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Schatten Class Toeplitz Operators on Generalized Fock Spaces

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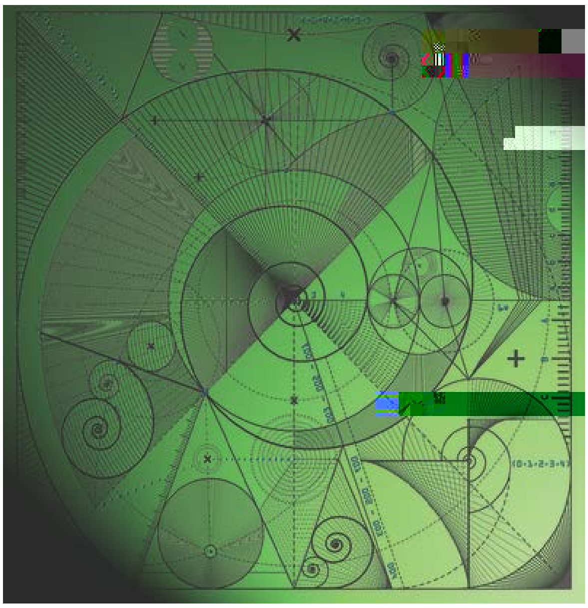

SCHATTEN CLASS TOEPLITZ OPERATORS ON GENERALIZED FOCK SPACES

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Abstract. In this paper we characterize the Schatten p class membership of Toeplitz operators with positive measure symbols acting on generalized Fock spaces for the full range $0 < p < \infty$.

1. Introduction

Let $d^c = \frac{i}{4}(\bar{\partial} - \partial)$ and let d be the usual exterior derivative. Throughout the paper, let $\varphi \in C^2(\mathbb{C}^n)$ be a real

In this paper we will provide very similar characterizations of the Schatten class membership of these Toeplitz operators. Note that unlike the classical Fock space setting where one can utilize explicit formulas for the reproducing kernel, we instead must rely on some known estimates on the behavior of the reproducing kernel (see the rst three lemmas in the next section). The proofs of our characterizations will

Lemma 2.4. Suppose that 0; $r >$ 0; and $0 < p$ 1: If satis es condition (2:1), then the following are equivalent:

(a) e 2 $L^p(C^n; dv)$
(b) $(B(; r))$ 2 $L^p(C^n; dv)$
(c) f $(B(a_m; r))$g 2 l^p

Proof. The equivalence of (b) and (c) for any $r >$ 0 was proved in [6], where it was also proved that (b) and (c) are in fact independent of $r >$ 0. Thus, we will complete the proof by showing that (a) () (c) for some $r >$ 0.

First assume that (c) is true. Then by Lemma 2.3 we have that

$$e(z) = \int_{C^n} jk_z(w)j^2 e^{\ 2\ (w)}\, d\ (w)$$

Thus, we have that

$$(2.5)\qquad \int_{\mathbb{C}^n \setminus B(a_m;2r)} \left(\int_{B(a_m;r)} e^{\ _0|z-u|}\, dv(u) \right)^p dv(z) \lesssim r^{2np} \int_{\mathbb{C}^n \setminus B(a_m;2r)} e^{-\frac{_0 p}{2}|z-a_m|}\, dv(z) \lesssim r^{2np}.$$

Finally, combining (2.3) with (2.4) and (2.5) we have that

$$\int_{\mathbb{C}^n} (e(z))^p\, dv(z) \quad C_r \sum_{m=1}^{\infty} ((B(a_m;2r)))^p < \infty$$

for some $C_r > 0$ since (c) is independent of $r > 0$.

We now complete the proof by showing that (a) $\Rightarrow$ (c) for $r = \frac{}{2}$ where is from Lemma 2.1. In particular,

$$\int_{\mathbb{C}^n} (e(z))^p\, dv(z) \gtrsim \sum_{m=1}^{\infty} \int_{B(a_m;\ /2)} (e(z))^p\, dv(z).$$

Moreover, if $z \in B(a_m;\ /2)$ then Lemma 2.2 gives us that

$$e(z) \quad \int_{B(a_m;\ /2)} |k_z(u)|^2 e^{-2\ (u)}\, d\ (u) \gtrsim \ (B(a_m;\ /2))$$

which immediately implies that (c) is true with $r = \frac{}{2}$.

Lemma 2.5. Suppose that 0 and satisfies condition (2.1). Then

(a) $T \in S_p$ if $e \in L^p(\mathbb{C}^n; dv)$ and $0 < p \quad 1$;
(b) $e \in L^p(\mathbb{C}^n; dv)$ if $T \in S_p$ and $1 \quad p < \infty$.

