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Darboux-Bäcklund transformations, dressing & impurities in multi-component vector NLS

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Contents

transformation i.e. the defect for the discrete and continuous vector NLS model. The sl_2 discrete NLS model was studied in [17], whereas the continuous generalized NLS in [19]. Here we generalize the study of the discrete vector NLS, and then we also consider the continuous vector NLS model. In the continuous case we mainly focus on the time evolution associated to the defect, and inspired by this we give some generic expressions on the Bäcklund transformations and dressing. A brief discussion of a novel class of BTs that associate solitonic with anti-solitonic solutions is also presented.

More precisely, the outline of this paper is as follows: in Section 2 we present the discrete vector NLS model, after a brief review we study the model in the presence of a local defect in section 3. The associated integrals of motion and the corresponding time components of the Lax pairs are presented. In Section 4 we focus on the dressing and Bäcklund transformations (BT) for the continuous vector NLS equation. We treat both the focusing and defocusing cases simultaneously using an appropriate symmetry of the related Lax pair. Such symmetry groups, known as reduction groups, were first introduced in [21, 22, 23] and later developed in, e.g. [24]. In particular, we present the dressing transformation and give the general higher rank 1-soliton solution, as well as the n-soliton solution as a ratio of determinants. Moreover, we obtain the Bäcklund transformation for the vector NLS model, which generalises the BT for the focusing and defocusing scalar NLS equation presented in [25, 26]. We also briefly discuss the existence of a novel class of BTs, which essentially relates each field to its conjugate or in other words solitonic solutions to anti-solitonic ones. In Section 5 we provide a generic description of the ZS dressing and the Darboux-Bäcklund transforms as integral representations. The novel case of different spectral parameters associated to each field even in the case of "one-soliton" solution is discussed. This picture is more in tune with the quantum picture and the nested Bethe ansatz formulation. Both the discrete and continuous spectrum are discussed for the vNLS model.

2 Discrete vector NLS

Let us first focus our analysis on the discrete vector NLS model, generalizing essentially the results presented in [17], where the sl_2 NLS model was studied in the presence of point-like defects. We shall focus mainly on the time evolution of the degrees of freedom associated to the defect obtained essentially as equations of motion of the system evaluated on the defect point. We consider the following linear system [5]

$$\begin{aligned} \Psi_{j+1} &= L_j \Psi_j \\ \frac{d}{dt}\Psi_j &= A_j \Psi_j \, , \end{aligned} \tag{2.1}$$

for an auxiliary function Ψ, with (L, A) the Lax pair. Here j denotes the lattice site on a one-dimensional N-site periodic lattice. The compatibility condition of the above system of equations reads

$$\frac{d}{dt} L_j = A_{j+1} L_j - L_j A_j \, , \tag{2.2}$$

and is equivalent to the discrete (differential-difference) equation at hand.

In the case of the discrete $\mathfrak{gl}_{\mathcal{N}}$ NLS model, the associated Lax operator L (acting on site j) is given by

$$L_j(\lambda) = \Big(1 + \lambda + \sum_{k=1}^{\mathcal{N}-1} x_j^{(k)} X_j^{(k)}\Big) e_{11} + \sum_{k=2}^{\mathcal{N}} e_{kk} + \sum_{k=2}^{\mathcal{N}} \Big(x_j^{(k-1)} e_{1k} + X_j^{(k-1)} e_{k1} \Big) \, , \tag{2.3}$$

where λ is the spectral parameter and e_{kl} are $N \times N$ matrices such that $(e_{kl})_{pq} = \delta_{kp}\delta_{lq}$. The Lax operator (2.3) satisfies the quadratic algebra [5]

Hence, expansion of the transfer matrix $\tau(\lambda)$ in powers of the spectral parameter λ provides the charges in involution. To obtain the local integrals of motion, expansion of $\ln\tau(\lambda)$ is required instead.

