useful to express more

Again, symmetry considerations imply that it is enough to consider one column of this function. For example, the second column of $\Phi(t;\lambda)$ satisfies, for $0 < t < T$ and $\lambda \in \mathbb{C}$, the following ODE:

$$\partial_t \begin{pmatrix} \Phi_{21}(t;\lambda) \\ \Phi_{22}(t;\lambda) \end{pmatrix} + 4i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi_{21}(t;\lambda) \\ \Phi_{22}(t;\lambda) \end{pmatrix} = \begin{pmatrix} -i|q(0;t)|^2 & iq_x(0;t) + 2\lambda q(0;t) \\ -i\overline{q_x}(0;t) + 2\lambda\overline{q}(0;t) & i|q(0;t)|^2 \end{pmatrix} \begin{pmatrix} \Phi_{21}(t;\lambda) \\ \Phi_{22}(t;\lambda) \end{pmatrix},$$

$$\begin{pmatrix} \Phi_{21}(0;\lambda) \\ \Phi_{22}(0;\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \qquad \lambda \in \mathbb{C}; \tag{5.3}$$

which again is equivalent to a linear Volterra integral equation, hence well defined.
Note that since in general only one boundary condition involving the two boundary values $q(0;t)$ and $q_x(0;t)$ can be prescribed, the boundary spectral functions $A(\lambda)$, $B(\lambda)$ are not fully characterised by the above ODE.

Nevertheless the solution $q(x;t)$ has a formal representation in terms of the solution $M(x;t;\lambda)$ of a Riemann-Hilbert problem defined on the real and imaginary axes, whose jump is defined in terms of the spectral functions $a(\lambda)$, $b(\lambda)$, $A(\lambda)$ and $B(\lambda)$. The function $q(x;t)$ exists uniquely, has explicit x and t dependence, and it represents a solution of the PDE satisfying the initial condition. However, in general, it will not satisfy prescribed boundary conditions. However, if the full set of boundary values is assumed *a priori* to satisfy the additional constraint given by the global relation, then the function $q(x;t)$ satisfies these boundary values.

The general picture is similar. For many interesting integrable PDEs of mathematical physics (for example NLS, KdV, mKdV, sine-Gordon), the representation of the solution $q(x;t)$ of the PDE is based on the unique solvability of the associated Riemann-Hilbert problem, which derive a solution representation that involves only the prescribed boundary conditions.

I will assume that the answer to the first question is the same as for the linearised version of the PDE. This assumption can be verified a posteriori by using the representation of the solution to prove existence and uniqueness for the given boundary value problem.

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Thus, given $q_0(x) \in S(\mathbb{R}^+)$ and a subset of set of all boundary values $\{f_k(t) = \partial_x^k q(0,t)\}_{k=0}^{n-1}$, the main problem becomes to show that the global relation characterises all other unknown boundary values. Namely, the last step in the full solution of a given, well-posed boundary value problem is the analysis of the invariance of the global relation in the complex plane to determine a representation depending only on the prescribed initial and boundary conditions. As discussed below, this step is fully successful in the nonlinear case only for certain special types of boundary conditions, called *linearisable boundary conditions* in the literature. For generic boundary conditions, the characterisation of the unknown boundary values via the global relation is itself a nonlinear problem, as it can be shown to be equivalent to solving a nonlinear system of equations [28].

5.1 The integral representation of the solution

In this section I summarise the steps to derive the main formal statement regarding integrable evolution PDE in the two independent variables $(x,t) \in \Omega$, where Ω is given by (5.1). Rather than specifying a set of boundary conditions, one assumes a-priori that the initial condition and the full set of boundary values satisfy the global relation. See [41] for the details.