Proof. Since $e \in L^p(\mathbb{C}^n; dv)$ implies that $\{ (B(a_m;r))\}$ is bounded by Lemma 2.4, we first of all have that T is bounded on F^2 by Theorem 1 in [4]. Furthermore, since $\sqrt{K(z;z)} \quad e^{\ (z)}$, one can repeat virtually word for word the arguments on pp. 96–97 in [6] to complete the proof.

We will need one more lemma before we prove the main result of this section. Note that this lemma is in fact a standard result in frame theory, though we will include its simple proof for the sake of completion.

Lemma 2.6. Let $r > 0$ and let f e_m ;n6 Tf 8.a(C)]TJ/F10 6.9738 Tf 7.195 3.615 Td [(n)]TJ/F11 9.9626 Tf 5.423

Proof. If $f, g \in F^2$, then the Cauchy-Schwarz inequality, Lemma 2.3, and the reproducing property gives us that

$$
\begin{aligned}
|\langle Af, g\rangle| &\leq \sum_{m=1}^{\infty} |\langle f, e_m\rangle \langle k_{\sigma_m}, g\rangle| \\
&\leq \|f\|_{F^2} \left(\sum_{m=1}^{\infty} |\langle k_{\sigma_m}, g\rangle|^2 \right)^{\frac{1}{2}} \\
&\leq \|f\|_{F^2} \left(\sum_{m=1}^{\infty} |g(\sigma_m) e^{-\phi(\sigma_m)}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{F^2} \left(\sum_{m=1}^{\infty} \int_{B(\sigma_m, r)} |g(u) e^{-\phi(u)}|^2 \, dv(u) \right)^{\frac{1}{2}} \lesssim \|f\|_{F^2} \|g\|_{F^2}
\end{aligned}
$$

Note that $\|\cdot\|_{S_p}$ is not a norm when $p < 1$. However, it is well known that if A and B are compact, then

$$s_{m+n-1}(A + B) \leq s_m(A) + s_n(B)$$

where $s_k(T)$ is the k^{th} singular value of a compact operator T. Thus, it is easy to see that for all $0 < p \leq 1$ we have

and clearly we have a similar estimate for $|u \quad _k|$.

Plugging this into (2:10) gives us that

$$\int_{C^n} e^{\,\frac{0}{2}|u \quad _m|} e^{\,\frac{0}{2}|u \quad _k|}\, d\;(u) \quad \sum_{j=1}^{\infty} (B(\;_j;\;))\, e^{\,\frac{0}{4}|\;_j \quad _m|} e^{\,\frac{0}{4}|\;_j \quad _k|}: \tag{2.11}$$

Thus, since $0 < p \quad 1$, we can plug (211) into (2:9) to get that

$$\begin{aligned} \|E\|_{S_p}^p &\lesssim e^{\,\frac{0pR}{2}} \sum_{j=1}^{\infty} (B(\;_j;\;))^p \sum_{m \neq k} e^{\,\frac{0p}{4}|\;_j \quad _m|} e^{\,\frac{0p}{4}|\;_j \quad _k|} \\ & \quad e^{\,\frac{0pR}{2}} \sum_{j=1}^{\infty} (B(\;_j;\;))^p \left(\sum_{m=1}^{\infty} e^{\,\frac{0p}{4}|\;_j \quad _m|} \right)^2 \\ &\lesssim e^{\,\frac{0pR}{2}} \sum_{j=1}^{\infty} (B(\;_j;\;))^p \end{aligned}$$

which means that there exists $C_2 > 0$ independent of N where

(2.12)

Theorem 3.2. Suppose 0, p 1, and satis es condition (2.1). Then the following are equivalent:

(a) T is in the Schatten class S_p;
(b) e 2 $L^p(C^n; dv)$;
(c) (B(; r)) 2 $L^p(C^n; dv)$;
(d) f (B(a_m; r))g 2 l^p.

Proof. That (c) and (d) are equivalent and that both conditions are independent of r > 0 was proved in [6] when n = 1, though the case n > 1 is analogous. Note that (a) implies (b) follows from Lemma 2.5 and the easy proof that (b) implies (d) is analogous to the case 0 < p 1.

We will nish the proof by showing that (c)) (a), so suppose that $\hat{}_r$ = (B(; r)) 2 $L^p(C^n; dv)$. Then by Fubini's theorem and the reproducing property

$$h T_{\hat{}_r} f; f i_{F^2} = \int_{C^n} \int_{C^n} {}_{B(w;r)}(z) \, jf(w)j^2 e^{-2 (w)} \ldots$$

(z) dv(w) 0 Td [(=)]TJ/F1 9.9626 Tf 7.4 0 Td [()]TJ -202.498 -30.497 Td [(=)]TJ/F1 9.9626 Tf 10.516

$$= \int_{C^n} \int \ldots z) \, dv =$$