In order to obtain the associated integrals of motion it is convenient to utilise the bra-ket notation for a $(\mathcal{N}-1)$-dimensional vector and co-vector, in other words,

$$|X\rangle := \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(\mathcal{N}-1)} \end{pmatrix}, \qquad \langle x| := \begin{pmatrix} x^{(1)} & x^{(2)} & \dots & x^{(\mathcal{N}-1)} \end{pmatrix}, \tag{2.9}$$

and also define

$$\mathrm{N}_j = 1 + \sum_{k=1}^{\mathcal{N}-1} x_j^{(k)} X_j^{(k)} := 1 + \langle x_j|X_j\rangle . \tag{2.10}$$

Then the Lax operator (2.3) can be written in block matrix form as

$$\mathrm{L}_j = \lambda D_j + A_j = \lambda \begin{pmatrix} 1 & \langle 0| \\ |0\rangle & 0 \end{pmatrix} + \begin{pmatrix} \mathrm{N}_j & \langle x_j| \\ |X_j\rangle & \mathbb{1} \end{pmatrix}, \tag{2.11}$$

where 0 and $\mathbb{1}$ denote the $(\mathcal{N}-1)\times(\mathcal{N}-$

3 Vector DNLS in the presence of defects

The integrals of motion take the form

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However, in order to derive the equations of motion for the fields in the neighbourhood of the defect, i.e. for $j = n \pm 1, n \pm 2$, one must take into account expressions (3.10). Hence, we obtain the following differential-difference equations for $\langle x|$, $|X\rangle$

$$
\begin{aligned}
\langle \dot{x}_{n-2}| &= N^2_{n-2}\langle x_{n-2}| - \langle x_{n-1}|X_{n-2}\rangle\langle x_{n-2}| - \langle x_{n}\text{[illegible]}X_{n-3}\rangle\langle x_{n-2}| - (N_{n-1} + N_{n-2})\langle x_{n-1}| \\
&\quad + \langle \beta_n| + \langle x_{n+1}|\,, \\
|\dot{X}_{n-2}\rangle &= -|X_{n-2}\rangle N^2_{n-2} + |X_{n-2}\rangle\langle x_{n-1}|X_{n-2}\rangle + \text{[illegible]}_{-2}\rangle\langle x_{n-2}|X_{n-3}\rangle + |X_{n-3}\rangle(N_{n-2} + N_{n-3}) \\
&\quad - |X_{n-4}\rangle\,,
\end{aligned}
\qquad (3.12)
$$

$$
\begin{aligned}
\langle \dot{x}_{n-1}| &= N^2_{n-1}\langle x_{n-1}| - \langle x_{n+1}|X_{n-1}\rangle\langle x_{n-1}| - \langle x_{n}\text{[illegible]}X_{n-2}\rangle\langle x_{n-1}| - \langle \beta_n|X_{n-1}\rangle\langle x_{n-1}| \\
&\quad - \alpha_n\langle \beta_n| - N_{n-1}\langle \beta_n| - (N_{n-1} + N_{n+1}\text{[illegible]}
\end{aligned}
$$

associated to the defect

$$\dot{\alpha}_n = (\alpha_n + N_{n-1})\langle\beta_n|X_{n-1}\rangle - (\alpha_n + N_{n+1})\langle x_{n+1}|\gamma_n\rangle + \langle x_{n+2}|\gamma_n\rangle - \langle\beta_n|X_{n-2}\rangle\,,$$

$$\langle\dot{\beta}_n| = -\langle x_{n+1}|X_{n-1}\rangle\langle\beta_n| - \langle x_{n+1}|\gamma_n\rangle\langle\beta_n| - \langle\beta_n|X_{n-1}\rangle\langle\beta_n| + \alpha_n^2\langle\beta_n|$$

$$+$$

where

$$Q = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\kappa \end{pmatrix}, \quad \kappa = \pm 1. \tag{4.4}$$

The $L(\lambda)$ in (4.3) denotes the formal adjoint operator of $L(\lambda)$, i.e. $L(\lambda) = -D$