For several of the most physically relevant PDEs in this class, the Lax pair takes the form (3.1). In this form, $M(x,t,\lambda)$ is a 2×2 matrix-valued function, while $f_1(\lambda)$, $f_2(\lambda)$ are given analytic (usually polynomial) functions of λ, encoding the dispersion relation of the PDE. The Lax pair (3.1) can be written in terms of a differential form $W(x,t,\lambda)$ as

$$d[e^{(if_1(\lambda)x+if_2(\lambda)t)\hat{\sigma}_3} M(x,t,\lambda)] = e^{(if_1(\lambda)x+f_2(\lambda)t)\hat{\sigma}_3} W(x,t,\lambda); \tag{5.4}$$

where the meaning of the notation $e^{\hat{\sigma}_3}$ is given in (2.10) and

$$W(x,t,\lambda) = \left[Q(x,t,\lambda)dx + \tilde{Q}(x,t,\lambda)dt\right] M(x,t,\lambda). \tag{5.5}$$

The direct problem

As for the linear case, the first step is constructing simultaneous solutions of the two ODEs in the Lax pair, in such a way that for each $\lambda \in \mathbb{C}$ there is only one solution bounded and analytic in a neighbourhood of Ω. These basic eigenfunctions are given by

$$M_j(x,t,\lambda) = I + \int_{(x_j,t_j)}^{(x,t)} e^{(-if_1(\lambda)(x-\xi)-if_2(\lambda)(t-\tau))\hat{\sigma}_3} W_j(\xi,\tau,\lambda); \quad (x,t);(x_j,t_j) \in \Omega : \tag{5.6}$$

In order to define a solution $M(x,t,\lambda)$ defined and analytic everywhere except on a contour, it is sufficient to consider the points (x_j,t_j) as the the vertices of the unbounded polygon Ω, namely

$$(x_1,t_1) = (0,T); \quad (x_2,t_2) = (0,0); \quad (x_3,t_3) = (\infty,t);$$

Thus one obtains three sectionally analytic basic eigenfunctions, M_1; M_2 and M_3. Their de nition is independently of the path of integration, and the column vectors are bounded and analytic in certain domains. On the common boundary of these domains, the eigenfunctions satisfy the following jump conditions:

$$M_3(x;t;\ ) = M_2(x;t;\ )e^{(\ if_1(\ )x\ \ if_2(\ )t)c_3}s(\ ); \quad 2\ (C\ ;C^+); \tag{5.7}$$
$$M_1(x;t;\ ) = M_2(x;t;\ )e^{(\ if_1(\ )x\ \ if_2(\ )t)c_3}S(\ ); \quad 2\ (D_2;D_3); \tag{5.8}$$

where $2\ (D;\bar{D})$ means the matrix identity is valid for the rst column in the domain D and for the second column in the domain $\bar{D}$,

$$D_1 = f\ : Imf_1(\ ) > 0 \text{ and } Imf_2(\ ) > 0g; \quad D_2 = f\ : Imf_1(\ ) > 0 \text{ and } Imf_2(\ ) < 0g;$$
$$D_3 = f\ : Imf_1(\ ) < 0 \text{ and } Imf_2(\ ) > 0g; \quad D_4 = f\ : Imf_1(\ ) < 0 \text{ and } Imf_2(\ ) < 0g; \tag{5.9}$$

and

$$s(\ ) = M_3(0;0;\ ); \qquad S(\ ) = [e^{if_2(\ )Tb_3}M_2(0;T;\ )]^{\ 1}: \tag{5.10}$$

The spectral functions

Letting

$$(\ x;\ ) = M_3(x;0;\ ); \quad 2\ (C\ ;C^+); \qquad (\ t;\ ) = M_2(0;t;\ ); \quad 2\ C: \tag{5.11}$$

one can write the matrices in (5.10) as

$$s(\ ) = (0\ ;\ ); \qquad 2\ (C\ ;C^+); \tag{5.12}$$
$$S(\ ) = \left[e^{if_2(\ )Tc_3}(\ T;\ )\right]^{\ 1}; \qquad 2\ C: \tag{5.13}$$

Since they solve the two ODEs in the Lax pair, these functions are the solutions of the following linear Volterra integral equations:

$$(\ x;\ ) = I\ \ \int_x^1 e^{\ if_1(\ )(x\ \ )c_3}Q(\ ;0;\ )(\ ;\ )d\ ; \quad x\ 2\ (0;1\ ); \quad 2\ (C\ ;C^+); \tag{5.14}$$

$$(\ t;\ ) = I + \int_0^t e^{\ if_2(\ )(t\ \ )c_3}\tilde{Q}(0;\ ;\ )(\ ;\ )d\ ; \quad t\ 2\ (0;T); \quad 2\ C; \tag{5.15}$$

which are respectively equivalent to ODE (5.2) and to the following analogue of (5.3):

$$@\ \begin{pmatrix} {}_{21}(t;\ ) \\ {}_{22}(t;\ ) \end{pmatrix} + 2if_2(\ )\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} {}_{21}(t;\ ) \\ {}_{22}(t;\ ) \end{pmatrix} = \tilde{Q}(0;t;\ )\begin{pmatrix} {}_{21}(t;\ ) \\ {}_{22}(t;\ ) \end{pmatrix};$$
$$\begin{pmatrix} {}_{21}(0;\ ) \\ {}_{22}(0;\ ) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \qquad 2\ C: \tag{5.16}$$

