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Competing Edge Networks

by

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the same group of nodes, where each edge type has its own discrete time dynamics, and series of adjacency matrices detailing its evolution. Since these edge types act upon the same group of nodes they may be superimposed onto a single graph, providing the different edge types are clearly differentiated. The work presented in this paper considers such a network where the different edge types are competing with one yneomod4i2y836k52-2.1.42439(k)8(e)3.3879(n)-1.42133(e)3.3

in the case of highly asymmetric competition.

2. Competing Edge Dynamics

First we introduce some terminology to define our competing evolving networks.

Following [8, 7, 10] we define an evolving network, over discrete time steps indexed by $k = 1, 2, ...$, via a sequence of adjacency matrices, say $\{A_k$. We shall assume that all edges are undirected and we do not allow any edges connecting a vertex with itself. Thus all of our adjacency matrices lie in the set S_n of binary, symmetric, $n \quad n$ matrices having zeros along their main diagonals. We assume the evolving network dynamic is first order in time: at the $(k+1)$th time step each edge in A_{k+1} will have a birth or death rate that is conditional on A_k. However no new vertices will enter, nor shall any existing vertices be permanently removed from the evolving network. At each time step the evolving network is thus a random network conditional on the evolving network at the previous time step, with a probability distribution $P(A_{k+1}|A_k)$, defined as A_{k+1} ranges over S_n.

We shall assume that presence of each each edge in A_{k+1} is determined independently of all other edges. This means that it is sufficient to specify the conditional expectation that each edge is present, given by

$$< A_{k+1}|A_k > = \sum_{A_{k+1} \in S_n} A_{k+1} P(A_{k+1}|A_k),$$

rather than dealing with full probability distribution. In fact for such edge-independent conditional random networks we may write

$$P(A_{k+1}|A_k) = \prod_{i<j} (< A_{k+1}|A_k >)_{ij}^{(A_{k+1})_{ij}} (1 - (< A_{k+1}|A_k >)_{ij})^{1-(A_{k+1})_{ij}},$$

demonstrating their equivalence.

Notice that since distinct edges may be conditionally dependent on some

Let the sequence $\{A_k$ within S_n denote a Red evolving network defined over a set of n vertices. Similarly let the sequence $\{B_k$ within S_n denote a Blue evolving network defined over the same set n vertices. Then, extending the above ideas, we will assume that both evolving networks have a first order edge-independent dynamic such that each network at each time step is a random network conditionally dependent upon both networks at the previous time step. Then such a c2(.8584.4998(o)-272r)-1.42541()-2.83857(ti)-2.83857(tie)3.38586(m)1.

Notice that since both A_{k+1} and B_{k+1} are dependent upon A_k and B_k, there is therefore no 'first/late mover advantage' [6] for the Red or Blue network.

Figure 1 shows the evolution of various synthetic networks in terms of the edge density for the Red and Blue networks, where each simulation starts from the same initial pair of matrices, A_1 and B_1. Their evolution is modelled according to (1) and (2), with $n = 39$, and the same parameter values for both Red and Blue networks: $\omega = 1/25$, $\square = 1/110$, $\mu = 1/17$, $\delta = 1/600$ and $\gamma = 1/600$. Notice that multiple apparently stable equilibria exist and that they are reachable from the same initial network pair at the first time step. This highlights the significance of identifying these equilibria for a given network, and motivates the analysis in the next section.

3. Mean Field Approximation

In order to identify and analyse the long term equilibria, we take the mean field approximation introduced in [8]. Symmetry of the dynamics implies there are no preferred vertices or edges (all edges satisfy the same rules since the birth and death rates have no explicit edge dependencies), so we assume that we may write $< A_k > \quad p_k\mathbf{1}$ and similarly $< B_k > \quad q_k\mathbf{1}$ where p_k and q_k represent the edge densities of the Red and Blue networks at the kth time step; and hence that these networks are approximated by Erdos-Renyi random graphs. Then the mean field approximation for the dynamics of this system is reduced to a nonlinear iteration of over the unit square:

$$p_{k+1} = p_k(1 - \omega_A - \mu_A q_k) + (1 - p_k)(\delta_A + \square_A(n-2)p_k^2 - \gamma_A q_k) \quad (3)$$

$$q_{k+1} = q_k(1 - \omega_B - \mu_B p_k) + (1 - q_k)(\delta_B + \square_B(n-2)q_k^2 - \gamma_B p_k). \quad (4)$$

