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by

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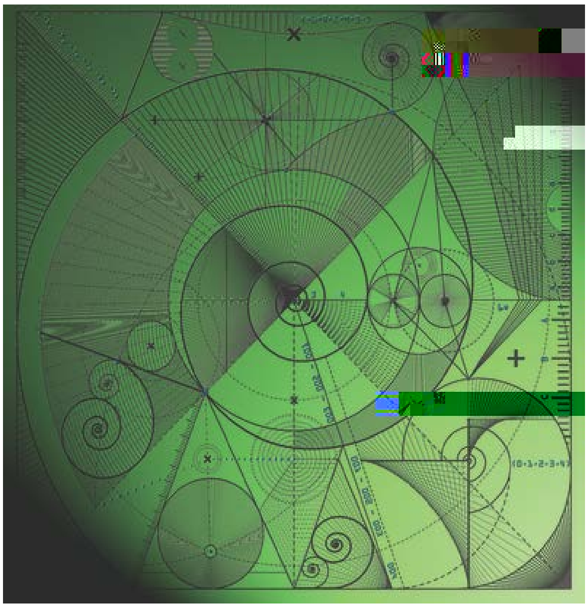

Interpolation of Hilbert and Sobolev Spaces: Quantitative Estimates and Counterexamples

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Abstract

This paper provides an overview of interpolation of Banach and Hilbert spaces, with a focus on establishing when equivalence of norms is in fact equality of norms in the key results of the theory. (In brief, our conclusion for the Hilbert space case is that, with the right normalisations, all the key results hold with equality of norms.) In the nal section we apply the Hilbert space results to the Sobolev spaces $H^s(\ )$ and

[4, 5, 23, 24] and the recent review paper [3] for the Hilbert space case), and it might be thought that there is littl to be said on the subject. The novelty of our presentation|this the perspective of numerical analysts as users of interpolation theory, are ultimately concerned with the computation of interpolation norm the computation of error estimates expressed in terms of interpolation norms|is that we pay parti attention to the question: \When is equivalence of norms in fact equality of norms in the interpolat Banach and Hilbert spaces?"

At the heart of the paper is the study, in Section 3, of the interpolation of Hilbert spaces embedded in a larger linear space V, in the case when the interpolating space is also so-called problem of quadratic interpolation, see, e.g., [2, 3, 10, 15, 17]. The one line sumn is that all the key results of interpolation theory hold with \equality of norms" in place of \ norms" in this Hilbert space case, and this with minimal assumptions, in particular we that our Hilbert spaces are separable (as, e.g., in [2, 3, 15, 17]).

Real interpolation between Hilbert spaces H_0

that, in general (we suspect, in fact, whenever $\Omega \subsetneq \mathbb{R}^n$), $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$ are not exact interpolation scales. Indeed, we exhibit simple examples where the ratio of interpolation norm to intrinsic Sobolev norm may be arbitrarily large. Along the way we give explicit formulas for some of the interpolation norms arising that may be of interest in their own right. We remark that our investigations, which are inspired by applications arising in boundary integral equation methods (see [9]), in particular are inspired by McLean [18], and by its appendix on interpolation of Banach and Sobolev spaces. However a result of §4 is that one result claimed by McLean ([18, Theorem B.8]) is false.

Much of the Hilbert space Section 3 builds strongly on previous work. In particular, our result that, with the right normalisations, the norms in the K- and J-methods of interpolation coincide in the Hilbert space case is a (corrected version of) an earlier result of Ameur [2] (the normalisations proposed and the definition of the J-method norm seem inaccurate in [2]). What is new in our Theorem 3.3 is the method of proof—all of our proofs in this section are based on the spectral theorem that every bounded normal operator is unitarily equivalent to a multiplication operator on $L^2(X;M;\mu)$, for some measure space $(X;M;\mu)$, this coupled with an elementary explicit treatment of interpolation on weighted L^2 spaces—which deals seamlessly with the general Hilbert space case without an assumption of separability or that $H_0 \cap H_1$ is dense in H_0 and H_1. Again, our result in Theorem 3.5 that there is only one (geometric) interpolation space of exponent θ, when interpolating Hilbert spaces, is a version of McCarthy's [17] uniqueness theorem. What is new is that we treat the general Hilbert space case by a method of proof based on the aforementioned spectral theorem. Our focus in this section is real interpolation, but we note in Remark 3.6 that, as a consequence of this uniqueness result (as noted in [17]), complex and real case where $X_1 \subset X_0$. In this case $\Delta = X_1$ and $\Sigma = X_0$ with equivalence of norms, indeed equality of norms if $\|\phi\|_{X_1} \geq \|\phi\|_{X_0}$, for $\phi \in X_1$.