In particular, the spectral functions satisfy the following:

$s(\ )$ is de ned by the values of the solution at $t = 0$ - the initial condition;

$S(\ )$ is de ned by the values of the solution at $x = 0$ - the boundary values.

The matrices Q, $\tilde{Q}$ for the integrable PDE considered have symmetry properties that imply that $s(\ )$, $S(\ )$ can be written as

$$s(\ ) = \begin{pmatrix} \overline{a(\ )} & b(\ ) \\ \overline{b(\ )} & a(\ ) \end{pmatrix}; \qquad S(\ ) = \begin{pmatrix} \overline{A(\ )} & B(\ ) \\ \overline{B(\ )} & A(\ ) \end{pmatrix}: \tag{5.17}$$

Hence the jump matrices depend on the four distinct functions of the spectral parameter de ned by (5.17).

The global relation

To obtain an additional relation involving the spectral functions $s(\ )$ and $S(\ )$, one observes that the function de ned by (5.11) and the function M_3

$$J_1 = \begin{pmatrix} 1 & 0 \\ (\ \ ) & 1 \end{pmatrix}; \quad J_3 = \begin{pmatrix} 1 & \overline{(\ \ )} \\ 0 & 1 \end{pmatrix};$$

$$J_4 = \begin{pmatrix} \frac{1}{(\ )} & 1 \\ & \frac{(\ )}{j\ (\ )j^2} \end{pmatrix}; \quad J_2 = J_3 J_4^{\ 1} J_1:$$

where

$$(\ \ ) = \frac{b(\ \ )}{a(\ \ )}; \quad (\ \ \ ) = \overline{B\,(}$$

This Riemann-Hilbert problem admits a unique solution $M(x;t; \ )$. In addition the function $q(x;t)$ defined in terms of of M by (5.24) satisfies either the NLS equation or mKdV equations, as well as

$$q(x;0) = q_0(x); \quad q(0;t) = f_0(t); \; q_x(0;t) = f_1(t); \; (q_{xx}(0;t) = f_2(t) \text{ for mKdV }):$$

Remark 5.1 The proof that $q(x;t)$ solves the given nonlinear PDE uses the standard arguments of the dressing method. The proof that $q(x;0) = q_0(x)$ is based on the fact that the Riemann-Hilbert problem satisfied by $M(x;0; \ )$ is equivalent to the Riemann-Hilbert problem defined by $s(\ )$, namely the Riemann-Hilbert problem which characterises $q_0(x)$. The proof that $\partial_x^k q(0;t)$, $k = 0;\ldots;n-1$ are the boundary values of the solution makes crucial use of the global relation [32]. Indeed, the Riemann-Hilbert problem satisfied by $M(0;t; \ )$ is equivalent to the Riemann-Hilbert problem defined by $S(\ )$, which characterises the boundary values, if and only if the spectral functions satisfy this global relation, hence this relation is a necessary and sufficient condition for the existence of a solution.

Remark 5.2 To simplify the exposition and stress the points that are of specific relevance

Linearisable boundary conditions are precisely the conditions such that the components of $(t; \;)$ admit this additional invariance property. A more precise statement is given in the following proposition, see [29].

Proposition 5.1 Suppose that the t part of the Lax pair of an integrable nonlinear PDE is characterised by the scalar function $f_2(\;)$ and by the $2 \quad 2$ matrix-valued function $\tilde{Q}(x; t; \;)$ given in (2.5). Let $(\;)$ be the transformations of complex $(\;)$-plane which leave $f_2(\;)$ invariant.
Define $U(t; \;)$ by

$$U(t; \;) = if_2(\;)\ _3 \quad \tilde{Q}(0; t; \;): \tag{5.25}$$

If it is possible to define a matrix-valued function $N(\;)$, in terms only of the prescribed boundary conditions, such that

$$U(t; \;(\;))N(\;) = N(\;)U(t; \;) \tag{5.26}$$

then the boundary spectral function $A(\;)$, $B(\;)$ defined in (5.17) possess explicit symmetry properties of the form

$$A(!(\;)) = L_1(A(\;); B(\;)); \qquad B(!(\;)) = L_2(A(\;); B(\;))$$

where L_1; L_2 are linear functions of $A(\;)$, $B(\;)$, $\overline{A(\;)}$, $\overline{B(\;)}$ with coefficients depending only on the entries of the matrix $N(\;)$.