Notice that $0 \quad p_k, q_k \quad 1$ for all k, and our parameters should satisfy several constraints. In (3) we require:

(a) $(1 - \omega_A - \mu_A q_k) \quad 0$, and hence, since $q_k \quad 1$, we must have $\omega_A + \mu_A \quad 1$;

(b) $(\delta_A + \square_A(n-2)p_k^2 - \gamma_A q_k) \quad 0$, and hence, since $p_k \quad 0$ and $q_k \quad 1$, we must have $\delta_A - \gamma_A \quad 0$;

(c) $(\delta_A + \square_A(n-2)p_k^2 - \gamma_A q_k) \quad 1$, and hence, since $p_k \quad 1$ and $q_k \quad 0$, we must haven $\delta_A + \square_A(n-2) \quad 1$.

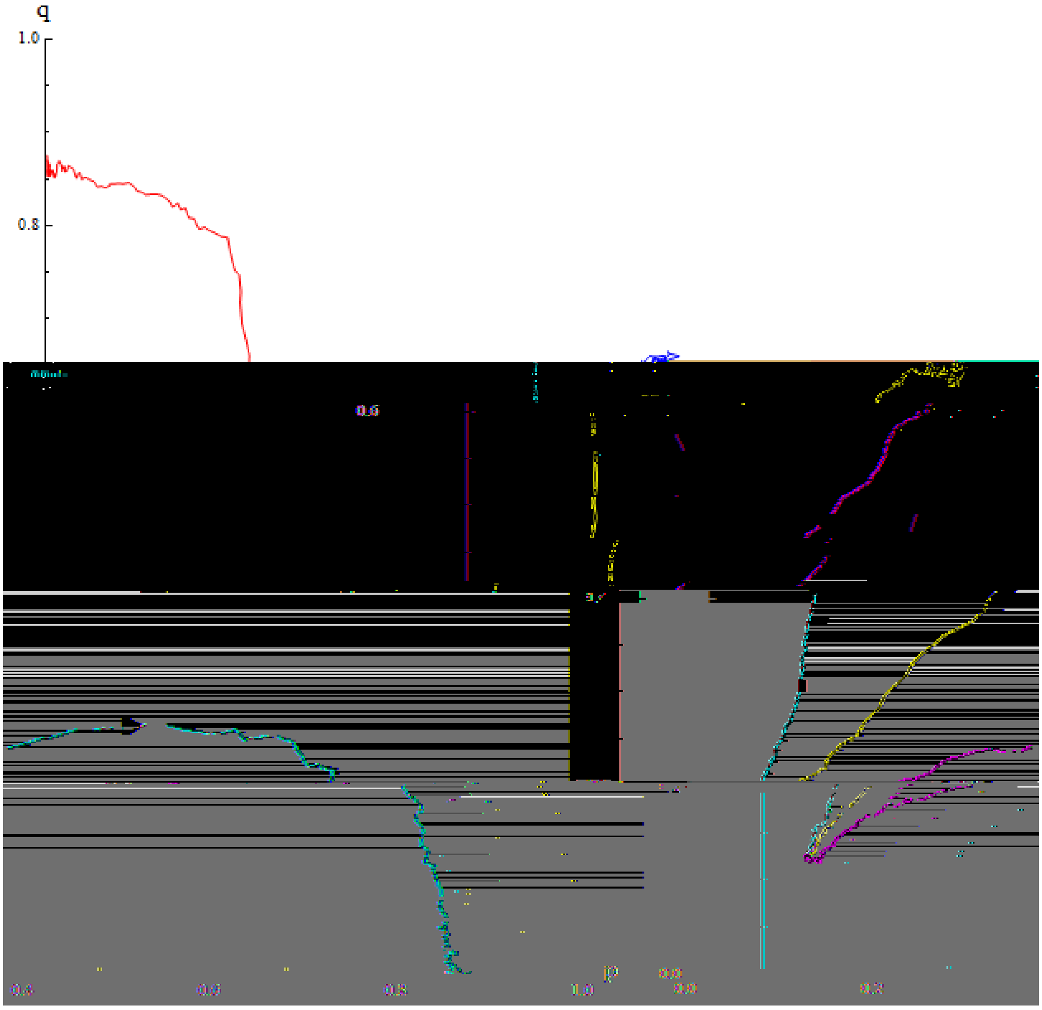

Figure 1: Three separate simulations of competing networks, modelled according to (1) and (2). In each case the edge densities of the competing networks are plotted against one another at each timestep. Notice that each simulation is performed with the same network parameter values and initial matrix pair, however evolve towards distinct network positions.