If X and Y are Banach spaces and $B : X \to Y$ is a bounded linear map, we will denote the norm of B by $\|B\|_{X,Y}$, abbreviated as $\|B\|_X$ when $X = Y$. Given compatible pairs $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ one calls the linear map $A : \Sigma(\overline{X}) \to \Sigma(\overline{Y})$ a couple map and writes $A : \overline{X} \to \overline{Y}$, if A_j, the restriction of A to X_j, is a bounded linear map from X_j to Y_j. Automatically $A : \Sigma(\overline{X}) \to \Sigma(\overline{Y})$ is bounded and A_Δ, the restriction of A to $\Delta(\overline{X})$, is also a bounded linear map from $\Delta(\overline{X})$ to $\Delta(\overline{Y})$. On the other hand, given bounded linear operators $A_j : X_j \to Y_j$, for $j = 0, 1$, one says that A_0 and A_1 are *compatible* if $A_0\phi = A_1\phi$, for $\phi \in \Delta(\overline{X})$. If A_0 and A_1 are compatible then there exists a unique couple map $A : \Sigma(\overline{X}) \to \Sigma(\overline{Y})$ which has A_0 and A_1 as its restrictions to X_0 and X_1, respectively.

Given a compatible pair $\overline{X} = (X_0, X_1)$ we will call a Banach space X an *intermediate space* between X_0 and X_1 [5] if $\Delta \subset X \subset \Sigma$ with continuous inclusions. We will call an intermediate space X an *interpolation space* relative to $\overline{X}$ if, whenever $A : \overline{X} \to \overline{X}$, it holds that $A(X) \subset X$ and $A : X \to X$ is a bounded linear operator. Generalising this notion, given compatible pairs $\overline{X}$ and $\overline{Y}$, and Banach spaces X and Y, we will call (X, Y) a pair of interpolation spaces relative to $(\overline{X}, \overline{Y})$ if X and Y are intermediate with respect to $\overline{X}$ and $\overline{Y}$, respectively, and if, whenever $A : \overline{X} \to \overline{Y}$, it holds that $A(X) \subset Y$ and $A : X \to Y$ is a bounded linear operator [5]. If (X, Y) is a pair of interpolation spaces relative to $(\overline{X}, \overline{Y})$ then [5, Theorem 2.4.2] there exists $C > 0$ such that, whenever $A : \overline{X} \to \overline{Y}$, it holds that

$$\|A\|_{X,Y} \leq C \max\left\{\|A\|_{X_0,Y_0}, \|A\|_{X_1,Y_1}\right\}. \tag{1}$$

If the bound (1) holds for every $A : \overline{X} \to \overline{Y}$ with $C = 1$, then (X, Y) are said to be *exact* interpolation spaces: for example the pairs $(\Delta(\overline{X}), \Delta(\overline{Y}))$ and $(\Sigma(\overline{X}), \Sigma(\overline{Y}))$ are exact interpolation spaces with respect to $(\overline{X}, \overline{Y})$, for all compatible pairs $\overline{X}$ and $\overline{Y}$ [5, Section 2.3]. If, for all $A : \overline{X} \to \overline{Y}$,

$$\|A\|_{X,Y} \leq \|A\|^{1-\theta}_{X_0,Y_0} \|A\|^{\theta}_{X_1,Y_1}, \tag{2}$$

then the interpolation space pair (X, Y) is said to be *exact of exponent* θ.