When the condition of this proposition is satisfied, the functions $A(\;)$, $B(\;)$ can be computed as effectively as in the linear case in terms of $a(\;)$, $b(\;)$ and the prescribed boundary conditions.
It follows from this proposition that a necessary condition for the existence of linearisable boundary conditions is that the determinant of the matrix $U(t; \;)$ defined by (5.25) is a function of only through $f_2(\;)$. However, this condition is not sufficient. In particular, since the function U depends on the particular choice of Lax pair, it follows that different Lax pairs allow one to uncover different linearisable conditions. An explicit example is given by the case of the sine-Gordon equation, derived following this strategy in [27]. Indeed, using the usual Lax pair for this equation, an invariance property can be established for a constant boundary condition, while an alternative Lax pair leads to the boundary condition (b) in (5.30) below, originally discovered by Sklyanin [59].

Particular conditions that are linearisable for some of the important integrable equations of mathematical physics are listed below:

NLS In this case, there are three linearisable boundary conditions satisfying the necessary condition on the determinant of $U(t; \;)$ with $\tilde{Q}$ defined by (2.5):

$$\text{(a) } q(0; t) = 0; \quad \text{(b) } q_x(0; t) = 0; \quad \text{(c) } q_x(0; t) \quad q(0; t) = 0; \quad 2 R^+; \tag{5.27}$$

KdV- This refers to the KdV equation with dominant surface tension, hence with a negative sign in front of the third derivative term:

$$q_t + q_x \quad q_{xxx} + 6qq_x = 0: \tag{5.28}$$

In this case, $N = 2$ so two boundary conditions must be prescribed at $x = 0$.

$$\text{(a) } q(0;t) = \ ; \ q_{xx}(0;t) = \ + 3 \ ^2; \quad 2 \ R; \tag{5.29}$$

<u>sG</u> I also mention the case of the sine-Gordon equation $q_{tt} \quad q_{xx} + \sin q = 0$, as in this case the two cases of linearisable conditions are obtained by considering the invariance (5.26) with respect to two distinct Lax pairs, [27, 29].

$$\text{(a) } q(0;t) = \ ; \quad 2 \ R; \quad \text{(b) } q_x(0;t) + \ _1 \cos(\frac{q(0;t)}{2}) + \ _2 \sin(\frac{q(0;t)}{2}) = 0\,; \quad _1; \ _2 \ 2 \ R: \tag{5.30}$$

5.3 General boundary conditions

For general boundary conditions, not necessarily linearisable, the invariance analysis of the global relation is not su cient to characterise the solution of the problem without involving the unknown boundary values. However, for general boundary conditions that decay for large t, the representation obtained through the Uni ed Transform yields useful asymptotic information even without the explicit characterisation of the spectral functions.
In addition, two di erent approaches for analysing the generalised Dirichlet to Neumann map for the case of the NLS equation, i.e. to express $q_x(0;t)$ in terms of the given boundary condition $f(t)$ and initial condition $q_0(x)$, have been presented recently in [41, 42].

For non-decaying boundary conditions, the computation of the large t behaviour of the solution and of its boundary values requires new ideas. The most signi cant example of this situation is the case of a time-periodic given boundary condition, an important condition in practice. For example, the KdV equation with given zero initial condition $q(x;0) = 0$ and a periodic boundary condition such as $q(0;t) = a\sin(!t)$, corresponds to the very realistic situation of shallow water waves in a tank, excited by a periodic wavemaker. The linear case of this model is studied in [15].
The rst results on the analysis of a periodic boundary value problem of this type, for the NLS equation, were obtained in [16, 17, 18] for the particular case that $f(t) = ae^{i!t}$. More recently, using the general approach described in this review, coupled with perturbation techniques, signi cant progress has been achieved for the physically signi cant case of the NLS, and mKdV equations given the boundary condition $f(t) = a\sin t$, $a\ 2\ R$, using a