Taking the poles at $\lambda = \mu$ and μ^* of equation $M(\lambda)M(\lambda)^{-1} = \mathbb{1}$ we obtain

$$M_0 \left(\mathbb{1} + \frac{Q M_0 Q}{\mu - \mu^*} \right) = 0, \qquad \left(\mathbb{1} + \frac{M_0}{\mu^* - \mu} \right) Q M_0 Q = 0 \tag{4.20}$$

respectively. Assuming that $\det(M_0) \neq 0$ we have that $\det\left(Q M_0 Q\right) \neq 0$ and thus the second equation of (4.20) implies that $M_0 = (\mu - \mu^*)\mathbb{1}$. In this case $M(\lambda) = \frac{\lambda-\mu^*}{\lambda-\mu}\mathbb{1}$ and so M is a trivial Darboux matrix. Hence we assume that M_0 is not of full rank and specifically we are interested in the case where $\operatorname{rank}(M_0) = 1$ and thus $M(\lambda)$ will be the simplest Darboux matrix (elementary Darboux matrix). ∎

4.2 Dressing and Bäcklund transformations

where $\Psi(\mu')$ is the fundamental solution of the linear problem

$$\Psi_x = U(\mu')\Psi, \quad \Psi_t = V(\mu')\Psi \tag{4.33}$$

and C is a constant matrix of dimension $N \times s$.

Using (4.32) and the expression for P (4.23), we can write the projector matrix P in terms of solutions of the linear system (4.33) that correspond to the vNLS potential $|u\rangle = (u_1, \dots, u_{\mathcal{N}-1})^T$. The first equation of (4.27) defines the transformation for the vNLS equation

$$\tilde{u}_j = u_j + \frac{i(\mu' - \mu)}{\sqrt{\kappa}} P_{\mathcal{N}j}, \quad j = 1, \dots, N-1 \tag{4.34}$$

and $P_{\mathcal{N}j}$ can be written as a ratio of two determinants

$$P_{\mathcal{N}j} = \frac{\begin{vmatrix} 0 & \kappa q_{\mathcal{N}}^{1*} & \cdots & \kappa q_{\mathcal{N}}^{s*} \\ q_j^1 & q^{1T} Q q^{1*} & \cdots & q^{1T} Q q^{s*} \\ \vdots & \vdots & \ddots & \vdots \\ q_j^s & q^{sT} Q q^{1*} & \cdots & q^{sT} Q q^{s*} \end{vmatrix}}{[illegible]}$$

arbitrary N. To this end we first use the rescaling symmetry (4.26) and write q in the following form

$$q = \begin{pmatrix} |q\rangle \\ 1 \end{pmatrix}.$$

Using relation (4.41) and (4.44) equation (4.47) takes the form

$$i\,(|\tilde{u}\rangle - |u\rangle)_x = -\mu\,(|\tilde{u}\rangle - |u\rangle) + \left(\frac{\mu' - \mu}{2} \pm \eta\right)|u\rangle - \frac{|\tilde{u}|^2 - \langle\tilde{u}|u\rangle}{|\tilde{u} - u|^2}\left(\frac{\mu' - \mu}{2} \mp \eta\right)(|\tilde{u}\rangle - |u\rangle) \tag{4.49}$$

and constitutes the $x-$part of the Bäcklund transformation of vNLS while (4.48) can be rewritten as

$$\begin{aligned} i(|\tilde{u}\rangle - |u\rangle)_t &= -i\mu(|\tilde{u}\rangle - |u\rangle)_x - i\left(\frac{\langle\tilde{u}_x|\tilde{u}\rangle - \langle\tilde{u}_x|u\rangle}{|\tilde{u} - u|^2}\right)\left(\frac{\mu' - \mu}{2} \mp \eta\right)(|\tilde{u}\rangle - |u\rangle) \\ &+ \kappa|\tilde{u}|^2(|\tilde{u}\rangle - |u\rangle) + \kappa\left(\langle\tilde{u}|u\rangle - |u|^2\right)|u\rangle + i\left(\frac{\mu' - \mu}{2} \pm \eta\right)|u_x\rangle. \end{aligned} \tag{4.50}$$

When $N = 2$ the Bäcklund transformation (4.49), (4.50) becomes the known BT for NLS equation with $\kappa = \pm 1$ (see [25]).

