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by

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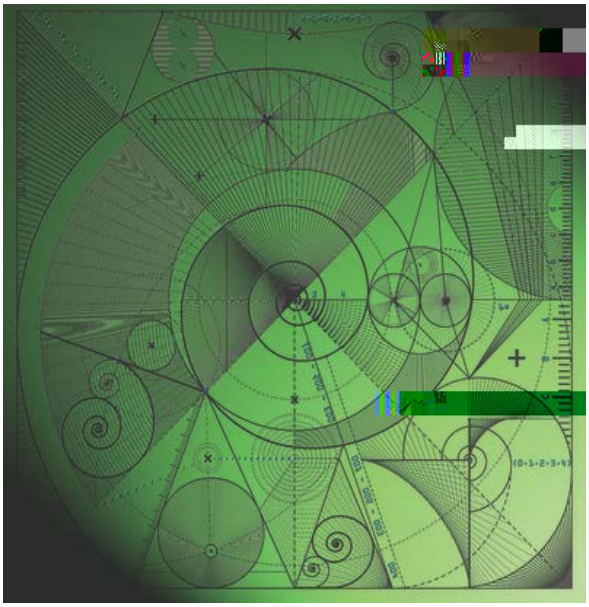

# Acoustic scattering by fractal screens: mathematical formulations and wavenumber-explicit continuity and coercivity estimates

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Abstract

We consider time-harmonic acoustic scattering by planar sound-soft (Dirichlet) and sound-hard (Neumann) screens. In contrast to previous studies, in which the domain occupied by the screen is assumed to be Lipschitz or smoother, we consider screens occupying an arbitrary

We give some motivation for and further explanation of objectives (i) and (ii) in §1.2 below. But first we provide a brief overview of the structure of the paper.

Our approach follows previous studies (e.g. [34, 33, 36]) in that it is based on the classical direct integral equation method, in which Green's theorem is used to reformulate the BVPs in the propagation domain as boundary integral equations (BIEs) on the screen. Specifically, the sound-soft BVP leads to the single-layer BIE, and the sound-hard BVP to the hypersingular BIE. To determine the solvability of these BIEs one needs to study the associated boundary integral operators (BIOs) as mappings between certain fractional Sobolev spaces defined on the screen. But while these Sobolev spaces are well-studied for Lipschitz screens, their properties for general non-Lipschitz screens do not seem to have been widely studied. We therefore begin in §2 by carefully documenting some of the properties of Sobolev spaces on an arbitrary open subset $\mathcal{O} \subset \mathbb{R}^n$. Of particular importance

sound-hard screens with bounded apertures. The analysis is similar to that of the screen problem, but with some small technical di erences. We note that for the aperture problem the sound-soft BVP gives rise to the hypersingular BIE and the sound-hard BVP gives rise to the single-layer BIE.

We remark that some of the results on the BVP and BIE formulations in this paper were stated without proof in the conference paper [20]. Further results concerning Sobolev spaces on non-Lipschitz domains will be presented in the future publication [9]. A BIE formulation for the case where is an arbitrary relatively closed subset of $_1$ is presented in [5], along with an application to screens de ned in terms of the \Cantor dust" fractal.

## 1.2 Motivation and background

We end this introduction by giving some motivation for objectives (i) and (ii) above.

Objective (i) above is motivated in part by the possible application to the design and simulation of antennas for electromagnetic wave transmission/reception whose geometry is based on fractal subsets of the plane (see e.g. [27, 31]). A key property of fractals is the fact that they possess structure on every lengthscale - this has been exploited to create antennas which can transmit/receive e ciently over a broad range of frequencies simultaneously. Although this seems to be a mature engineering technology, as far as we are aware no analytical framework is currently available for such problems. Of course, quantitative modelling of such problems would involve a study of the full electromagnetic wave scattering problem, but the acoustic case considered in the current paper represents a rst step in this direction.

Objective (ii) forms part of a wider e ort in the rigorous mathematical analysis of BIE methods for high frequency acoustic scattering problems (for a recent review of this area see e.g. [6]). Typically (and this is the approach adopted in the current paper) one reformulates the scattering problem as an integral equation, which can be written in operator form as

$$A = f; \tag{7}$$

where and f are complex-valued functions de ned on the boundary of the scatterer. A standard and appropriate functional analysis framework is that the solution is sought in some Hilbert space $V$, with $f \in V$, the dual space of $V$ (the space of continuous antilinear functionals), and $A : V \to V$ a bounded linear boundary integral operator[2]. Equation (7) can be restated in weak (or variational) form as

$$a( ; ' ) = f (' ); \quad \text{for all } ' \in V; \tag{9}$$

[2] A concrete example is the standard Brakhage-Werner formulation [4, 6] of sound-soft acoustic scattering by a bounded, Lipschitz obstacle, in which case $V = V = L^2( )$, and

$$A = \frac{1}{2}I + D$$

(c) Developing, and justifying by a complete numerical analysis, novel BEMs for high frequency scattering problems based on the so-called `hybrid numerical-asymptotic' (HNA) approach, the idea of which is to use an approximation space enriched with oscillatory basis functions, carefully chosen to capture the high frequency solution behaviour. The aim is to develop algorithms for which the number of degrees of freedom $N$ required to achieve any desired accuracy be fixed or increase only very mildly as $k \to \infty$. This aim is provably achieved in certain cases, mainly 2D so far; see, e.g., [22, 8] and the recent review [6]. For 2D screen and aperture problems we recently proposed in [21] an HNA BEM which provably achieves a fixed accuracy of approximation with $N$ growing at worst like $\log^2 k$ as $k \to \infty$, our numerical analysis using the wavenumber-explicit continuity and coercivity estimates of the current paper. Numerical experiments demonstrating the effectiveness of HNA approximation spaces for a 3D screen problem have been presented in [7, 6].

# 2 Sobolev spaces

We now define the Sobolev spaces that we will use throughout. Our analysis is mostly in the context of the Bessel potential spaces $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$, defined in §2.1 below. In line with other analyses of high frequency scattering we use a wavenumber-dependent norm $H^s_{(k)}(\mathbb{R}^n)$ which is equivalent to the standard norm, but allows easier derivation of wavenumber-explicit estimates.

where the support of a distribution is understood as in [26]. Clearly $\tilde{H}^s_F$ is a closed subspace of $H^s(\mathbb{R}^n)$.