2.1 The K-method for real interpolation

To explain the K-method, for every compatible pair $\overline{X} = (X_0, X_1)$ define the K-functional by

$$K(t, \phi) = K(t, \phi, \overline{X}) := \inf\left\{\left(\|\phi_0\|^2_{X_0} + t^2\|\phi_1\|^2_{X_1}\right)^{1/2} : \phi_0 \in X_0, \ \phi_1 \in X_1, \ \phi_0 + \phi_1 = \phi\right\}, \tag{3}$$

for $t > 0$ and $\phi \in \Sigma(\overline{X})$; our definition is precisely that of [15, p. 98], [6,18]. (More usual, less suited to the Hilbert space case, but leading to the same interpolation spaces and equivalent norms, is to replace the 2-norm $\left(\|\phi_0\|^2_{X_0} + t^2\|\phi_1\|^2_{X_1}\right)^{1/2}$ by the 1-norm $\|\phi_0\|_{X_0} + t\|\phi_1\|_{X_1}$ in this definition, e.g. [5].) Elementary properties of this K-functional are noted in [18, p. 319]. An additional elementary calculation is that, for $\phi \in \Delta$,

$$K(t, \phi) \leq K_1(t, \phi) := \inf_{a \in \mathbb{C}} \left(|a|^2\|\phi\|^2_{X_0} + t^2|1-a|^2\|\phi\|^2_{X_1}\right)^{1/2} = \frac{t\|\phi\|_{X_0}\|\phi\|_{X_1}}{\left(\|\phi\|^2_{X_0} + t^2\|\phi\|^2_{X_1}\right)^{1/2}}, \tag{4}$$

this infimum achieved by the choice $a = t^2\|\phi\|^2_{X_1}/(\|\phi\|^2_{X_0} + t^2\|\phi\|^2_{X_1})$.

Next we define a weighted L^q norm by

$$\|f\|_{\theta,q} := \left(\int_0^\infty |t^{-\theta} f(t)|^q \frac{dt}{t}\right)^{1/q}, \quad \text{for } 0 < \theta < 1 \text{ and } 1 \leq q < \infty,$$

with the modification when $q = \infty$, that

$$\|f\|_{\theta,\infty} := \operatorname*{ess\,sup}_{t>0} |t^{-\theta} f(t)|. \tag{5}$$

Now define, for every compatible pair $\overline{X} = (X_0, X_1)$, and for $0 < \theta < 1$ and $1 \leq q \leq \infty$,

$$K_{\theta,q}(\overline{X}) := \left\{\phi \in \Sigma(\overline{X}) : \|K(\cdot, \phi)\|_{\theta,q} < \infty\right\}, \tag{6}$$

this a normed space (indeed a Banach space [5, Theorem 3.4.2]) with the norm

$$\|\phi\|_{K_{\theta,q}(\overline{X})} := N_{\theta,q}\|K(\cdot, \phi)\|_{\theta,q}. \tag{7}$$

Here the constant $N_{;q} > 0$ is an arbitrary normalisation factor. We can, of course, make the (usual) choice $N_{;q}$

Theorem 2.2. Suppose that $\overline{X} = (X_0; X_1)$ and $\overline{Y} = (Y_0; Y_1)$ are compatible pairs. Then:

(i) For $0 < \theta < 1$, $1 \le q \le \infty$, $(K_{\theta;q}(\overline{X}); K_{\theta;q}(\overline{Y}))$ is a pair of interpolation spaces with respect to $(\overline{X}; \overline{Y})$ that is exact of exponent θ.

(ii) For $0 < \theta < 1$, $1 \le q \le \infty$, $(X_0; X_1)_{\theta;q} = (X_1; X_0)_{1-\theta;q}$, with equality of norms if $N_{\theta;q} = N_{1-\theta;q}$ (which holds for the choice (8)).

(iii) For $0 < \theta_1 < \theta_2 < 1$ and $1 \le q \le \infty$, if $X_1 \subset X_0$, then $X_1 \subset K_{\theta_2;q}(\overline{X}) \subset K_{\theta_1;q}(\overline{X}) \subset X_0$, and the inclusion mappings are continuous. Furthermore, if $\|\phi\|_{X_0} \le \|\phi\|_{X_1}$, for $\phi \in X_1$, then, with the choice of normalisation (8), $\|\phi\|_{K_{\theta_1;q}(\overline{X})} \le \|\phi\|_{K_{\theta_2;q}(\overline{X})}$ for $\phi \in K_{\theta_2;q}(\overline{X})$,

$$\|\phi\|_{X_0} \le \|\phi\|_{K_{\theta_1;q}(\overline{X})}; \quad \text{for } \phi \in K_{\theta_1;q}(\overline{X}); \quad \text{and} \quad \|\phi\|_{K_{\theta_2;q}(\overline{X})} \le \|\phi\|_{X_1}; \quad \text{for } \phi \in X_1.$$

(iv) For $0 < \theta < 1$, $1 \le q < \infty$, $\Delta(\overline{X})$ is dense in $K_{\theta;q}(\overline{X})$.