Bäcklund transformations: solitons to anti-solitons

We shall briefly discuss here the existence of a novel type of Bäcklund transformations that associate solitonic to anti-solitonic solutions. This idea is essentially inspired by the existence of certain boundary conditions in high rank gl_N integrable systems that force a soliton to reflect as an anti-soliton. In the language of representation theory in

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$$T(b,a,\lambda) = \exp\left(\int_a^b dx\, U(x,\lambda)\right). \tag{4.53}$$

In the case where r is the Yangian matrix the $r_{ab}^{T_a T_b} = r_{ab}$. Working out the BT for the setting above we end up to structurally similar BTs as the ones defined earlier in the text, but now the following identifications hold:

$$\lambda \to -\lambda, \quad |\tilde{u}\rangle \to |u'\rangle, \quad |\tilde{u}'\rangle \to |u\rangle. \tag{4.54}$$

In the vNLS case the situation is quite straightforward, however more interesting and presumably richer scenarios could arise in more involved models, such the affine Toda field theories or higher rank Landau-Lifshitz models. Also, this setting naturally applies to discrete integrable modes associated to higher rank algebras. All these are significant issues that will be discussed in detail in future investigations, given that our main purpose here is to provide a brief introduction to the soliton anti-soliton type BTs.

4.3 Higher Darboux transformation

In this section we investigate Darboux-dressing transformations that correspond to multi-soliton solutions. In principle, in order to obtain higher soliton solutions one can consider compositions of elementary Darboux transformations of the form (4.18) with several different poles in λ, see [5, 31]. However, here we are interested in a non-elementary Darboux matrix, which has n poles and is of the form

$$M(\lambda) = \mathbb{1} + \sum_{i=1}^{n} \frac{M_i}{\lambda - \mu_i}. \tag{4.55}$$

Moreover, we assume that $M(\lambda)$ has the same structure as the 1-soliton Darboux matrix, that is it satisfies relation (4.14). It follows that the inverse matrix is of the form

$$M(\lambda)^{-1} = \mathbb{1} + \sum_{i=1}^{n} \frac{QM_iQ}{\lambda - \mu_i'}. \tag{4.56}$$

Comparing the asymptotic expansions of $M(\lambda)$ and $M(\lambda)^{-1}$ at $\lambda \to \infty$ we also obtain the following relation

$$\sum_{i=1}^{n} M_i = -\sum_{i=1}^{n} QM_iQ\,. \tag{4.57}$$

Taking the residue at $\lambda = \mu_j$ and μ_j' of equation $M(\lambda)M(\lambda)^{-1} = \mathbb{1}$ we have that

$$M_j M(\mu_j)^{-1} = 0, \quad M(\mu_j')QM_jQ = 0, \quad j = 1, \dots n, \tag{4.58}$$

respectively. The above equations imply that all $\mathbf{M_j}$ are not of full rank. In general we can proceed assuming that $\mathrm{rank}(\mathbf{M_j}) = s_j$ with $1 \leq s_j \leq N - 1$ but instead we will treat only the case where $\mathrm{rank}(\mathbf{M_j}) = 1$ for all j.

As in the 1-soliton case, we can express all $\mathbf{M_j}$ in the form $\mathbf{M_j} = \mathbf{p_j} \mathbf{q_j^T}$ where $\mathbf{p_j}$ and $\mathbf{q_j}$ are N-vectors. Then, equations (4.58) imply that

$$\mathbf{q_j^T} \mathbf{M}(\mu_j)^{-1} = 0, \quad \mathbf{M}(\mu_j) Q \mathbf{q_j} = 0, \quad j = 1, \ldots, n \tag{4.59}$$

respectively. As the relations in (4.59) are equivalent to each other, using one of them we have that

$$\sum_{i=1}^{n} \frac{\mathbf{q_i^T} Q \mathbf{q_j}}{\mu_i - \mu_j} \mathbf{p_i} = Q \mathbf{q_j}, \quad j = 1, \ldots, n \tag{4.60}$$

with $\mu_i \neq \mu_j$ for all i, j. We define the scalar quantities (q q where the C_j are constant N-vectors and $\Psi(\mu_j)$ is a fundamental solution to the linear problem at $\lambda = \mu_j$.