These three constraints hold similarly for the Blue network's parameters.

At equilibrium, where, say, $p_k = p$ and $q_k = q$ for all k, we may rearrange (3) and (4) into the following form,

$$q = \frac{\square_A(n-2)(1-p)p^2 + (1-p)\delta_A - p\omega_A}{\gamma_A(1-p) + p\mu_A} = f_A(p), \tag{5}$$

$$p = \frac{\square_B(n-2)(1-q)q^2 + (1-q)\delta_B - q\omega_B}{\gamma_B(1-q) + q\mu_B} = f_B(q), \tag{6}$$

where the functions f_A and f_B differ only in their (suppressed) parameter values.

The mean field approximation retains the nonlinear nature of the full stochastic iteration, but it is itself a deterministic iteration (over (p, q) space), since the stochastic evolution has been smoothed away by projecting the expected value of the adjacency matrix into its mean field representation. This approximation is likely to become unreliable where the original evolution is sensitive to small perturbations within the network structures (see [8] for further reading). This certainly would include situations where one or other network is very sparse and also where the pair are close to any unstable equilibrium or other regions of instability, for the mean field dynamics.

4. Identifying System Equilibria for Symmetrical Competition

Before locating the equilibria for our system we first make the following simplification that equalizes the competition: we shall assume that the parameter values for both the Red and Blue networks are equal, i.e., $\delta_A = \delta_B = \delta$ for every parameter in (3) and (4). Hence (5) and (6) become

$$q = f(p), \quad p = f(q), \tag{7}$$

where

$$f(y) = \frac{\square(n-2)(1-y)y^2 + (1-y)\delta + y\omega}{\gamma(1-y) + y\mu}.$$

Now consider the one dimensional iteration defined on [0,1], indexed by $t = 1, 2, \ldots,$

$$y_{t+1} = f(y_t). \tag{8}$$

Then equilibria for the mean field iteration (3) and (4), necessarily satisfying (7), are represented by either fixed point equilibria, y^* say, for (8); or a period two, or "flip", solution for (8), say $y_1 = f(y_2)$, $y_2 = f(y_1)$ $(y_1 \neq y_2)$. The first case leads to a symmetrical equilibrium for (3) and (4), with $p = q = y^*$; the second case leads to a mirror image pair of non-symmetric equilibria for (3) and (4), with $(p, q) = (y_1, y_2)$ and (y_2, y_1).

Once an equilibrium is identified, its stability is determined by the spectral radius of the Jacobian obtained by linearizing (3) and (4) about that point,(h)0.97oearao1