There are a number of ways to define Sobolev spaces on $\Omega$ when $\Omega$ is a non-empty open subset of $\mathbb{R}^n$. First, we can consider the space $H^s_0(\Omega)$, defined as in (16). Second, we can consider the closure of $\mathcal{D}(\Omega)$ in the space $H^s(\mathbb{R}^n)$, denoted

$$\tilde{H}^s(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^s(\mathbb{R}^n)}.$$

By definition, $\tilde{H}^s(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^n)$, and it is easy to see that $\tilde{H}^s(\Omega) \subset H^s_0(\Omega)$. Third, let

$$H^s(\Omega) := \{u \in \mathcal{D}'(\Omega) : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\},$$

where $U|_\Omega$ denotes the restriction of the distribution $U$ to $\Omega$ in the sense defined in [26], with norm

$$\|u\|_{H^s(\Omega)} := \inf_{\substack{U \in H^s(\mathbb{R}^n) \\ U|_\Omega = u}} \|U\|_{H^s(\mathbb{R}^n)}.$$

We note that the restriction operator $|_\Omega : (H^s_{\mathbb{R}^n\setminus\Omega})^\perp \to H^s(\Omega)$ is a unitary isomorphism (see [26, p. 77]); here $^\perp$ denotes the orthogonal complement in $H^s(\mathbb{R}^n)$.

where $\nabla u$ is the weak gradient. Note that $W^1(\mathbb{R}^d) = H^1(\mathbb{R}^d)$ with

$$\|u\|^2_{H^1_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\nabla u(x)|^2 + k^2 |u(x)|^2 \, dx.$$

Further [26, Theorem 3.30], $W^1(\Omega) = H^1(\Omega)$ whenever $\Omega$ is a Lipschitz open set, in the sense of, e.g., [28, 6]. It is convenient to use the notation

$$W^1_{\mathrm{loc}}(\Omega) := \{ u \in L^2_{\mathrm{loc}}(\Omega) : \nabla u \in L^2_{\mathrm{loc}}(\Omega) \},$$

where $L^2_{\mathrm{loc}}$

where $U \in H^s(\mathbb{R}^n)$ is any extension of $u$ with $U|_\Omega = u$. Indeed, $I_s$ can be viewed as the Banach space adjoint (or transpose) of $\gamma_s$ in the sense, e.g., of Kato [23]. In particular, note that

$$\langle v, u \rangle_{\widetilde{H}^s(\Omega),\, H^s(\Omega)} = \overline{\langle u, v \rangle}_{H^s(\Omega),\, \widetilde{H}^s(\Omega)}, \quad v \in \widetilde{H}^s(\Omega),\ u \in H^s(\Omega).$$

In fact, as Theorem 2.1 below states, the natural embeddings $\gamma_s$ and $I_s$ are unitary isomorphisms, and in this sense it holds that

$$H^s(\Omega) = (\widetilde{H}^s(\Omega))' \quad \text{and} \quad \widetilde{H}^s(\Omega) = (H^s(\Omega))'. \tag{23}$$

We remark that the representations (23) for the dual spaces are well known when $\Omega$ is sufficiently regular. However, it is not widely appreciated, at least in the numerical PDEs community, that (23) holds without any constraint on the geometry of $\Omega$. For example, $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$ are defined precisely as above for an arbitrary open set $\Omega$ in [32], but (23) is -9at1openH

## 2.3 Nullity and capacity

In order to compare Sobolev spaces defined on different open sets (which we do in §2.4), and to study the relationship between the spaces $H^s(\Omega)$ and $\tilde{H}^s$ on a given open set $\Omega$ (which we do in §2.5), we require the concept of $s$-nullity of subsets of $\mathbb{R}^n$, which can be thought of as an indicator of negligibility in the sense of Sobolev regularity.

**Definition 2.2.** For $s \in \mathbb{R}$ we say that a set $E \subset \mathbb{R}^n$ is *$s$-null* if there are no non-zero elements of $H^s(\mathbb{R}^n)$ supported entirely inside $E$ (equivalently, if $H^s_{E'} = \{0\}$ for every closed set $E' \subset E$).

**Remark 2.3.** While the term "$s$-null" appears to be new, the concept it describes is very natural and has been considered elsewhere in different contexts (see, e.g., the book by Maz'ya [25]). For integer $s < 0$ our definition of $s$-nullity coincides with the notion of "$(2, -s)$-polarity" defined in [25, §13.2]. For integer $s > 0$, our notion of $s$-nullity is related to the concept of "sets of uniqueness" (cf. [25, p692]). The reason Maz'ya uses two different terminologies for the positive and negative order spaces is not explained in [25], but we expect this is due to the fact that Maz'ya works primarily with the Sobolev spaces $W^s$, where the positive order spaces are defined using weak derivatives, and the negative order spaces are defined by duality. By contrast, in the Bessel potential framework of the current paper, the spaces $H^s$ are defined in the same way for all $s \in \mathbb{R}$; hence it seems natural to define the notion of "negligibility" in the same way for all $s \in \mathbb{R}$. Our choice of "$s$-nullity" as the terminology for this concept over Maz'ya's teminology "$(-s)$-polarity" was made simply to simplify the presentation of the results which follow and make the arguments easier to read (we find it more natural to say that a set which does not support an $H^s(\mathbb{R}^n)$ distribution is "$-s$-null" rather than "$(-s)$-polar"). But the difference is essentially semantic, so readers familiar with the concept of polarity should read "$(-s)$-polar" for "$s$-null" throughout.

The following lemma collects a number of basic facts about $s$-nullity, relating the concept to topological and geometrical properties of a set. In particular, the results in parts (vii) and (viii) provide a partial characterization of the $s$-null sets for $n=2$, $s < 0$ in terms of Hausdorff dimension, (defined e.g. in [2, §5.1]), which we denote here, for an arbitrary set $E \subset \mathbb{R}^n$, by $\dim_H(E)$. We remark that the results in [2, Chapter 5] in fact allow a slightly more precise characterization of $s$-null sets for $n=2$, $s < 0$ in terms of Hausdorff measures, but the results in parts (vii) and (viii) seem sufficient for the applications of scattering by fractal screens that motivate the current study (and moreover we note that Adams and Hedberg's remark [2, §5.6.4] implies that no complete characterization in terms of Hausdorff measure is possible).

**Lemma 2.4.** Let $E$ be any subset of $\mathbb{R}^n$, and $s \in \mathbb{R}$. Then:

(i) If $E$ is $s$-null then any subset $E'$ of $E$ is also $s$-null.

(ii) If $E$ is $s$-null then $E$ is also $t$-null for all $t > s$.

(iii) If $E$ is $s$-null then $E$ must have empty interior.

(iv) If $s > n/2$ then $E$ is $s$-null if and only if $E$ has empty interior.

(v) If $s \geq 0$ then any set $E$ with zero Lebesgue measure is $s$-null.

(vi) $E$ is 0-null if and only if $E$ has zero Lebesgue measure.

(vii) For $n/2 < s < 0$, if $E$ is non-empty, Borel and $s$-null, then $\dim_H(E) \geq n + 2s$.

(viii) For $n/2 < s < 0$, if $E$ is non-empty and Borel and $\dim_H(E) < n + 2s$, then $E$ is $s$-null.

(ix) If $s = -n/2$, any finite set is $s$-null.

(x) If $s < -n/2$ then there are no non-empty $s$-null sets.

(xi) If $\Omega$ is a $C^0$ open set then $\partial\Omega$ is $s$-null if $s \geq 0$.

(xii) If $\Omega$ is a $C^0$ open set then $\partial\Omega$ is not $s$-null if $s < 1/2$.

(xiii) If $\Omega$ is a $C^{0,\alpha}$ open set for some $0 < \alpha < 1$ then $\partial\Omega$ is $s$-null if $s > -\alpha/2$.

(xiv) If $\Omega$ is a Lipschitz open set then $\partial\Omega$ is $s$-null if and only if $s \geq 1/2$.

(xv) Let $F_1$ and $F_2$ be closed, $s$-null subsets of $\mathbb{R}^n$. Then $F_1 \cup F_2$ is $s$-null.

