A POINTWISE CHARACTERISATION OF THE PDE SYSTEM OF VECTORIAL CALCULUS OF VARIATIONS IN L^1

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Abstract. Let $n; N$ 2 N with R^n open. Given H 2 C^2(R^N R^{Nn}); we consider the functional

(1) E_1 (u; O) := ess sup H(; u; Du); u 2 $W^{1;}$ H_P (; u; Du) H (; u; Du) = 0 ;

where $[\![A]\!]^?$:= Proj $_{R(A)^?}$. Herein we establish that generalised solutions to (2) can be characterised as local minim1ll3r lof-4196(21ou 930(cal)r1(cte)-1(r)1(i193l(6i050)olutiono

of (regular) maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ are valued, whilst subscripts of H denotes derivatives with respect to the respective variables (x, η, P). We use the symbolisations $x = (x_1, \ldots, x_n)^\top$, $u = (u_1, \ldots, u_N)^\top$, $D_i \equiv \partial/\partial x_i$ whilst Latin indices $i, j, k, \ldots$ will run in $\{1, \ldots, n\}$ and Greek indices $\alpha, \beta, \gamma, \ldots$ will run in $\{1, \ldots, N\}$. Further, for any linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$, the notation $[\![A]\!]^\perp$ used above denotes the orthogonal projection onto the orthogonal complement of its range $R(A) \subseteq \mathbb{R}^N$:

$$[\![A]\!]^\perp := \mathrm{Proj}_{R(A)^\perp}. \tag{1.4}$$

Also, "$O \Subset \Omega$" means that O is open and $\overline{O} \subseteq \Omega$. In index form, F_∞ reads

$$
\begin{aligned}
F_\infty{}_\alpha(x, \eta, P, X) := {} & \sum_i H_{P_{\alpha i}}(x, \eta, P) \left(\sum_{\beta, j} H_{P_{\beta j}}(x, \eta, P) X_{\beta ij} + \sum_\beta H_{\eta_\beta}(x, \eta, P) P_{\beta i} \right. \\
& \left. + H_{x_i}(x, \eta, P) \right) + H(x, \eta, P) \sum_\beta [\![H_P(x, \eta, P)]\!]^\perp_{\alpha\beta} \\
& \left(\sum_{\gamma, i, j} H_{P_{\beta i} P_{\gamma j}}(x, \eta, P) X_{\gamma ij} + \sum_{\gamma, i} H_{P_{\beta i} \eta_\gamma}(x, \eta, P) P_{\gamma i} \right. \\
& \left. + \sum_i H_{P_{\beta i} x_i}(x, \eta, P) - H_{\eta_\beta}(x, \eta, P) \right),
\end{aligned}
$$

where $\alpha = 1, \ldots, N$. Note that, although H is C^2, the projection map $[\![H_P(\cdot, u, Du)]\!]^\perp$ is discontinuous when the rank of $H_P(\cdot, u, Du)$ changes. Further, we remark that because of the perpendicularity of H_P and $[\![H_P]\!]^\perp$, the system can be decoupled into two independent systems which we write in a contracted fashion:

$$
\left\{
\begin{aligned}
& H_P(\cdot, u, Du)\, D\big(H(\cdot, u, Du)\big) = 0, \\
& H(\cdot, u, Du)\, [\![H_P(\cdot, u, Du)]\!]^\perp \Big(\mathrm{Div}\big(H_P(\cdot, u, Du)\big) - H_\eta(\cdot, u, Du) \Big) = 0.
\end{aligned}
\right.
$$

When $H(x, \eta, P) = |P|^2$ (the Euclidean norm on $\mathbb{R}^{Nn}$ squared), the system (1.2)-(1.4) simplifies to the so-called ∞-Laplacian:

$$\Delta_\infty u := \Big(Du \otimes Du + |Du|^2 [\![Du]\!]^\perp \otimes I \Big) : D^2 u = 0. \tag{1.5}$$

The scalar case $N = 1$ first arose in the work of G. Aronsson in the 1960s [A1, A2] who initiated the area of Calculus of Variations in the space L^∞. The field is fairly well-developed today and the relevant bibliography is vast. For a pedagogical introduction to the topic accessible to non-experts, we refer to [K8]. We just mention that in the scalar case, generalised solutions to the respective PDE which is commonly referred to as the Aronsson equation and simplifies to

$$H_P(\cdot, u, Du) \cdot \Big(H_P(\cdot, u, Du)^\top D^2 u + H_\eta(\cdot, u, Du) Du + H_x(x, \eta, P) \Big) = 0$$

are understood in the viscosity sense (see [C, CIL, K8]). The study of the vectorial case $N \geq 2$ started much more recently and the full system (1.2)-(1.4) first appeared in the work [K1] of one of the authors in the early 2010s and it is being studied quite systematically ever since (see [K2]-[K7], [K9]-[K13], as well as the joint works of the second author with Abugirda, Pryer, Croce and Pisante [AK], [CKP], [KP, KP2]).