(v) For $0 < \theta < 1$, $1 \le q < \infty$, where X_j° denotes the closure of $\Delta(\overline{X})$ in X_j,

$$(X_0; X_1)_{\theta;q} = (X_0^{\circ}; X_1)_{\theta;q} = (X \ \overset{1}{\text{and}}1$$

A major motivation for introducing the J-method is the following duality result. Here, for a Banach space X, X^* denotes the dual of X.

Theorem 2.4. If $\overline{X} = (X_0, X_1)$ is a compatible pair and $\Delta(\overline{X})$ is dense in X_0 and X_1, then $\Delta(\overline{X})$ is dense in $\Sigma(\overline{X})$ and $\overline{X}^* := (X_0^*, X_1^*)$ is a compatible pair, and moreover

$$(\Delta(\overline{X}))^* = \Sigma(\overline{X}^*) \quad \text{and} \quad (\Sigma(\overline{X}))^* = \Delta(\overline{X}^*), \tag{15}$$

with equality of norms. Further, for $0 < \theta < 1$, $1 \le q < \infty$, with q^* defined by (14),

$$(X_0, X_1)^*_{\theta,q} = (X_0^*, X_1^*)_{\theta,q^*},$$

with equivalence of norms: precisely, if we use the normalisation (8), for $\phi \in (X_0, X_1)^*_{\theta,q}$,

$$\|\phi\|_{K_{\theta,q^*}(\overline{X}^*)} \le \|\phi\|_{J_{\theta,q}(\overline{X})^*} \quad \text{and} \quad \|\phi\|_{K_{\theta,q}(\overline{X})^*} \le \|\phi\|_{J_{\theta,q^*}(\overline{X}^*)}.$$

Proof. We embed X_j^* in $(\Delta(\overline{X}))^*$, for $j = 0, 1$, in the obvious way, mapping $\phi \in X_j^*$ to its restriction to $\Delta(\overline{X})$, this mapping injective since $\Delta(\overline{X})$ is dense in X_j. That (15) holds is shown as Theorem 2.7.1 in [5]. The remainder of the theorem is shown in the proof of [18, Theorem B.5]. □

The above theorem has the following corollary that is one motivation for our choice of normalisation in (13) (cf., the corresponding result for K-norms in Lemma 2.1 (iii)).

Corollary 2.5. If $\overline{X} = (X, X)$ then $J_{\theta,q}(\overline{X}) = X$ with equality of norms.

Proof. It is clear, from Lemma 2.1 and Theorem 2.3, that $J_{\theta,q}(\overline{X}) = X$. It remains to show equality of the norms which we will deduce from Theorem 2.4 for $1 < q \le \infty$.

We first observe (cf. part (vi) of Theorem 2.2) that, for $0 < \theta < 1$, $1 \le q \le \infty$, it follows immediately from the definitions that if Z_j is a closed subspace of Y_j, for $j = 0, 1$, and $\overline{Z} = (Z_0, Z_1)$, $\overline{Y} = (Y_0, Y_1)$, then $\|\phi\|_{J_{\theta,q}(\overline{Y})} \le \|\phi\|_{J_{\theta,q}(\overline{Z})}$, for $\phi \in J_{\theta,q}(\overline{Z})$. We will apply this result in the case that, for some Banach space X and $j = 0, 1$, $Z_j = X$, and $Y_j = X^{**}$, the second dual of X, recalling that X is canonically and

3 Interpolation of Hilbert spaces

We focus in this section on so-called *quadratic interpolation*, meaning the special case of interpolation where the compatible pairs are pairs of Hilbert spaces and the interpolation spaces are also Hilbert spaces. For the remainder of the paper we assume the normalisations (8) and (13) for the K- and J-methods, and focus entirely on the case

Now we show below that this in mum is achieved for the choice

$$f(t) = \frac{t^2}{(w_0 + w_1 t^2) \int_0^1 s^2 \quad 1 = (w_0 + w_1 s^2)\, ds} = \frac{w \; N^2_{;2} t^2}{w_0 + w_1 t^2}\;; \tag{17}$$

to get the second equality we use that, from (10),

$$\int_0^1 \frac{s^2 \quad 1}{w_0 + w_1 s^2}\, ds = \int_0^1 \frac{s^{1 \quad 2}}{w_0 s^2 + w_1}\, ds = \frac{w_1}{N^2_{;2} w_0 w_1} = \frac{1}{w \; N^2_{;2}}:$$