Using equation (4.61), relation (4.62) is the dressing transformation which can be written as

$$\tilde{u}_i = u_i - \frac{j}{\sqrt{\kappa}} \frac{\tau_i}{\tau}, \quad i = 1, \cdots, N - 1, \tag{4.66}$$

where τ, τ_i stand for the following determinants

$$\tau = \begin{vmatrix} (q_1, q_1) & \cdots & (q_1, q_n) \\ \vdots & \ddots & \vdots \\ (q_n, q_1) & \cdots & (q_n, q_n) \end{vmatrix}, \quad \tau_i = \begin{vmatrix} 0 & q_{1,i} & \cdots & q_{n,i} \\ \kappa q_{1,\mathcal{N}} & (q_1, q_1) & \cdots & (q_1, q_n) \\ \vdots & \vdots & \ddots & \vdots \\ \kappa q_{n,\mathcal{N}} & (q_n, q_1) & \cdots & (q_n, q_n) \end{vmatrix}, \tag{4.67}$$

with $q_{k,m}$ denoting the m-th component of the q_k vector. Recently, bright and dark soliton solutions were obtained using the dressing method, see [35, 36].

$$\Psi = \mathbb{1} + \hat{K}, \qquad \Psi_0 = \mathbb{1} + F, \qquad B = \mathbb{1} + K. \tag{5.10}$$

We come now to the main objective, which is the solution of the GLM equation (5.2) for the vector NLS system. Given the form of the solution F (5.7), and also considering the generic expression $K(x,z) = \sum_{i,j} K_{ij}(x,z) e_{ij}$ we end up to the following set of equations (see also [39, 37] for the sl_2 NLS case):

$$K_{1j}(x,z) + f_j(x,z) + \int_x^{\infty} K_{11}(x,y) f_j(y,z)\, dy = 0$$
$$K_{11}(x,z) + \int_x^{\infty} \sum_j K_{1j}(x,y) \hat{f}_j(y,z)\, dy = 0, \tag{5.11}$$

$$K_{i1}(x,z) + \hat{f}_i(x,z) + \sum_j \int_x^{\infty} K_{ij}(x,y,t) \hat{f}_j(y,z)\, dy = 0$$
$$K_{ij}(x,z) + \int_x^{\infty} K_{i1}(x,y,t) f_j(y,z)\, dy = 0. \quad j \in \{2, \ldots, N\}. \tag{5.12}$$

The two sets of equations above can be independently solved to provide K_{1j}, K_{11} and K_{i1}, K_{ij} respectively. Moreover, given the form of the dressed operators it is clear that K_{1j} and K_{j1} provide the fields u_{j-1} and u^*_{j-1} respectively, i.e the components of $\langle u|$ and $|u^*\rangle$ (see also [

Notice that the obvious choice

$$X_j^{(\alpha)}(x,t) = b_j e^{i\Lambda_j^{(\alpha)} t + i\lambda_j^{(\alpha)} x}, \qquad Z_j^{(\alpha)} = e^{i\mu_j^{(\alpha)} z}$$
$$\hat{X}_j^{(\alpha)}(x,t) = \hat{b}_j e^{i\hat{\Lambda}_j^{(\alpha)} t + i\hat{\lambda}_j^{(\alpha)} x}, \qquad \hat{Z}_j^{(\alpha)} = e^{i\hat{\mu}_j^{(\alpha)} z} \tag{5.18}$$

leads to simple expressions for $P,\ \hat{P}$ after integration (see also [38] for relevant expressions in the context of the inverse scattering transform):

$$P_{jj}^{\beta\gamma} = -\hat{b}_j^{(\gamma)} \frac{e^{i\hat{\Lambda}_j^{(\gamma)} t + i\hat{\lambda}_j^{(\gamma)} x + i\mu_j^{(\beta)} x}}{i(\hat{\lambda}_j^{(\gamma)} + \mu_j^{(\beta)}}$$

Moreover, the quantities K_{11} and K_{ij} may be also derived via (5.11), (5.12), hence we obtain:

$$K_{11}(x,z) = -\sum_j \sum_{\beta,\gamma} L_j^{(\beta)}(x) P_{jj}^{\beta\gamma}(x) \hat{Z}_j^{(\gamma)}(z)$$
$$K_{ij}(x,z) = -\sum_\beta \hat{X}_i^{(\beta)}(x) \hat{Z}_i^{(\beta)}(z) - \sum_j \sum_{\beta,\gamma} L_j^{(\beta)}(x) P_{jj}^{\beta\gamma}(x) \hat{Z}_j^{(\gamma)}(z). \qquad (5.25)$$