equations. Indeed, for the particular example of NLS, this system is given explicitly as follows

$$A(\ ) = \overline{\varphi_2(T;\ )};\quad B(\ ) = \ e^{4i\ ^2T}\varphi_1(T;\ );$$

where $\varphi_1(t;\ )$, $\varphi_2(t;\ )$ are solutions of

$$\varphi_1(t;\ ) = \int_0^t e^{4i\ ^2(s\ t)}[\ j f_0(t)j^2 \varphi_1 + (2\ f_0(t) + i f_1(t))\varphi_2](s;\ )ds \tag{5.31}$$

$$\varphi_2(t;\ ) = 1 \int_0^t [(2\ f_0(t)\ \ i f_1(t))\varphi_1 + i j f_0(t)j^2 \varphi_2](s;\ )ds \qquad 0 < t < T; \qquad 2\ C:$$

where I use the notation

$$f_0(t) = q(0;t); \qquad f_1(t) = q_x(0;t):$$

By substituting into the equations above the expression for f_0 and f_1 given below in equations (5.32)-(5.33), it becomes apparent that this system is itself nonlinear.

Indeed the following result summarises the situation for the general (non-homogeneous) Dirichlet or Neumann case, directly in terms of boundary functions in physical variables.

Proposition 5.2 Let $T < 1$. Consider the NLS equation on the positive half-line

$$iq_t + q$$

(b) For $f_1(t)$ given,

$$f_0(t) = \frac{1}{} \int_{@D_3} \sim_1 (_3$$

For the semistrip, the analysis is fully carried out in [33] but only for a simple example of linearisable boundary conditions.

The half-plane problem

Consider the Dirichlet problem for the elliptic sine-Gordon equation in the half plane $\{(x; y) \in \mathbb{R}^2 : y > 0\}$. In [37], it is shown that the solution $q(x; y)$ can be expressed in terms of a Riemann-Hilbert problem whose jump matrix is uniquely defined by a certain function $b(\ )$, $\ \in \mathbb{R}$, explicitly expressed in terms of the given Dirichlet datum $g_0(x) = q(x; 0)$ and the unknown Neumann boundary value $g_1(x) = q_y(x; 0)$, where $g_0(x)$ and $g_1(x)$ are related via the global relation, which in this case is the following constraint:

$$b(\ ) = 0 \quad \text{for} \qquad 0: \tag{6.1}$$

Furthermore, it is shown that the latter relation can be used to characterise the Dirichlet to Neumann map, i.e. to express $g_1(x)$ in terms of $g_0(x)$. It appears that this provides the first case that such a map is explicitly characterised for a nonlinear integrable elliptic PDE, as opposed to an evolution PDE.
I give first the main theorem on the representation of the solution under the assumption that the global relation holds. This theorem, analogous to Theorem 5.1 for the evolution case, is based on the analysis of the Lax pair (3.5).

Theorem 6.1 Let the functions $g_0(x)$, $g_1(x)$ be such that $g_0 \quad 2m \ \in H^1(\mathbb{R})$, $m \in \mathbb{Z}$, and $g_1(x) \in H^1(\mathbb{R})$.
Let

$$\omega(\ ) = \frac{1}{4i} \qquad \frac{1}{\ } \ ; \quad (\ ) = \frac{1}{4} \qquad + \frac{1}{\ } \ : \qquad g\ (x$$

Define $a(\ )$ and $b($ H

De ne the following Riemann-Hilbert problem in terms of $b(\ )$:

$$(x;y;\ ) = \ {}^{+}(x;y;\ )J(x;y;\ ); \qquad 2\ R; \qquad = I + O\left(\frac{1}{}\right); \quad !1 \quad ; \tag{6.6}$$

where

$$J = \begin{pmatrix} 1 & b(\ )e^{(x;y;\ )} \\ b(\ )e^{(x;y;\ )} & 1 \end{pmatrix} \qquad (x;y;\ ) = 2(\,!\,(\ )x + (\ )y: \tag{6.7}$$