Substituting from (17) in (16) gives that

$$k \;\; k^2_{J\,;\,2(\overline{H})} = N^2_{;2} \int_X w^2 j \;\; j^2 \int_0^1 \frac{t^{\;1+2}}{w_0 + w_1 t^2}\, dt \quad d \;\; = \int_X w \; j \;\; j^2\, d \;\; = k \;\; k^2_H \;\; :$$

It remains to justify that the in mum is indeed attained by (17). We note rst that the de nition of f implies that $\int_0^1$ (f (t)=t)dt = , so that (11) holds. Now suppose that g is another eligible function such that (11) holds, and let = g f . Then $\int_0^1$ ((t)=t) dt =) d $\int$

Proof. For $j = 0;1$, de ne the non-negative bounded, injective operator A_j : ($\overline{H}$) ! ($\overline{H}$) by the relation $(A_j\ ;\ )_{(\ \overline{H})} = (\ ;\ )_{H_j}$, for ; 2 ($\overline{H}$), where $(\ ;\ )_{(\ \overline{H})}$ denotes the inner product induced by the norm k $k^0_{(\ \overline{H})}$. By the spectral theorem [11, Corollary 4, p. 911] there exists a measure space $(X;M;\ )$, a bounded -measurable function w_0, and a unitary isomorphism U : ($\overline{H}$) ! $L^2(X;M;\ )$ such that

$$A_0 = U\ ^1 w_0 U\ ; \quad \text{for} \quad 2\ (\ \overline{H});$$

and $w_0 > 0$ -almost everywhere since A_0 is non-negative and injective. De ning $w_1 := 1\quad w_0$ we see that $A_1 = U\ ^1 w_1 U$, for 2 ($\overline{H}$), so that also $w_1 > 0$ -almost everywhere.

For 2 ($\overline{H}$),

$$k\ k^2_{H_j} = (U\ ^1 w_j U\ ;\ )_{(\ \overline{H})} = (w_j U\ ; U\ )_{L^2(X;M;\ )} = kU\ k^2_{L^2(X;M;w_j\ )}; \quad \text{for } j = 0;1:$$

Thus, where (similarly to H_1) H_0 denotes the closure of ($\overline{H}$) in H_0, U extends to an isometry U : H_j ! $L^2(X;M;w_j\ )$ for $j = 0;1$. These extensions are unitary operators since their range contains $L^2(X;M;\ )$, which is dense in $L^2(X;M;w_j\ )$ for $j = 0;1$. Where $\overline{H}$:= $(H_0;H_1)$, U extends further to a linear operator U : ($\overline{H}$) ! Y , the space of -measurable functions de ned on X. Thus, applying Corollary 3.2 and noting part (v) of Theorem 2.2, we see that $H = K_{;2}(\overline{H}) = J_{;2}(\overline{H})$, with equality of norms, where

$$H := \quad 2\ (\ \overline{H}) : k\ k_H := kU\ k_{L^2(X;M;w\ )} < 1 \quad ;$$

and $w := w_0^1\quad w_1$. Moreover, for 2 ($\overline{H}$), the unbounded operator T : H_1 ! H_1 satis es $T = U\ ^1(w_0{=}w_1)U$ so that $kS^1\quad k^2_{H_1} = (T^1\quad ;\ )_{H_1} = (A_1 T^1\quad ;\ )_{(\ \overline{H})} = (w\ U\ ; U\ )_{L^2(X;M;\ )} = k\ k^2_H$, for $0 < \ < 1$, and $kS\ k^2_{H_1} = (w_0 U\ ; U\ )_{L^2(X;M;\ )} = k\ k^2_{H_0}$. □

In the special case, considered in [15], that H_0 is densely and continuously embedded in H_1, when ($\overline{H}$) = H_0 and ($\overline{H}$) = H_1, the above theorem can be interpreted as stating that $(H_0;H_1)_{;2}$ is the domain of the unbounded self-adjoint operator S^1 : H_1 ! H_1 (and H_0 the domain of S), this a standard characterisation of the K-method interpolation spaces in this special case, see, e.g., [15, p. 99] or [6]. The following theorem (cf., [6, Theorem B.2]), further illustrating the application of Corollary 3.2, treats the special case when H_1 H_0, with a compact and dense embedding (which implies that both H_0 and H_1 are separable).