In this paper we are interested in the characterisation of appropriately defined generalised vectorial solutions $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$ to (1.2)-(1.4) in terms of the

and

$$O(u) := \mathrm{Argmax} \quad H(\ ; u; Du) \ : \ \overline{O} \ :$$

We conclude the introduction with some rudimentary facts about generalised

and hence

$$\frac{1}{\ }\, r(\ ) \quad r(0) \qquad \frac{1}{\ }\, R(\ ; y^0) \quad R(0; y^0) \ ;$$

where $y^0 \in \overline{O}$ is any point such that $R(0; y^0) = \max_{\overline{O}} R(0;\ )$. Hence, we have

$$\begin{aligned}
\underline{D}r(0^+) = {} & \liminf_{\ \to 0^+} \frac{1}{\ }\, r(\ ) \quad r(0) \\
& \max_{y^0 \in \overline{O}} \ \liminf_{\ \to 0^+} \frac{1}{\ }\, R(\ ; y^0) \quad R(0; y^0) \\
= {} & \max_{y \in O(u)} \ \liminf_{\ \to 0^+} \frac{1}{\ }\, R(\ ; y) \quad R(0; y) \\
= {} & \max_{O(u)} \ \liminf_{\ \to 0^+} \frac{1}{\ }\, H \quad ; u + \ A; \ Du + \ DA \qquad H \ ; u; Du) \\
= {} & \max_{O(u)} \ \liminf_{\ \to 0^+} \frac{1}{\ }\, H(\ ; u; Du) + \quad H \quad ; u; Du \quad A + \quad H_P(\ ; u; Du) : DA \\
& + O \ |\ DA|^2 \ )
\end{aligned}$$

Note that $F_1^?(x; ;P; X)$ 2 R^N, A

for any $(x, \ , X) \in \ \ \mathbb{R}^N \ \ \mathbb{R}^{Nn^2}_s$.

(B) In view of the mutual perpendicularity of the two components of F_∞ (see (3.1)-(3.2)), (A) is a consequence of the following particular results:

$$H_P(\ ; u; Du)\, F^{\parallel}_{\infty}(\ ; u; Du; D^2u) = 0 \quad \text{in} \quad ;$$

in the D-sense, if and only if

$$E_\infty(u; O) \quad E_\infty(u + A; O); \quad \forall\, O \Subset \ ; \ \forall\, A \in \mathcal{A}^{\parallel,\infty}_O(u)$$

and also

$$H(\ ; u; Du) \ \ H_P(\ ; u; Du)^{\perp} \ F^{\perp}_{\infty}(\ ; u; Du; D^2u) \quad H(\ ; u; Du) \ = 0 \quad \text{in} \quad ;$$

in the D-sense, if and only if

$$E_\infty(u; O) \quad E_\infty(u + A; O); \quad \forall\, O \Subset \ ; \ \forall\, A \in \mathcal{A}^{\perp,\infty}_O(u):$$

We note that in the special case of C^2 solutions, Corollary 1 describes the way that classical solutions $u : \mathbb{R}^n \ \ \to \ \mathbb{R}^N$ to (1.2)-(1.4) are characterised.

Remark 8 (About pointwise properties of C^1 D-solutions). Let $u : \mathbb{R}^n \ \ \to \ \mathbb{R}^N$ be a D-solution to (1.2)-(1.4) in $C^1(\ ; \mathbb{R}^N)$. By Definition 3, this means that for any $D^2u \in \mathscr{Y}(\ ; \overline{\mathbb{R}}^{Nn^2}_s)$,

$$F_\infty\big(x; u(x); Du(x); X_x\big) = 0; \quad \text{a.e. } x \in \ \ \text{and all } X_x \in \operatorname{supp}\big(D^2u(x)\big):$$

By Definition 2, every diffuse hessian of a putative solution is defined a.e. on as is clearly satisfied at x. If $H_P\big(x, u(x), Du(x)\big) \neq 0$, then we select any direction normal to the range of $H_P\big(x, u(x), Du(x)\big) \in \mathbb{R}^{N \times n}$, that is

$$n_x \in R\Big(H_P\big(x, u(x), Du(x)\big)\Big)^{\perp} \subseteq \mathbb{R}^N$$

which means $n_x^\top H_P\big(x, u(x), Du(x)\big) = 0$. Of course it may happen that the linear map $H_P\big(x, u(x), Du(x)\big) : \mathbb{R}^n$

and by applying Lemma 5a), we have

$$0 \quad \max_{z 2 \overline{O_{"}(x)}} \Big\{ H_P\big(z; u(z); Du(z)\big) : DA(z) + H\big(z; u(z); Du(z)\big) \; A(z) \Big\}$$

$$\xrightarrow{" ! \, 0} \quad H_P\big(x; u(x); Du(x)\big) : \quad F_1^k\big(x; u(x); Du(x); X_x\big)$$

and hence

$$H_P\big(x; u(x); Du(x)\big) \; F_1^k\big(x; u(x); Du(x); X_x\big) \qquad 0;$$

for any $\;2\;\mathbb{R}^N$. By the arbitrariness of $\;$ we deduce that

$$H_P\big(x; u(x); Du(x)\big) \; F_1^k\big(x; u(x); Du(x); X_x\big) = 0$$

for any $D^2u \; 2 \; Y\big(\; ; \overline{\mathbb{R}}_s^{Nn^2}\big)$, $x \; 2 \;$ and $X_x \; 2 \; \mathrm{supp}\big(D^2u(x)\big)$, as desired.

Conversely, we x O

Proof of Corollary 1. If u 2 $C^2(\; ;R^N)$, then by Lemma 4 any di use hessian of u satis es $D^2u(x) = \; _{D^2u(x)}$ for a.e. x 2 . By Remark 8, we may assume this happens for all x 2 . Therefore, the reduced support of $D^2u(x)$ is the singleton set f $_{D^2u(x)}$