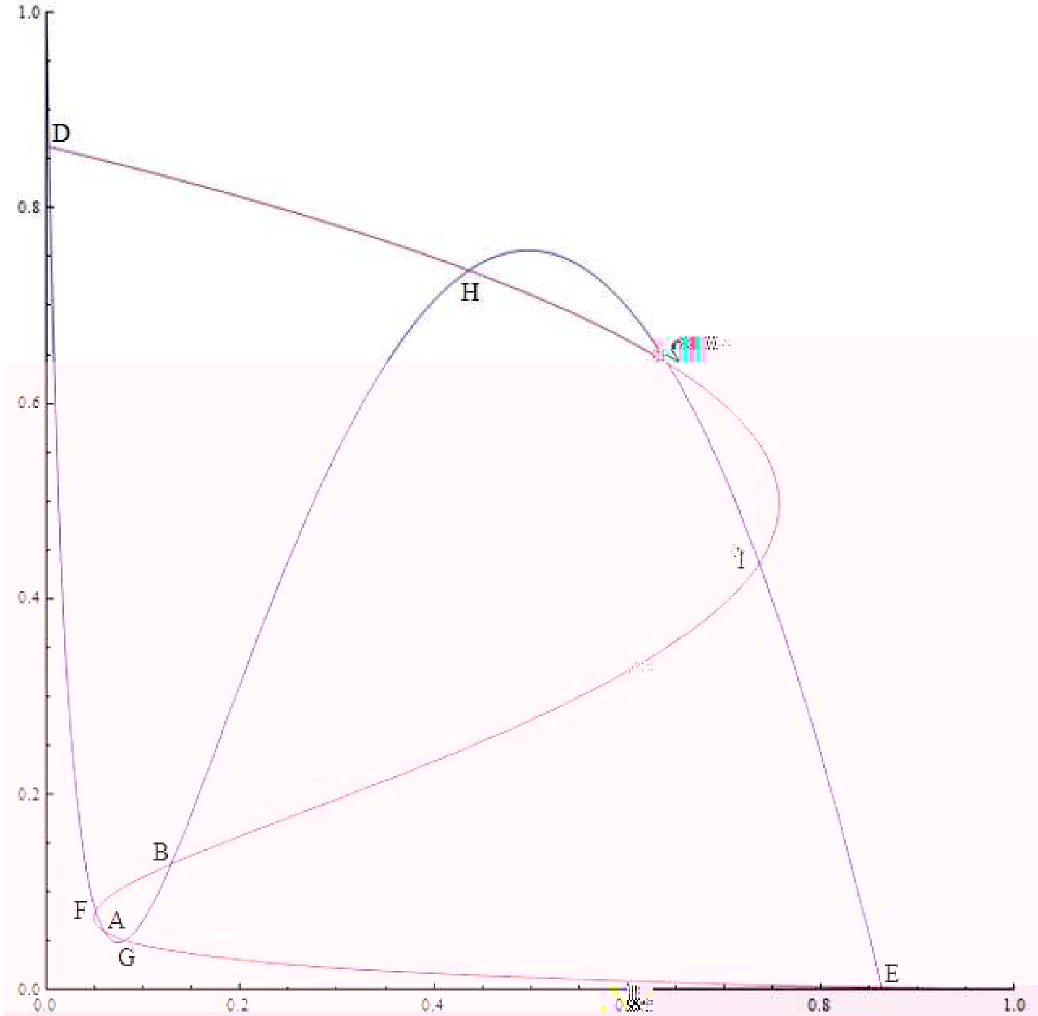

Figure 2: An example of fixed point curves for a mean field approximation of a competing edge network with specified parameters resulting in nine intersections.

The remaining non symmetric equilibria for (3) and (4) may be identified by considering two-periodic “flip-solutions” for (8). These may occur specifically when any fixed point for (8) (corresponding to a symmetric equilibrium for (3) and (4)) undergoes a flip bifurcation, when the system parameters change so that the slope of f goes from above to below -1 at that equilibrium, whence a pair of period-two solutions will be born. For (3) and (4) this evento/I8(o)-2.836519(d)

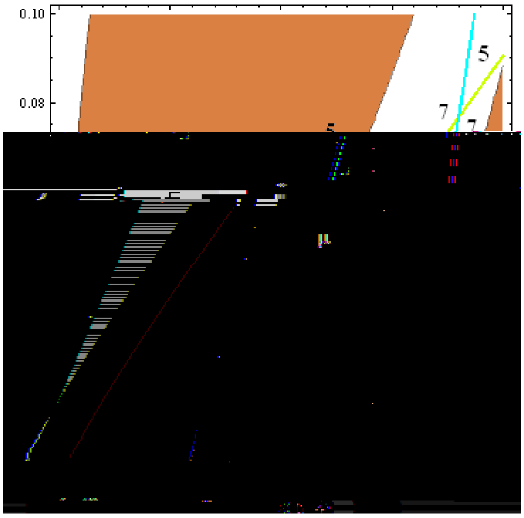

Figure 3: A bifurcation map for varying values of $\epsilon(n-2)$ (x-axis) and μ (y-axis). The