Theorem 3.4. Suppose that $\overline{H} = (H_0;H_1)$ is a compatible pair of Hilbert spaces, with H_1 densely and compactly embedded in H_0. Then the operator T : H_1 ! H_1, de ned by

$$(T\ ;\ )_{H_1} = (\ ;\ )_{H_0}; \quad ; \quad 2\ H_1;$$

is compact, self-adjoint, and injective, and there exists an orthogonal basis,

3.2 Uniqueness of interpolation in the Hilbert space case

Theorem 3.3 is a statement that, in the Hilbert space case, three standard methods of interpolation produce the same interpolation space, with the same norm. This is illustrative of a more general result. It turns out, roughly speaking, that all methods of interpolation between Hilbert spaces that produce, for $0 < \quad < 1$, interpolation spaces that are Hilbert spaces and that are exact of exponent , must coincide. To make a precise statement we need the following de nition: given a Hilbert space compatible pair $\overline{H} = (H_0; H_1$

so that

$$p^{n(1 \quad)} k \;\; {}_{n} k^{2}_{H_1} \quad k \quad {}_{n} k^{2}_{H} \qquad p^{(n+1)(1 \quad)} k \;\; {}_{n} k^{2}_{H_1} :$$

Combining these inequalities with (18) (taking $a = p^{n}$, $b = p^{n+1}$) and (19), we see that

$$p^{\;(1 \quad)} k \;\; k^{2}_{G} \quad k \quad k^{2}$$

4 Interpolation of Sobolev spaces

In this section we study Hilbert space interpolation, analysed in Section 3, applied to the classical Sobolev spaces $H^s(\Omega)$ and $\tilde{H}^s(\Omega)$, for $s \in \mathbb{R}$ and an open set Ω. (Our notations here, which we make precise below, are those of [18].) This is a classical topic of study (see, e.g., notably [15]). Our results below provide a more complete answer than hitherto available to the following questions:

(i) Let H_s, for $s \in \mathbb{R}$, denote $H^s(\Omega)$ or $\tilde{H}^s(\Omega)$. For which classes of Ω and what range of s is $\{H_s\}$ an (exact) interpolation scale?

(ii) In cases where $\{H_s\}$ is an interpolation scale but not an exact interpolation scale, how different are the H_s norm and the interpolation norm?

Our answers to (i) and (ii) will consist mainly of examples and counterexamples. In particular, in the course of answering these questions we will write down, in certain cases of interest, explicit expressions for interpolation norms that may be of some independent interest. Our investigations in this section are in very large part prompted and inspired by the results and discussion in [18, Appendix B], though we will exhibit a counterexample to one of the results claimed in [18].

We talk a little vaguely in the above paragraph about "Hilbert space interpolation". This vagueness is justified in Section 3.2 which makes clear that, for $0 < \theta < 1$, there is only one method of interpolation of a pair of compatible Hilbert spaces $\overline{H} = (H_0, H_1)$ which produces an interpolation space $\overline{H}_\theta$ that is a geometric interpolation space of exponent θ (in the terminology of §3.2). Concretely this intermediate space is given both by the real interpolation methods, the K- and J-methods with $q = 2$, and by the complex interpolation method: to emphasise, these methods give the identical interpolation space with identical norm (with the choice of normalisations we have made for the K- and J-methods). We will, throughout this section, use $\overline{H}_\theta$ and $(H_0, H_1)_\theta$ as our notations for this interpolation space and $\|\cdot\|_{\overline{H}_\theta}$ as our notation for the norm, so that $\overline{H}_\theta = (H_0, H_1)_\theta$ and $\|\cdot\|_{\overline{H}_\theta}$ are abbreviations for $(H_0, H_1)_{\theta,2} = K_{\theta,2}(\overline{H}) = J_{\theta,2}(\overline{H})$ and $\|\cdot\|_{K_{\theta,2}(\overline{H})} = \|\cdot\|_{J_{\theta,2}(\overline{H})}$, respectively, the latter defined with the normalisation (9).