If the H^1 norm of the data $g_0(x)$, $g_1(x)$ is su ciently small, the above Riemann-Hilbert problem admits a unique solution $(\ x;y;\ )$.
Let the function $q(x;y)$, $x\ 2\ R$, $0 < y <\ 1$, be de ned in terms of this unique solution by

$$iq_x + q_y = \lim_{!1} (i\ )_{12}; \quad \cos q(x;y) = 1 \quad \lim_{!1} 4i \frac{@}{@x}{}_{22} \quad 2 \lim_{!1} (\ )^2_{12}: \tag{6.8}$$

Then $q(x;y)$ solves the elliptic sine-Gordon equation (3.4) in the half plane $y > 0$, and furthermore

$$q(x;0) = g_0(x); \qquad q_y(x;0) = g_1(x); \qquad x\ 2\ R: \tag{6.9}$$

It ,is(6.9)dent th!cFe;9R-s76.372 0 Td [(y2126 Tor)]TJ420(5.1[(y212)]TJF11(y212ev.3724 Td [(x)420(e5 0 Td s:

where the vectors $(m_1(x; \ ); m_2(x; \ ))$ and $(n_1(x; \ ); n_2(x; \ ))$ satisfy the ODEs

$$\begin{cases} (m_1)_x = \frac{i}{\ }(1 \quad \cos g_0(x))\, m_1 \quad [\frac{1}{\ } \sin g_0(x) \quad (x)]\, m_2 \\ (m_2)_x + 2!\,(\ )\, m_2 = [\frac{1}{\ } \sin g_0(x) + \quad (x)]\, m_1 \quad \frac{i}{\ }(1 \quad \cos g_0(x))\, m_2 \end{cases} \quad \lim_{x!1} \quad (m_1; m_2) = (1\,; 0)$$

$$x\ 2\ R; \quad 2\ C^+;$$

$$\begin{cases} (n_1)_x = \frac{i}{\ }(1 \quad \cos g_0(x))\, n_1 \quad [\frac{1}{\ } \sin g_0(x) \quad (x)]\, n_2 \\ (n_2)_x + 2!\,(\ )\, n_2 = [\frac{1}{\ } \sin g_0(x) + \quad (x)]\, n_1 \quad \frac{i}{\ }(1 \quad \cos g_0(x))\, n_2 \end{cases} \quad \lim_{x!\,1} \quad (n_1; n_2) = (1\,; 0)$$

$$x\ 2\ R; \quad 2\ C \ :$$

The semistrip problem

In the case of the sine-Gordon equation posed in a semistrip, there are three unknown boundary values to be determined, one on each of the three boundaries $y = 0$, $y = L$ and $x = 0$. In general, the complexity of this problem appears out of reach of the current techniques.

In [33], the problem is analysed for one particularly simple example of linearisable boundary conditions, namely the case that the prescribed boundary conditions are zero along the unbounded sides of a semistrip and constant along the bounded side. A major di culty for this problem is the existence of non-integrable singularities of the function q_y at the two corners of the semistrip; these singularities are generated by the discontinuities of the boundary condition at these corners. Following the spirit of the recent solution of the analogous problem for the modi ed Helmholtz equation [9], it is possible to introduce an appropriate regularisation which overcomes this di culty. Furthermore, by mapping the basic Riemann-Hilbert problem to an equivalent modi ed Riemann-Hilbert problem, it can be shown that the solution can be expressed in terms of a 2 2 matrix Riemann-Hilbert problem whose jump matrix depends explicitly on the width of the semistrip L, on the constant value d of the solution along the bounded side, and on the residues at the given poles of a certain spectral function denoted by $h(\ )$. The explicit determination of the function h remains open, even for this simplest case of boundary conditions.

7 Conclusions

This review is intended as a summary of the most recent results obtained by the Uni ed Transform, or Fokas Transform, for solving boundary value problems for linear evolution and integrable nonlinear PDEs in two variables.

This method is truly unifying, in the sense that the construction of the formal solution representation follows the same steps in all cases, and is based on solving simultaneously the system given by the Lax pair via a Riemann-Hilbert problem. This leads not only to a formal solution representation involving an integral in the complex plane, but also to the formulation of a global constraint among the boundary values. This constraint, although elementary (it can be derived by a straightforward argument appealing to Green's Theorem in the plane), holds the key to the e ective characterisation of the problem in terms only of the prescribed boundary data. The important conceptual step to unlock the potential of

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