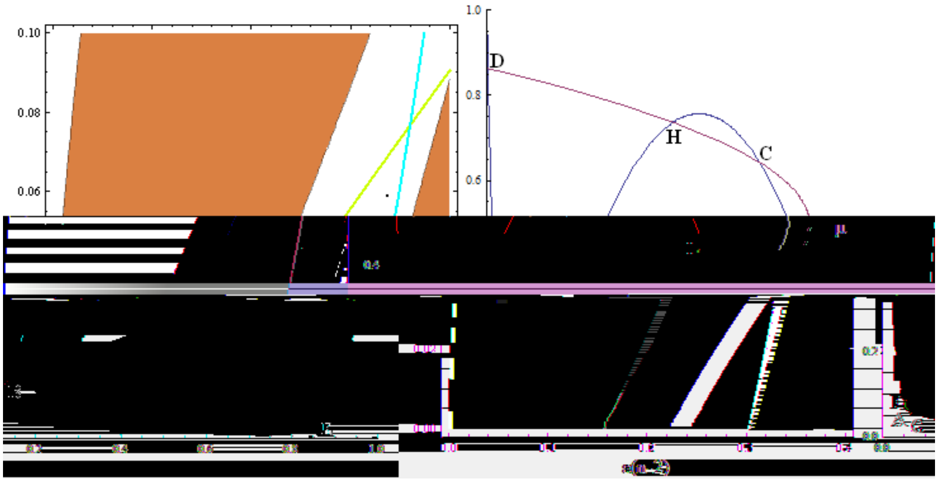

Figure 4: Parameter values chosen are $\epsilon(n-2) = 37/110$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with nine applicable roots. Of these roots, points

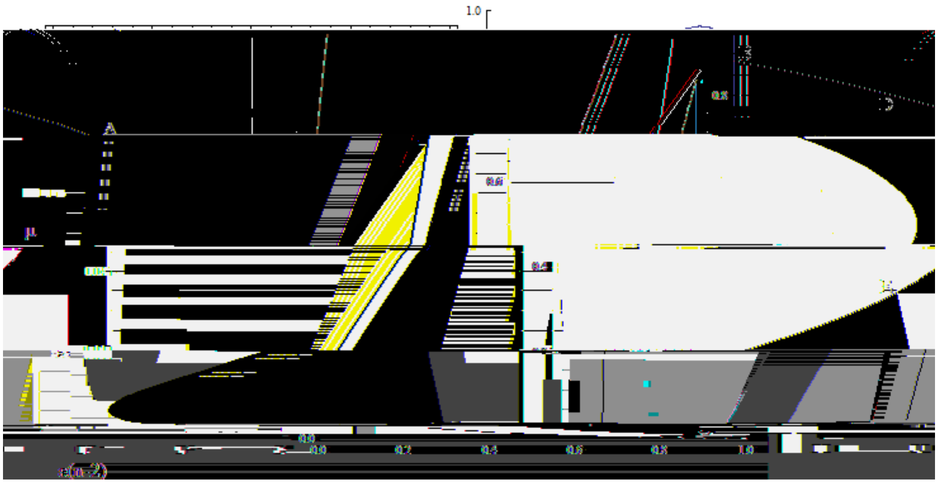

Figure 6: Parameter values chosen are $\epsilon(n-2) = 17/44$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with five applicable roots. With respect to Figure 5, two more roots are lost due to only possessing a single first order solution. Of these roots, points A, B and C are found to be stable, whereas the others are unstable.

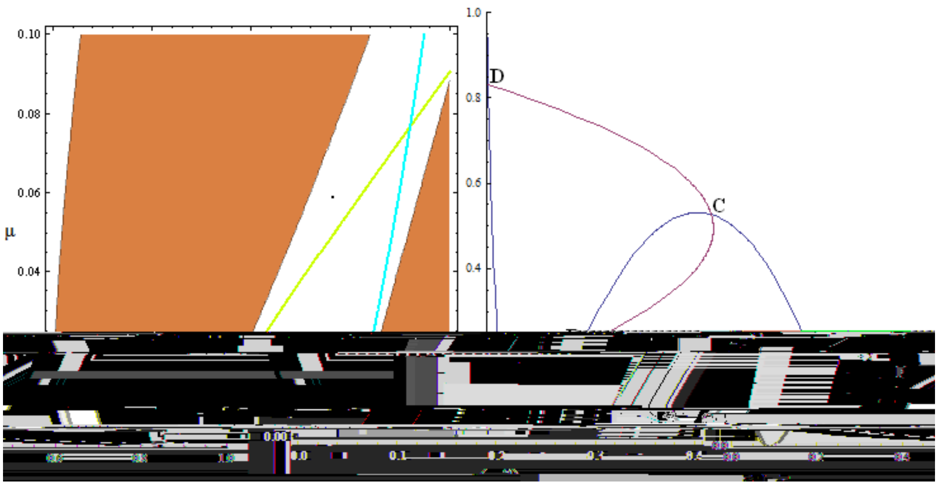

Figure 7: Parameter values chosen are $\epsilon(n-2) = 31/110$ and $\mu = 1/17$, resulting in an associated mean field fixed points graph with seven applicable roots. With respect to Figure 4, two roots are lost due to non-bifurcation of point C. Of these roots, points A, D and E are found to be stable, whereas the others are unstable.