4.1 The spaces $H^s(\mathbb{R}^n)$

Our function space notations and definitions will be those in [18]. For $n \in \mathbb{N}$ let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of smooth rapidly decreasing functions, and $\mathcal{S}^*(\mathbb{R}^n)$ its dual space, the space of tempered distributions. For $u \in \mathcal{S}(\mathbb{R}^n)$, $v \in \mathcal{S}^*(\mathbb{R}^n)$, we denote by $\langle u, v\rangle$ the action of v on u, and we embed $L^2(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}^*(\mathbb{R}^n)$ in the usual way, i.e., $\langle u, v\rangle := (u, v)$, where $(\cdot, \cdot)$

4.2 The spaces $H^s()$

For $\quad R^n$ there are at least two de nitions of $H^s()$ in use (equivalent if is su ciently regular). Following [18] (or see [24, Section 4.2.1]), we will de ne

$$H^s() := \quad u\ 2\ D\ () : \ u = Uj \ ; \text{ for some } U\ 2\ H^s(R^n) \ ;$$

where $D\ ()$ denotes the space of Schwartz distributions, the continuous linear functionals on $D()$ [18, p. 65], and Uj denotes the restriction of $U\ 2\ D\ (R^n) \quad S\ (R^n)$ to . $H^s()$ is endowed with the norm

$$kuk_{H^s()} := \inf \quad kUk_{H^s(R^n)} : Uj = u \ ; \quad \text{for } u\ 2\ H^s() \ :$$

With this norm, for $s\ 2\ R$, $H^s()$ is a Hilbert space, $D(\overline{\ }) := f\,Uj : U\ 2\ D(R^n) g$ is dense in $H^s()$, and $H^t()$ is continuously and densely embedded in $H^s()$ with $kuk_{H^s()} \quad k\ uk_{H^t()}$, for $s < t$ and $u\ 2\ H^t()$ [18]. Further $L^2() = \ H^0()$ with equality of norms, so that $H^s() \quad L^2$

Proof. Let $\overline{H}$:= $(H^0(\ )\ ; H^2(\ ))$, for $0 < \ < 1$. Choose an even function 2 C^1 (R) such that 0 (t) 1 for t 2 R, with (t) = 0 if jtj > 1, and (t) = 1 if jtj < 1=2. For 0 < h 1 de ne $_h$ 2 $H^2(\ )$ by $_h(x)$ = $(x_1$=h), x 2 .((We observe that

Except in the case $\Omega = \mathbb{R}^n$, it appears that $\{\widetilde{H}^s(\Omega) : s \in \mathbb{R}\}$ is not an exact interpolation scale. Example 4.15 below shows, for the simple one-dimensional case $\Omega = (0,1)$, that $\{\widetilde{H}^s(\Omega) : 0 \le s \le 1\}$ is not an exact interpolation scale, using a representation for the norm for interpolation between $L^2(\Omega) = \widetilde{H}^0(\Omega)$ and $\widetilde{H}^1(\Omega)$ given in the following lemma that illustrates the abstract Theorem 3.4 (cf., [14, Chapter 8]). For the cusp domain example of Lemma 4.8, by Lemma 4.8 and Corollary 4.9, $\{\widetilde{H}^s(\Omega) : -2 \le s \le 0\}$ is not an interpolation scale at all.

Lemma 4.11. Let Ω be bounded and set $H_0 := \widetilde{H}^0(\Omega) = L^2(\Omega)$, $H_1 := \widetilde{H}^1(\Omega) =$

Lemma 4.13. If = (0 ; a), with a > 0, then f H ^{s}( ) : 0 s 2g is not an exact interpolation scale. In particular, where