Proof. (i) and (ii) These follow straight from the definition of $s$-nullity. (iii) If $E$ has non-empty interior one can trivially construct a non-zero element of $C_0^1(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ supported inside $E$. (iv) In this case $H^s(\mathbb{R}^n)$ consists of continuous functions (by the Sobolev embedding theorem [26, Theorem 3.26]). (v) and (vi) follows from the fact that $H^s(\mathbb{R}^n)$ is continuously embedded inside $L^2(\mathbb{R}^n)$ for $s \geq 0$. (vii) and (viii) are proved below. (ix) If $E$ is a finite set, then any distribution supported on $E$ is necessarily a linear combination of delta functions and their derivatives supported on the points of $E$ [26, Theorem 3.9], but delta functions are not contained in $H^{-n/2}(\mathbb{R}^n)$ (see (15)). (x) In this case $H^s(\mathbb{R}^n)$ does contain delta functions, so any non-empty set $E$ supports non-zero elements of $H^s(\mathbb{R}^n)$. (xi) follows from (v) and the fact that the graph of a continuous function has zero Lebesgue measure (which can be seen by considering the measure of the union of finitely many vertical translates of its graph). (xii) The case $n = 1$ is covered by (x), so assume that $n \geq 2$. Suppose, for a contradiction, that $\partial\Omega$ is $s$-null for some $s < 1/2$. Without loss of generality we can, by (ii), assume that $-n/2 \leq s < 1/2$. Then, by (vii), $\dim_H \partial\Omega \geq n + 2s < n - 1$. But this contradicts the fact that $\dim_H \partial\Omega \geq n - 1$. Hence no such $s$ can exist. (xiii) follows from (viii) and the fact that $\dim_H \partial\Omega \geq n - \alpha$. (xiv) is proved at the end of this section. (xv) We first notice the following fact: if $U \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$, and if there exists $x \in \operatorname{supp} U$ such that $\psi(x) \neq 0$, then $U\psi \neq 0$ as a distribution on $\mathbb{R}^n$. To prove this, suppose that there exists $x \in \operatorname{supp} U$ such that $\psi(x) \neq 0$. Then for $\varepsilon > 0$ let $B_\varepsilon(x)$ be the ball of radius $\varepsilon$ centred on $x$. Choose $\varepsilon$ such that $\psi$ is non-zero in $B_\varepsilon(x)$. Then, since $x \in \operatorname{supp} U$, $U|_{B_\varepsilon(x)} \neq 0$ and so $U(\phi) \neq 0$ for some $\phi \in \mathcal{D}(B_\varepsilon(x))$. But then, defining $\phi' \in \mathcal{D}(\mathbb{R}^n)$ by $\phi'(x) := \phi/\psi$, for $x \in B_\varepsilon(x)$, and $\phi'(x) := 0$ otherwise, we have $(U\psi)(\phi') = U(\phi) \neq 0$, so $U\psi$ is non-zero. To prove (xv) we argue again by contrapositive. Suppose that $F_1 \cup F_2$ is not $s$-null, i.e. there exists $0 \neq u \in H^s(\mathbb{R}^n)$ with $\operatorname{supp} u \subset F_1 \cup F_2$. Then if $\operatorname{supp} u \subset F_1$ or $\operatorname{supp} u \subset F_2$ we are done. Otherwise, suppose w.l.o.g. that there exists $x \in \operatorname{supp} u \cap (F_1 \setminus F_2)$. Then since $F_2$ is closed, $\varepsilon := \operatorname{dist}(x, F_2) > 0$. Let $\phi \in \mathcal{D}(B_\varepsilon(x))$ with $\phi(x) \neq 0$. Then by the result mentioned above $0 \neq u\phi \in H^s(\mathbb{R}^n)$ with $\operatorname{supp} u\phi \subset F_1$, so that $F_1$ is not $s$-null. □

Remark 2.5. Parts (vii) and (viii) of Lemma 2.4 imply that if $E$ is Borel and has zero Lebesgue measure, then

$$\dim_H(E) = \inf\{d : E \text{ is } (d, n)/2\text{-null}\},$$

which is similar to [35, Theorem 17.8] in the special case $p = q = 2$.

Remark 2.6. Part (xv) of Lemma 2.4 is not true for general subsets. A simple counterexample is where $F_1$ comprises the rational elements of $[0,1]$ and $F_2$ the irrational elements. Then for $s > 1/2$ both $F_1$ and $F_2$ are $s$-null, since they both have empty interior. But $F_1 \cup F_2 = [0,1]$, which is not $s$-null for any $s$ (since it has non-empty interior).

The remainder of this section is dedicated to the proof of parts (vii), (viii) and (xiv) of Lemma 2.4. For this we require an alternative characterisation of $s$-null sets in terms of a set function called capacity. Since the notion of capacity is not used elsewhere in the paper, readers wishing to skip the remainder of this section can safely proceed to §2.5.

Definition 2.7. For $s \in \mathbb{R}$ and $K \subset \mathbb{R}^n$ compact, define (cf. [25, §13.1], where Cap

(ii) For $0 < s < n = 2$, if $\dim_H(E) < n \quad 2s$ then $\overline{Cap}_s(E) = 0$.

Proof. This follows from Corollary 3.3.4, Theorem 5.1.9 and Theorem 5.1.13 of [2]. □

Remark 2.10.

From Theorem 2.11 one can deduce a corresponding result about spaces de ned on open subsets.

Theorem 2.12. Let $_1$, $_2$ be non-empty, open subsets of $\mathbb{R}^n$, and let $s \in \mathbb{R}$. Then the following are equivalent:

(i) $_1 \quad _2$ is s-null.

(ii) $_1 \setminus \quad _2$ and $_2 \setminus \quad _1$ are both s-null.

(iii) $_1 \cap \quad _2$ is non-empty and $H^s(\ _1 \cap \quad _2) = H^s(\ _1) = H^s(\ _2) = H^s(\ _1 \cup \quad _2)$, in the sense that $H^s_{\mathbb{R}^n \setminus (\ _1 \cap \ _2)}{}^{?} = H^s_{\mathbb{R}^n \setminus \ _1}{}^{?} = H^s_{\mathbb{R}^n \setminus \ _2}{}^{?} = H^s_{\mathbb{R}^n \setminus (\ _1 \cup \ _2)}{}^{?}$ (recall the identi cation $H^s(\ ) = (H^s_{\mathbb{R}^n \setminus \ })^{?}$ discussed in x2.1).

(iv) $_1 \cap \quad _2$ is non-empty and $\dot{H}^s(\ _1 \cap \quad _2) = \dot{H}^s(\ _1) = \dot{H}^s(\ _2) = \dot{H}^s(\ _1 \cap \quad _2)$.

Proof. The result follows from applying Theorem 2.11 with $F_j := \mathbb{R}^n \setminus \ _j$, $j = 1; 2$, and from the duality Theorem 2.1. □

Remark 2.13. We note that for non-empty, open sets $_1$, $_2$, the symmetric di erence $_1 \quad _2$ has empty interior if and only if

$$\overline{\ _1 \cap \quad _2} = \overline{\ _1} = \overline{\ _2} = \overline{\ _1 \cup \quad _2}: \tag{25}$$

Thus, by Lemma 2.4(iii),(iv), (25) is a necessary condition for the statements (i){(iv) of Theorem 2.12 to hold; (25) is also a su cient condition when $s > n=2$, but not in general for smaller $s$.