It is easy now to extract for instance the one-soliton solution given the description above. Indeed expressions (5.20),(5.24) still hold, but now M is defined as

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and P is expressed as a bi-vector

$$\mathbf{P} = \mathbf{H}(x,t)\ \hat{\mathbf{B}}^{\mathbf{T}}\mathbf{B} \tag{5.33}$$

and is also a projector:

$$\mathbf{P}^{2} = \mathbf{C}\ \mathbf{H}(x,t)\ \mathbf{P}, \tag{5.34}$$

which leads to the immediate identification of the inverse $\mathbf{M}^{-1}$

$$\mathbf{M}^{-1} = \mathbb{1} - \frac{1}{1 + \mathbf{C}\ \mathbf{H}(x,t)}\mathbf{P}. \tag{5.35}$$

The identification of $\hat{L}_{\mathbf{i}}$ is then straightforward

$$L_{\mathbf{j}}(x,t) = -\frac{b_{\mathbf{j}}\, e^{\mathbf{i}\Lambda\mathbf{t}+\mathbf{i}\lambda\mathbf{x}}}{1 + \mathbf{C}\ \mathbf{H}(x,t)}, \tag{5.36}$$

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The fundamental linear equation (5.15) is then written as

$$\sum_i \int_{-} d\tilde{k}\; L_i(x,t;\tilde{k})\; M_{ij}(x,t;\tilde{k},k) = -X_j(x,t;k), \tag{5.40}$$

where we define

$$M_{ij}(\tilde{k},\ k) = \delta_{ij}\delta(\tilde{k},k) - \int dk\; P_{ii}(\tilde{k},\ k\ )\ \hat{P}_{ij}(k\ ,k) \tag{5.41}$$

and $P,\ \hat{P}$ are then defined as the continuum analogues of (5.19), i.e.

$$P_{ii}(x,t;k,\tilde{k}) = \hat{b}_j(\tilde{k})e^{i\hat{\Lambda}(\tilde{k})t+i\hat{\lambda}(\tilde{k})x+i\mu}$$

in the previous sections as will become apparent below. Let $\mathbf{G}$ be the global Darboux transformation such that:

$$\hat{\Psi} = \mathbf{G}\,\Psi \tag{5.46}$$

and Ψ, $\tilde{\Psi}$ satisfy:

$$\begin{aligned} ia\,\partial_t\hat{\Psi} &= \hat{\mathbf{D}}\,\hat{\Psi} \\ ia\,\partial_t\Psi &= \mathbf{D}\,\Psi \end{aligned} \tag{5.47}$$

where

$$\mathbf{D} = \mathbf{D}_0 + M, \qquad \hat{\mathbf{D}} = \hat{\mathbf{D}}_0 + \hat{M}. \tag{5.48}$$

From the latter equations immediately follows the typical time part of a Darboux-Bäcklund transformation

$$\partial_t\mathbf{G} = \hat{\mathbf{D}}\,\mathbf{G} - \mathbf{G}\,\mathbf{D}. \tag{5.49}$$

Taking into account (5.48), (5.49) and setting $\mathbf{G} = \mathbb{1} + \mathbf{K}$, we obtain the following global expression:

$$ia\,\partial_t\mathbf{K} = \hat{M}\,\mathbf{K} - \mathbf{K}\,M + \mathbf{D}_0\,\mathbf{K} - \mathbf{K}\,\mathbf{D}_0 + \hat{M} - M \tag{5.50}$$

M, $\hat{M}$ are $N \times N$ matrices, and $\mathbf{D}_0 = \mathbb{1}\partial_x^2$. The integral representation of the expression above becomes (K are also $N \times N$ matrices)

$$\begin{aligned} ia\int_{-\infty}^{x} dy\,\partial_t K(x,y)\mathbf{f}(y) &= \int_{-\infty}^{x} dy\,M(x)K(x,y)\mathbf{f}(y) - \int_{-\infty}^{x} dy\,K(x,y)M(y)\mathbf{f}(y) \\ &+ \int_{-\infty}^{x} dy\,\Big(\partial_x^2 K(x,y) - \partial_y^2 K(x,y)\Big)\,\mathbf{f}(y) \end{aligned}$$

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