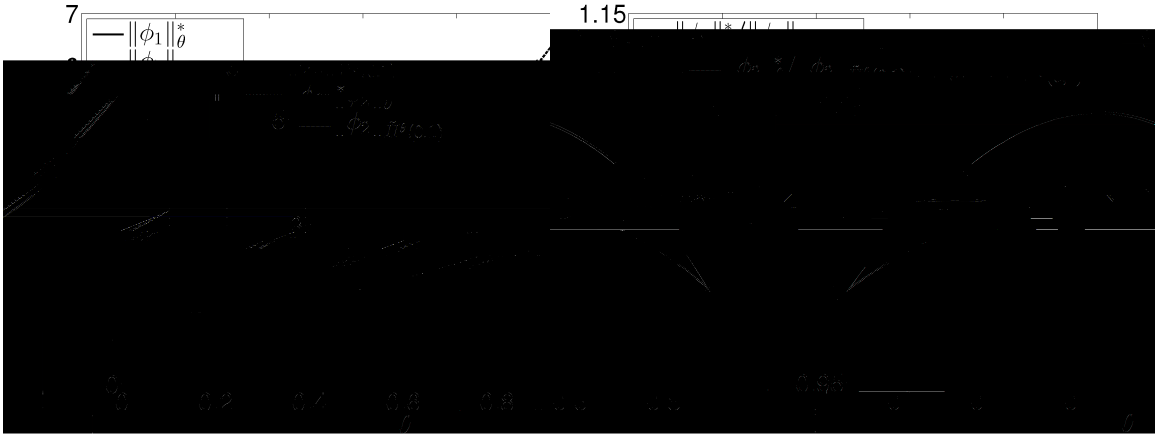

Figure 1: Comparison of Sobolev and interpolation norms in $\widetilde{H}^\theta(\Omega)$, for the functions ϕ_1 and ϕ_2.

Our last example uses the results of Lemma 4.11, and shows that $\{\widetilde{H}^s(0,1) : 0 \le s \le 1\}$ is not an exact interpolation scale by computing values of the Sobolev and interpolation norms for specific functions. This example also demonstrates that no normalisation of the interpolation norm can make the two norms equal.

Example 4.15. Let $\Omega = (0,1)$, $H_0 = \widetilde{H}^0(\Omega) = L^2(\Omega)$ and $H_1 = \widetilde{H}^1(\Omega) = H^1_0(\Omega)$. The eigenfunctions and eigenvalues in Lemma 4.11 are $\phi_j(x) = \sqrt{2}\sin(j\pi x)$ and $\lambda_j = j^2\pi^2$, so that, for $0 < \theta < 1$, the interpolation norm on $\overline{H}_\theta = \widetilde{H}^\theta(\Omega)$ is given by (24). In particular,

$$\|\phi_j\|_\theta = (1 + j^2\pi^2)^{\theta/2}, \quad \text{for } j \in \mathbb{N}.$$

Noting that

$$\hat{\phi}_j(\xi) = \frac{1}{\sqrt{\pi}}\int_0^1 \sin(j\pi x)e^{-i\xi x}\,dx = \frac{j\sqrt{\pi}\,\big(1 - (-1)^j e^{-i\xi}\big)}{j^2\pi^2 - \xi^2} = \frac{2j\sqrt{\pi}\,e^{-i\xi/2}}{j^2\pi^2 - \xi^2}\begin{cases}\cos\xi/2, & j \text{ odd},\\ i\sin\xi/2, & j \text{ even},\end{cases}$$

it holds that

$$\|\phi_j\|_{\widetilde{H}^\theta(\Omega)} = \left(\int_{-\infty}^{\infty}(1+\xi^2)^\theta\,|\hat{\phi}_j(\xi)|^2\,d\xi\right)^{1/2} = 2j\sqrt{2\pi}\left(\int_0^\infty \frac{(1+\xi^2)^\theta}{(j^2\pi^2-\xi^2)^2}\begin{Bmatrix}\cos^2(\xi/2)\\ \sin^2(\xi/2)\end{Bmatrix}d\xi\right)^{1/2}.$$

A comparison of $\|\phi_j\|_\theta$ and $\|\phi_j\|_{\widetilde{H}^\theta(\Omega)}$ for $j = 1,2$ and $\theta \in (0,1)$ is shown in Figure 1(a). It is clear from Figure 1(a) that the interpolation and Sobolev norms do not coincide in this case. In particular, for $\theta = 1/2$ we have

$$\|\phi_1\|_{1/2} \approx 1.816, \quad \|\phi_1\|_{\widetilde{H}^{1/2}(\Omega)} \approx 1.656, \quad \|\phi_2\|_{1/2} \approx 2.522, \quad \|\phi_2\|_{\widetilde{H}^{1/2}(\Omega)} \approx 2.404.$$

The ratio between the two norms is plotted for both ϕ_1 and ϕ_2 in Figure 1(b). In particular,

$$\|\phi_1\|_{1/2}/\|\phi_1\|_{\widetilde{H}^{1/2}(\Omega)} \approx 1.096, \quad \|\phi_2\|_{1/2}/\|\phi_2\|_{\widetilde{H}^{1/2}(\Omega)} \quad 1\ 1$$

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