## 2.5 The relationship between $\dot{H}^s(\ )$ and $H^s$

(iii) If $s > n/2$ and $\Omega \neq \operatorname{int}(\overline{\Omega})$ then $H^s(\Omega) \not\subset H^s_0$.

Proof. (i) holds because $H^0(\Omega) = L^2(\Omega)$ and $H^0_0 = L^2(\overline{\Omega})$. For (ii) and (iii), if either $0 < s \leq n/2$ and $\dim_H(\operatorname{int}(\overline{\Omega}) \setminus \Omega) > n - 2s$ or $s > n/2$ and $\Omega \neq \operatorname{int}(\overline{\Omega})$, then (by Lemma 2.4(vii) or (x) respectively) $\operatorname{int}(\overline{\Omega})$ is $(-s)$-null, and hence $H^s(\Omega) \not\subset H^s(\operatorname{int}(\overline{\Omega})) \subset H^s_0$

is any element of $D_{1;}(R^n)$ with $x \in \mathrm{supp}$, and

$$(x;y) := \begin{cases} \dfrac{e^{ikjx\ \ yj}}{4\ jx\ \ yj}; & n = 3; \\ \dfrac{i}{4} H_0^{(1)}(kjx\ \ yj); & n = 2; \end{cases} \qquad x;y \in R^n; \tag{27}$$

is the fundamental solution of the Helmholtz equation.

The following properties of $S_k$ and $D_k$ are well-known when the densities lie in $H^{-1=2}(\ )$ and $H^{-1=2}(\ )$ respectively. The extension to $H_-^{1=2}$ can be carried out with the help of Remark 2.17.

Theorem 3.1. (i) For any $\in H_-^{1=2}$ and $\in H_-^{1=2}$ the potentials $S_k$ and $D_k$ are twice-continuously differentiable in $D$, satisfy the Helmholtz equation in $D$, and satisfy the Sommerfeld radiation condition at infinity;

(ii) for any $\in D(R^n)$ the following mappings are bounded:

$$S_k : H_-^{1=2} \to W^1(D); \qquad D_k : H_-^{1=2} \to W^1(D);$$

(iii) the following jump relations hold for all $\in H_-^{1=2}$, $\in H_-^{1=2}$ and $\in D_{1;}(R^n)$:

$$[S_k] = 0; \tag{28}$$

$$\partial_n(S_k) = \ = 2; \quad \text{so that } [\partial(S_k)=\partial n] = \ ; \tag{29}$$

$$(D_k) = \ = 2; \quad \text{so that } [D_k] = \ ; \tag{30}$$

$$[\partial(D_k)=\partial n] = 0; \tag{31}$$

(iv) for $\in D(\ )$ the following integral representations are valid:

$$S_k(x) = \int (x;y)(y)\,ds(y); \qquad x \in D; \tag{32}$$

$$D_k(x) = \int \frac{\partial(x;y)}{\partial n(y)}(y)\,ds(y); \qquad x \in D: \tag{33}$$

Proof. (Sketch) The proof mostly follows standard arguments, but the extension to $H_-^{1=2}$ requires us to use the fact (cf. Remark 2.17) that $H_-^{1=2} \to H^{-1=2}(\ ")$ for any neighbourhood $"$ of , so that elements of $H_-^{1=2}$ can be approximated arbitrarily well by smooth functions whose support is arbitrarily close to .

To show the claimed regularity, one uses continuity of the trace operators to show that the potentials converge uniformly on compact subsets of $D$, and that they are bounded in terms of the norm of their argument. The potentials are clearly infinitely differentiable in $D$ and satisfy the Helmholtz equation and the SRC for densities $\in D(\ ")$, then one can apply standard elliptic regularity results (e.g. [14, Lemma 3.9]), along with the uniform boundedness, to deduce that the same is also

true for general densities. To show that the potentials are in $W^1_{loc}(D)$, one first shows that they are in $W^1_{loc}(U)$. To do this, one shows that the potentials, after multiplication by a cut-off function, live in $H^1(U) = \tilde{H}^{-1}(U)$. To show that they define elements of $\tilde{H}^{-1}(U)$, first consider a smooth argument, then use Fubini's Theorem to rewrite the duality pairing in terms of the trace operator and the Newton potential. Finally, use the fact that the Newton potential maps $H^{-1}(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. For the double layer potential one has to use [26, Lemma 4.3] to get a bound on the Neumann trace in a different norm to the one on $W^1(U;)$. Once boundedness has been shown, one can then extend to non-smooth densities. Since we first work locally rather than the whole of $D$, the modification of the proof from $\tilde{H}^{-1/2}(\Gamma)$ to $H^{-1/2}$ is straightforward. One then appeals to smoothness across $\Gamma_1$ to conclude that the potentials map into $W^1_{loc}(D)$ using (26).

The jump relations and integral representations are standard and can be derived from the corresponding results for Lipschitz domains (see, e.g., [6]). We also make use of the fact that the double layer and adjoint double layer operators vanish on a flat screen. □

## 3.3 Boundary value problems

We now recall the boundary value problems D and N introduced in Definitions 1.1 and 1.2; for ease of reference we restate them here, with the boundary conditions (3) and (5) stated more precisely in terms of traces:

**Definition 3.3** (Problem D). Given $g_D \in H^{1/2}(\Gamma)$, find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$\Delta u + k^2 u = 0, \quad \text{in } D; \tag{37}$$

$$(\gamma u)|_\Gamma = g_D, \quad \text{for any } \gamma \in \mathcal{D}_{1,\Gamma}(\mathbb{R}^n); \tag{38}$$

and $u$ satisfies the Sommerfeld radiation condition.

**Definition 3.4** (Problem N). Given $g_N \in H^{-1/2}(\Gamma)$, find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$\Delta u + k^2 u = 0, \quad \text{in } D; \tag{39}$$

$$\partial_n(\gamma u)|_\Gamma = g_N, \quad \text{for any } \gamma \in \mathcal{D}_{1,\Gamma}(\mathbb{R}^n); \tag{40}$$

and $u$ satisfies the Sommerfeld radiation condition.

Following the standard direct boundary integral equation approach, we would like to use Theorem 3.2 to represent the solution of problem D (assuming it exists) in the form

$$u(x) = S_k[\partial u/\partial n](x), \qquad x \in D. \tag{41}$$

Of course, deriving (41) from the general representation formula (34) requires us to show that

$$[u] = 0. \tag{42}$$

Now, by definition we have that $[u] \in H^{1/2}_-$, and the boundary condition (38) implies that $[u]|_\Gamma = 0$. So it must hold that $[u] \in H^{1/2}_\Gamma$. If $\Gamma$ is $1/2$-null, i.e. $H^{1/2}_\Gamma = \{0\}$, then (42) immediately follows. This holds, for example, if $\Gamma$ is $C^0$ (in particular if $\Gamma$ is Lipschitz), by Lemma 2.4(xi). But if $\Gamma$ is not $1/2$-null then any non-zero $\varphi \in H^{1/2}_\Gamma \cap H^{1/2}_-$ provides, by Theorem 3.1, a non-trivial solution (namely $D_k\varphi$) of the homogeneous Dirichlet problem (i.e. problem D with $g_D = 0$). So the solution to problem D for general $\Gamma$ and general $g_D \in H^{1/2}(\Gamma)$ is not unique.