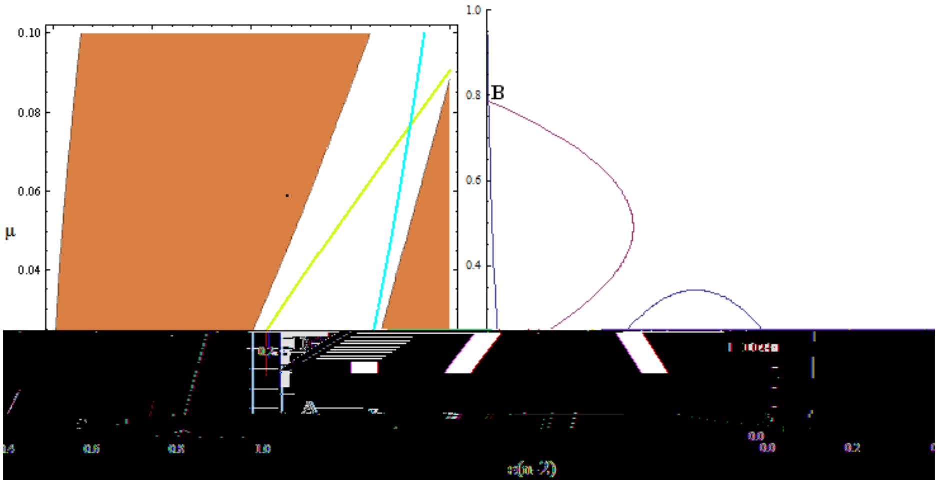

Figure 8: Parameter values chosen are

Given $f_B(f_A(p)) - p = 0$, from (5) and (6), this occurs where p is the root of a ninth order polynomial $r(p)$, say. Then by direct calculation

$$r(0) = \delta_A{}^2 \square_B(\delta_A - \gamma_A) + \gamma_A(\delta_B(\delta_A - \gamma_A) + \delta_A\omega_B). \tag{10}$$

Recall both that all parameters are positive and $\gamma_A \quad \delta_A$: hence $r(0) > 0$. Notice also that,

$$\begin{aligned} r(1) \;=\; & -\mu_A{}^3(\delta_B - \gamma_B) - \mu_A{}^2\omega_A(\delta_B - \gamma_B) - \mu_A{}^2\mu_B\omega_A \\ & -\square_B\mu_A\omega_A{}^2 - \square_B\omega_A{}^3 - \mu_A{}^2\omega_A\omega_B. \end{aligned} \tag{11}$$

Then similarly $r(1) < 0$.

It follows that there exists an odd number of applicable roots satisfying (5) and (6) in $[0, 1]$, even without equality between network parameters by the intermediate value theorem. We would expect this to be the case since applicable roots can only be lost in pairs, through the coalescence of two roots or a pitchfork bifurcation.

6. Discussion

In this paper we have introduced a model for networks that compete edge-wise over a fixed set of vertices. Both networks inhibit the other's growth (through lower edge birth rates) and encourage the other's demise (through greater edge death rates).

The nonlinear stochastic competition equations yield to a mean field analysis that results in an associated nonlinear deterministic system. This in turn indicates there may be multiple dynamic steady states; regions of stability, with some sensitivity to the stochastic details found close to unstable equilibria; and a sensitivity to sparse initial conditions.

The applications we have in mind are situations where one peer-to-peer communication network competes and gradually displaces the other. For example where the emergence of BlackBerry Messenger has created a competing network against SMS messages, resulting in a decreased edge density for SMS communication over this userbase [2]. Our analysis illustrates how the ultimate fate of such competitions may depend upon early sensitive and stochastic behaviour.

Acknowledgements

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