Similarly, we would like to represent the solution of problem N (assuming it exists) in the form

$$u(x) = D_k[u](x), \qquad x \in D; \tag{43}$$

and deriving (43) from (34) requires us to show that

$$[\partial u/\partial n] = 0. \tag{44}$$

By definition we have that $[\partial u/\partial n] \in H^{-1/2}_-$, and the boundary condition (40) gives $(\partial u/\partial n)|_\Gamma = 0$. So it must hold that $[\partial u/\partial n] \in H^{-1/2}_\Gamma$. If $\Gamma$ is $(-1/2)$-null, i.e. $H^{-1/2}_\Gamma = \{0\}$, then (44) immediately

follows. This holds, for example, if is Lipschitz, by Lemma 2.4(xiv). But if @ is not ( 1=2)-null then any non-zero 2 $H_{@}^{1=2}$ $H_{-}^{1=2}$ provides, by Theorem 3.1, a non-trivial solution (namely $S_k$ ) of the homogeneous Neumann problem (i.e. problem N with $g_N = 0$). So the solution to problem N for general and general $g_N$ 2 $H^{1=2}$( ) is not unique.

To deal with this possible nonuniqueness we modify the BVPs D and N by requiring that their solutions satisfy the jump conditions (42) and (44), respectively.

De nition 3.5. Let $D^0$ and $N^0$ denote problems D and N, supplemented respectively with the additional constraints (42) and (44).

It then follows straight from Theorems 3.1 and 3.2 that the modi ed BVPs $D^0$ and $N^0$ are equivalent to the usual boundary integral equations involving the operators $S_k$ and $T_k$, respectively.

De nition 3.6 (Problem S). Given $g_D$ 2 $H^{1=2}$( ) , nd 2 $H_{-}^{1=2}$ such that

$$S_k = g_D: \tag{45}$$

De nition 3.7 (Problem T). Given $g_N$ 2 $H^{1=2}$( ) , nd 2 $H_{-}^{1=2}$ such that

$$T_k = g_N: \tag{46}$$

Theorem 3.8. Suppose that u is a solution of problem $D^0$. Then the representation formula (41) holds, and [@u=@ ] 2 $H_{-}^{1=2}$ satis es problem S. Conversely, suppose that 2 $H_{-}^{1=2}$ satis es problem S. Then u := $S_k$ satis es problem $D^0$, and [@u=@ ] = .

Theorem 3.9. Suppose that u is a solution of problem $N^0$. Then the representation formula (43) holds, and [u] 2 $H_{-}^{1=2}$ satis es problem T. Conversely, suppose that 2 $H_{-}^{1=2}$ satis es problem T. Then u := $D_k$ satis es problem $N^0$, and [u] = .

The question of the unique solvability (or otherwise) of problems S and T, and hence (by Theorems 3.8 and 3.9) of problems $D^0$ and $N^0$, is answered by the following two theorems, which follow from Theorems 5.2, 5.3, 6.1 and 6.2 (where the dependence of the continuity and coercivity constants on both k and is stated explicitly, the continuity results being shown to hold on Sobolev spaces of arbitrary real index s). We emphasize that these results all hold with an arbitrary non-empty open subset of $_1$ .

Theorem 3.10. For every k > 0 the single-layer operator $S_k : H_{-}^{1=2} \to H^{1=2}$( ) is continuous, and is coercive as an operator $S_k : H_{\sim}^{1=2}$( ) ! $H^{1=2}$( ) = ($H_{\sim}^{1=2}$( ))$^*$ .

Theorem 3.11. For every k > 0 the hypersingular operator $T_k : H_{-}^{1=2} \to H^{1=2}$( ) is continuous, and is coercive as an operator $T_k : H_{\sim}^{1=2}$( ) ! $H^{1=2}$( ) = ($H_{\sim}^{1=2}$( ))$^*$ .

Thus, by the Lax-Milgram Lemma, the operator $S_k : H_{\sim}^{1=2}$( ) ! $H^{1=2}$( ) is invertible. Hence if $H_{\sim}^{1=2}$( ) = $H_{-}^{1=2}$, then problem S (and hence also problem $D^0$) is uniquely solvable. This holds,

for example, if $\Gamma$ is $C^0$, by Lemma 2.14. But if $\tilde{H}^{1/2}(\Gamma) \simeq H^{-1/2}(\Gamma)$ then $S_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is surjective but not injective, i.e., a solution to problem $S$ (and hence also problem $D^0$) exists, but this solution is not unique.

Hence the Fourier transform (with respect to $x \in \mathbb{R}^{n-1}$) of $S_k$ is given by the product

$$\hat{S}_k(\xi; x_n) = (2\pi)^{(n-1)/2} c_c(|\xi|; x_n)\hat{\nu}(\xi).$$

To evaluate $c_c$ we note that for a function $f(x) = F(r)$, where $r = |x|$ for $x \in \mathbb{R}^d$, $d = 1, 2$, the Fourier transform of $f$ is given by (cf. [16, xB.5]) [4]

$$\hat{f}(\xi) = \begin{cases} 2\pi \int_0^\infty F(r) J_0(|\xi| r)\, r\, dr, & d = 2, \\ \sqrt{\frac{2}{\pi}} \int_0^\infty F(r) \cos(\xi r)\, dr, & d = 1. \end{cases} \tag{63}$$

This result, combined with the identies [15, (6.677), (6.737)] and [1, (10.16.1), (10.39.2)], gives (see also [7, eqn (4.17)] for the case $n = 3$)

$$c_c(|\xi|; x_n) = \frac{i e^{i|\xi| x_n|Z(\xi)}}{2(2\pi)^{(n-1)/2} Z(\xi)},$$

where $Z(\xi)$ is de ned as in (57). The representation (58) is then obtained by Fourier inversion.

Lemma 5.1. Given $L > 0$ let

$$_L(\cancel{x}; x_n) := \quad ( \quad {}^c$$

and by (66) we get $|I_1| \le (C/k)(1 + R)$ $\le C/k$. If $\le 1$ then $R = 1$ and $I_2 = 0$. If $> 1$ then $R = 1/$ and $I_2 \ne 0$, and a similar integration by parts to that used for $I_1$ above, combined with the bound (69), gives the estimate

$$|I_2| \le \frac{C}{k}\left(1 + \int_{1/}^{1} \frac{dr}{\sqrt{r}}\right) \le \frac{C}{k}(1 + k^{1/2}): \tag{73}$$

Finally, since $\sqrt{k^2 + \ ^2} \ge \sqrt{2}k$ in this case, we conclude that

$$|c_1(\ ; x_3)| \sqrt{k^2 + \ ^2} \le C(1 + k^{1/2}); \qquad k > 1; \qquad k: \tag{74}$$

When $> 1$ and $k < $ , $R = 1/$ and $|I_1| \le 1/$ by (66). Integration by parts, using the relation (71), gives

$$I_2 = \frac{1}{} \left[ \frac{re^{ik\sqrt{r^2 + x_3^2}}}{\sqrt{r^2 + x_3^2}} J_1(r) \right]_{1/}^{"} - \frac{1}{} \int_{1/}^{1} r^2 e^{ik\sqrt{r^2 + x_3^2}} \left( \frac{ik}{r^2 + x_3^2} - \frac{1}{(r^2 + x_3^2)^{3/2}} \right) J_1(r)\, dr;$$

and using (69) we have

$$|I_2| \le \frac{C}{} \left(1 + \frac{1}{1/2} \int_{1/}^{1} \left( \frac{k}{\sqrt{r}} + \frac{1}{r^{3/2}} \right) dr \right) \le \frac{C}{}(1 + k^{1/2}):$$

Then, since $\sqrt{k^2 + \ ^2} \ge \sqrt{2}$ in this case, we conclude that

$$|c_1(\ ; x_3)| \sqrt{k^2 + \ ^2} \le C(1 + k^{1/2}); \qquad > 1; \ k < : \tag{75}$$

Combining (72), (74) and (75) gives the result (65) in the case $n = 3$.

(ii) In the case $n = 2$, $|c_1(\ ; x_2)| \ |I| \le (2\sqrt{2})$, where now

$$I := \int_0^1 H_0^{(1)}(k\sqrt{r^2 + x_2^2}) \cos(r)\, dr = \underbrace{\int_0^R H_0^{(1)}(k\sqrt{r^2 + x_2^2}) \cos(r)\, dr}_{:= I_1} + \underbrace{\int_R^1 H_0^{(1)}(k\sqrt{r^2 + x_2^2}) \cos(r)\, dr}_{:= I_2};$$

with $R = \min\{1; 1/k\}$ and $I_2 := 0$ for $k \le 1$. We distinguish the same three subcases as before:

When both $k \le 1$ and

When $k > 1$ and $k$, $R = 1=k$ and $|I_1| \quad C=k$ after use of (67). Integration by parts, using the relation (71), gives

$$I_2 = \frac{1}{k}\left[\frac{\sqrt{r^2+x_2^2}}{r}H_1^{(1)}(k\sqrt{r^2+x_2^2})\cos r\right]_{1=k} + \frac{1}{k}\int_{1=k}^{1}\frac{\sqrt{r^2+x_2^2}}{r}H_1^{(1)}(k\sqrt{r^2+x_2^2})\quad \sin r + \frac{\cos r}{r}\quad dr$$

and using (69) we have

$$|I_2| \quad \frac{C}{k}\quad 1+\sqrt{|kx_2|}+\frac{1}{k^{1=2}}\int_{1=k}^{1}\quad \frac{k}{r}+\frac{1}{r^{3=2}}+\frac{k|x_2|^{1=2}}{r}+\frac{|x_2|^{1=2}}{r^2}\quad dr$$

$$\frac{C}{k}\quad 1+k^{1=2}+\sqrt{|kx_2|}\log(2+k)\quad :$$

Then, since $\sqrt{k^2+\quad^2}\quad \sqrt{2}k$ in this case, we have

$$|c_1(\ ;x_2)|\sqrt{k^2+\quad^2}\quad C\quad 1+k^{1=2}+\sqrt{|kx_2|}\log(2+k)\quad ;\qquad k>1;\qquad k: \qquad (77)$$

When $\quad > 1$ and $k < \quad$, integration by parts, using the relation (70), gives

$$I_1 = \frac{1}{}\quad H_0^{(1)}(k\sqrt{r^2+x_2^2})\sin r\quad{}_0^R + \frac{k}{}\int_0^R \frac{r}{\sqrt{r^2+x_2^2}}H_1^{(1)}(k\sqrt{r^2+x_2^2})\sin r\ dr: \qquad (78)$$

Noting that (cf. [1, x10.8 and x10.17]) there exist $c_0; c_1 > 0$ such that $H_1^{(1)}(z) = c_0=z + F_1(z)$ for $z > 0$, where $|F_1(z)| \quad c_1$ for $z > 0$, we can rewrite the second term in (78) as

$$\frac{1}{}\int_0^R \quad \frac{c_0 r}{r^2+x_2^2}+\quad kF_1 r$$

Proof. We rst prove (80) for $\tilde{H}^s(\ )$. By the density of $D(\ )$ in $\tilde{H}^s(\ )$ it su ces to prove (80) for $2\ D(\ )$. For $2\ D(\ )$ we rst note that $S_k = (S_k^L\ )j$, where $S_k^L : D(R^{n\ 1})\ !\ D(R^{n\ 1})$ is the convolution operator de ned by $S_k^L\ ' := (\ _L(\ ;0)\ '\ )$, for $'\ 2\ C_0^1\ (R^{n\ 1})$, where $_L$ is de ned as in (64). While $S_k^1\ '\ 2\ C^1\ (R^{n\ 1})$ for $'\ 2\ D(R^{n\ 1})$, the fact that $_L$ has compact support means that $S_k^L\ '\ 2\ D(R^{n\ 1})\ \ H^{s+1}(R^{n\ 1})$ (cf. [37, Corollary 5.4-2a]). Therefore, for $2\ D(\ )$ we can estimate $kS_k\ k_{H_k^{s+1}(\ )}\ \ k\ S_k^L\ k_{H_k^{s+1}(R^{n\ 1})}$, and since $\widehat{S_k^L\ '}(\ ) = (\ _L(\ ;0)\ '\ )(\ ) = (2\ )^{(n\ 1)=2}c_L(\ ;0)'\hat{}(\ )$ for any $'\ 2\ D(R^{n\ 1})$, the bound (80) follows from Lemma 5.1.

Finally, we can extend the bound (80) to $2\ H^{\underline{s}}$ (without changing the constant) by appealing to Remark 2.17 (we can approximate $2\ H^{\underline{s}}$ arbitrarily well by an element of $\tilde{H}^s(\ ")$ for $"$ an arbitrarily small neighbourhood of ). □

Theorem 5.3. The sesquilinear form on $\tilde{H}^{\ 1=2}(\ )\ \ \tilde{H}^{\ 1=2}(\ )$ de ned by

$$a(\ ;\ \ ) := hS_k\ ;\ \ i_{H^{1=2}(\ )\ \ \tilde{H}^{\ 1=2}(\ )}\ ;\ \ ;\ \ 2\ \tilde{H}^{\ 1=2}(\ )\ ;$$

satis es the coercivity estimate

$$ja(\ ;\ \ )j\ \ \frac{1}{2\ \overline{2}}\ k\ k^2_{\tilde{H}_k^{\ 1=2}(\ )}\ ;\ \ 2\ \tilde{H}^{\ 1=2}(\ )\ ;\ k > 0: \tag{81}$$

Proof. By the density of $D(\ )$ in $\tilde{H}^{\ 1=2}(\ )$ it su ces to prove (81) for $2\ D(\ )$. For such a , formula (60) from Theorem 4.1 gives

$$ja(\ ;\ \ )j = \frac{1}{2}\ \int_{R^{n\ 1}} \frac{j\ b(\ )j^2}{Z(\ )}\ d\ \ \frac{1}{2\ \overline{2}}\ \int_{R^{n\ 1}} \frac{j\ b(\ )j^2}{\overline{jk^2\ \ j\ \ j^2j}}\ d\ \ \frac{1}{2\ \overline{2}}\ \int_{R^{n\ 1}} \frac{j\ b(\ )j^2}{\overline{k^2 + j\ \ j^2}}\ d\ ; \tag{82}$$

as claimed. □

Remark 5.4. We can show that the bounds established in Theorem 5.2 are sharp in their dependence on $k$ as $k\ !\ 1$, at least for the cases $s = \ 1=2$. For simplicity of presentation we assume that diam $= 1$ and $k > 1$. Let $(\tilde{x}) := e^{ik\tilde{d}\ \tilde{x}}\ (\tilde{x})$ for $\tilde{x}\ 2\ R^{n\ 1}$, where $\tilde{d}\ 2\ R^{n\ 1}$ is a unit vector and $0\ \overset{6}{=}\ \ 2\ D(\ )$ is independent of $k$, depending only on the shape of. Then $b(\ ) = \hat{}(\ )$, where $= \ \ k\tilde{d}$, and for any $\ \ 1$ the rst inequality in (82) gives that

$$ja(\ ;\ \ )j\ \ \frac{1}{2\ \overline{2}k}\ \int_{R^{n\ 1}} \frac{j\ b(\ )j^2}{\overline{j1\ \ j\ \ \tilde{d} + \ \ =kj^2j}}\ d\ \ \frac{1}{2\ \overline{6}\ \ k^{1=2}}\ \int_{j\ j}\ j\ b(\ )j^2\ d\ ; \tag{83}$$

since $j1\ \ j\ \ \tilde{d} + \ \ =kj^2j\ \ (1=k)j2\tilde{d}\ \ + j\ \ j^2=kj\ \ 3\ ^2=k$, for $j\ \ j$ . Also, for the same choice of ,

$$k\ \ k^2_{\tilde{H}_k^{\ 1=2}(\ )} = \frac{1}{k}\ \int_{R^{n\ 1}} \frac{j\ b(\ )j^2}{\overline{j1 + j\tilde{d} + \ \ =kj^2j}}\ d\ \ \frac{1}{k}\ \int_{R^{n\ 1}}\ j\ b(\ )j^2\ d\ \ \frac{2}{k}\ \int_{j\ j}\ j\ b(\ )j^2\ d\ ; \tag{84}$$

where $L := \operatorname{diam}$ , $C > 0$ is a constant depending only on $k_0$, and

$$= \begin{cases} \frac{2}{3}; & n = 3; \\ \frac{1}{2}; & n = 2: \end{cases}$$

Proof. As in the proof of Lemma 5.1, for ease of presentation we present the proof only for the case $L = 1$, but a simple rescaling deals with the general case. By the density of $D(\ )$ in $H^{-1=2}(\ )$ it su ces to prove (87) for $2\ D(\ )$. For such a , equation (61) from Theorem 4.1 gives that

$$jb(\ ;\ )j = \frac{1}{2} \int_{R^{n\ 1}} Z(\ )j\,b(\ )j^2\,d \qquad \frac{I}{2^p\ 2}; \tag{88}$$

where

$$I := \int_{R^{n\ 1}} jZ(\ )jj\,b(\ )j^2\,d\ :$$

De ning

$$J := k\ k^2_{H^{-1=2}_k(\ )} = \int_{R^{n\ 1}} (k^2 + j\ j^2)^{1=2} j\,b(\ )j^2\,d\ ;$$

the problem of proving (87) reduces to that of prcF14 11.ving 51 reduces [(01 Tf 4.234 092F2901 0 1 38

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and we have used the fact that, for $0 < |\xi| < k - \varepsilon$,

$$\frac{k^2 + |\xi|^2}{k^2 - |\xi|^2} \leq \frac{k^2 + (k-\varepsilon)^2}{k^2 - (k-\varepsilon)^2} \leq C_1^2.$$

Similarly,

$$J_4 := \int_{|\xi| > k+\varepsilon} (k^2 + |\xi|^2)^{1/2} |b(\xi)|^2 \, d\xi = \int_{|\xi| > k+\varepsilon} \frac{(k^2 + |\xi|^2)^{1/2}}{(}$$

where $a_n := k\ \ "=2$ if $n = 3$ and $a_n := 1$ if $n = 2$. We now note that, for any $0 < " < c < d$ ,

$$\int_{c<|\xi|<d} f(\xi\ \ "\hat{}(\xi)\,d\xi = \begin{cases} \int_{c\ "<|\xi|<d\ "} f(\xi)\left(1 - \frac{"}{|\xi|}\right)d\xi\ ; & n = 3; \\ \int_{c\ "<|\xi|<d\ "} f(\xi)\,d\xi\ ; & n = 2: \end{cases} \tag{94}$$

Assume $0 < " < k = 3$. Then for $k\ \ 2" \le |\xi| < k\ \ "$, we have $1 + "=|\xi|\ \ 2$, so that, using (94),

$$\int_{k\ "<|\xi|<k} |b(\xi\ \ "\hat{}(\xi)|^2\,d\xi \quad 2 \int_{k\ 2"<|\xi|<k\ "} |b(\xi)|^2\,d\xi$$

$$\frac{2}{(k^2\ \ (k\ \ ")^2)^{1=2}} \int_{k\ 2"<|\xi|<k\ "} (k^2\ \ |\xi|^2)^{1=2} |b(\xi)|^2\,d\xi$$

$$\frac{2}{("(2k\ \ "))_|}$$

Remark 6.3. We can show that the bounds established in Theorem 6.1 are sharp in their dependence on $k$ as $k \to 1$, at least for the cases $s = 1/2$. As before, for simplicity of presentation we assume that $\operatorname{diam} \Omega = 1$ and $k > 1$. Let $0 \neq \varphi \in D(\Omega)$ be independent of $k$. Then, by (88),

We are assuming that we have an estimate of the quantity $\|v\|_{H_k^{1/2}(\Omega)}$. It therefore remains to bound $\|w\|_{H_k^{1/2}(\Omega)}$.

Lemma 7.1. Let $k > 0$, let $\Omega$ be an arbitrary nonempty relatively open subset of $\mathbb{R}^1$, and let $L := \operatorname{diam} \Omega$.

(i) Let $d \in \mathbb{R}^n$ with $|d| \leq 1$. Then there exists $C > 0$, independent of $k$ and $d$, such that

singularities in $u_\eta(y)$ and therefore no restriction on the choice of $\eta$ in this case). Explicitly,

$$\begin{aligned} \|L_\eta u_\eta\|^2_{H^{1/2}_k(\mathbb{R}^{n-1})} &= L^{2(n-1)} \int_{\mathbb{R}^{n-1}} (k^2 + |\xi|^2)^{1/2} |\hat{b}(L(\xi - k\eta))|^2 \, d\xi \\ &= L^{n-1} \int_{\mathbb{R}^{n-1}} (k^2 + |\xi/L + k\eta|^2)^{1/2} |\hat{b}(\xi)|^2 \, d\xi \\ &\leq L^{n-2}(1 + kL) \int_{\mathbb{R}^{n-1}} (2 + |\xi|) |\hat{b}(\xi)|^2 \, d\xi \, ; \end{aligned}$$

after the change of variable $\xi = L(\xi - k\eta)$. Using this in (102) gives the desired result (100), where $C > 0$ is a constant independent of $k$, and proportional to

since the de nition of " = min f 1; d=(2L)g ensures that" d=(2L), which in turn ensures that r(s) d=2 for s 2 $F_{L''}$. We can estimate $jH_0^{(1)}(kd=2)j$ C log (2 + 1=(kd)), and also

$$\frac{1}{kL''} \quad 2 \quad \frac{1}{kL} + \frac{1}{kd} \quad 2 \quad 1 + \frac{1}{kL} \quad 1 + \frac{1}{kd} \quad ; \tag{105}$$

so that

$$k^{1=2} k_L^0 \text{ ʉ} k_{L^2(R)} \quad C \quad 1 + \text{ƿ}\frac{1}{kL} \quad 1 + \text{ƿ}\frac{1}{kd} \quad \log \quad 2 + \frac{1}{kd} \quad : \tag{106}$$

For the term involving $k_L \text{ ʉ}^0 k_{L^2(R)}$ we rst note that $j\text{ʉ}^0(s)j$ (k=4)$jH_1^{(1)}(kr(s))j$. The fact that $jH_1^{(1)}(z)j$ is decreasing implies that

$$k_L \text{ ʉ}^0 k^2_{L^2(R)} \quad \overset{Z}{\underset{F_{L''}}{}} \quad j\text{ʉ}^0 j^2$$

Finally, using (104), (106) and (107) in (103) gives the required estimate (101).

When n = 3 we have, in place of (103),

$$\|u\|_{H^{1/2}_k(R^2)} \le k^{1/2}\|u\|_{H^{1}_k(R^2)} \le k^{1/2}\|u\|_{L^2(R^2)} + k^{-1/2}\sum_{j=1}^{2}\left\|\frac{\partial u}{\partial y_j}\right\|_{L^2(R^2)} + \left\|\frac{\partial u}{\partial y_3}\right\|_{L^2(R)}. \tag{108}$$

Now, with $y = (\tilde{y}, 0)$ and $\tilde{y} = (y_1, y_2) \in R^2$, we have $u(y) - u(\tilde{y}) = e^{ikr(\tilde{y})}/(4\pi r(\tilde{y}))$, where $r(\tilde{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}$. Then $(\partial u/\partial y_j)(\tilde{y}) = (y_j - x_j)(ikr(\tilde{y}) - 1)e^{ikr(\tilde{y})}/(4\pi r(\tilde{y})^3)$. To avoid the singularity at $y = x$ we define $d := \mathrm{dist}(x, \Gamma)$ and take our closed set $F = B \setminus B_d$, where $B$ is the smallest closed ball containing $\Gamma$ and $B_d = \{\tilde{y} \in R^2 : r(\tilde{y}) < d\}$. Again we set $\varepsilon = \min\{1, d/(2L)\}$, and note that in th793 TFTJ 09 11.9552 Tf 17.18 0 T( )]TJ/F43 7.97(2

Then [10]

$$L \frac{@dr}{@y}^{2}_{L^2(R)} \quad Z_{F_{L''}} \frac{@dr}{@y}^{2} \quad Ck^2 \; Z_0^{l} \frac{1 + \frac{p}{r^2 + d^2}^{2} r\,dr}{(r^2 + d^2)^2}$$

$$Ck^2 \quad 1 + l$$

# 8 Aperture problems

**Definition 8.2** (Problem H). Given $g_H \in H^{-1/2}(\Gamma)$, find $u \in C^2(U) \cap W$ such that

$$\Delta u + k^2 u = 0; \quad \text{in } U; \tag{115}$$

$$u = 0; \quad \text{on } \Gamma_1 \subset \partial D; \tag{116}$$

$$\left[\frac{\partial u}{\partial \nu}\right] = g_H; \tag{117}$$

$$[u] = 0; \tag{118}$$

$$\{u\} \in H^{1/2}(\Gamma); \tag{119}$$

and $u$ satisfies the Sommerfeld radiation condition.

By (116) we mean, precisely, that $(\gamma u)|_{\Gamma_1} = 0$, where $\gamma$ is any element of $\mathcal{D}(R^n)$.

**Definition 8.3** (Problem I). Given $g_I \in H^{1/2}(\Gamma)$, find $u \in C^2(U) \cap W$ such that

$$\Delta u + k^2 u = 0; \quad \text{in } U; \tag{120}$$

$$\frac{\partial u}{\partial \nu} = 0; \quad \text{on } \Gamma_1 \subset \partial D; \tag{121}$$

$$[u] = g_I; \tag{122}$$

$$\left[\frac{\partial u}{\partial \nu}\right] = 0; \tag{123}$$

$$\{\partial u/\partial \nu\} \in H^{-1/2}(\Gamma); \tag{124}$$

and $u$ satisfies the Sommerfeld radiation condition.

By (121) we mean, precisely, that $\frac{\partial}{\partial \nu}(\gamma u)|_{\Gamma_1} = 0$, where $\gamma$ is any element of $\mathcal{D}(R^n)$.

Example 8.4. Consider the aperture problem of scattering by $\Gamma_1 \subset \partial D$ of a plane wave

$$u^i(x) := e^{ikx \cdot d}; \qquad x \in U^+; \tag{125}$$

incident from $U^+$, where $d = (\hat{d}, d_n) \in R^n$ is a unit direction vector with $\hat{d} \in R^{n-1}$ and $d_n < 0$. The cases of a `sound soft' and a `sound hard' screen are modelled respectively by problem H (with $g_H = -2\partial u^i/\partial \nu$) and problem I (with $g_I = -2u^i$). In both cases $u$ represents the diffracted field, the total field being given by

$$u^{tot} = \begin{cases} u + u^i + u^r; & \text{in } U^+; \\ u; & \text{in } R^n \setminus \end{cases}$$

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here, we note that the conditions (118) and (123) in problems H and I are required to ensure that (112) and (113) hold for both problems, so that the representation theorem is valid. Specifically, if $u$ is a solution of problem H then set $w(x) := u(x) - u(x^0)$, where $x^0$ denotes the reflection of $x$ in $\Gamma_1$. From (116) and (118) and the uniqueness of the solution of the Helmholtz equation in a half-space with Dirichlet boundary conditions it follows that $w = 0$, i.e. that $u(x) = u(x^0)$, which implies (113). Similarly, one can prove that if $u$ is a solution of problem I then $u(x) = -u(x^0)$, which implies (112). These results are summarised in the following theorems.

Theorem 8.5. For any $g_H \in H^{1/2}(\Gamma)$ problem H has a unique solution given by formula

$$u(x) = D_k\{u\}(x), \qquad x \in U,$$

where $\{u\}$ is the unique solution in $H^{-1/2}(\Gamma)$ of equation (46) with $g_N = g_H/2$.

Theorem 8.6. For any $g_I \in H^{1/2}(\Gamma)$ problem I has a unique solution given by formula

$$u(x) = S_k\{\partial u/\partial n\}(x), \qquad x \in H^{1-}$$

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