Department of Mathematics and Statistics

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The full infinite dimensional moment problem on semi-algebraic sets of

THE FULL INFINITE DIMENSIONAL MOMENT PROBLEM ON SEMI-ALGEBRAIC SETS OF GENERALIZED FUNCTIONS

M. INFUSINO [;+], T. KUNA [+], A. ROTA [+]

Abstract. We consider a generic basic semi-algebraic subset S of the space of generalized functions, that is a set given by (not necessarily countably many) polynomial constraints. We derive necessary and su cient conditions for an in-

result completely characterizes the support of the realizing measure in terms of its moments. As concrete examples of semi-algebraic sets of generalized functions, we consider the set of all Radon measures and the set of all the measures having bounded Radon-Nikodym density w.r.t. the Lebesgue measure.

Introduction

It is often more convenient to consider characteristics of a random distribution instead of the random distribution itself and try to extract information about the distribution from these characteristics. In this paper, we are more concretely interested in distributions on functional objects like random elds, random points, random sets and random measures. The characteristics under study are polynomials of these objects like the density, the pair distance distribution, the covering function, the contact distribution function, etc.. This setting is considered in numerous areas of applications: heterogeneous materials and mesoscopic structures [44], stochastic geometry [29], liquid theory [14], spatial statistics [43], spatial ecology [30] and neural spike trains [7, 16], just to name a few.

The subject of this paper is the full power moment problem on a pre-given subset S of $\mathcal{D}'(\mathbb{R}^d)$, the space of all generalized functions on $\mathbb{R}^d$. This framework choice is mathematically convenient and general enough to encompass all the aforementioned applications. More precisely, our paper addresses the question of whether certain prescribed generalized functions are in fact the moment functions of some nite measure concentrated on S. If such a measure does exist, it will be called realizing. The main novelty of this paper is to investigate how one can read o support properties of the realizing measure directly from positivity properties of its moment functions.

To be more concrete, homogeneous polynomials are de ned as powers of linear functionals on $\mathcal{D}'(\mathbb{R}^d)$ and their linear continuous extensions. We denote by $\mathcal{P}_{C_c^\infty}(\mathcal{D}'(\mathbb{R}^d))$ the set of all polynomials on $\mathcal{D}'(\mathbb{R}^d)$ with coe cients in $C_c^\infty(\mathbb{R}^d)$, which is the set of all in nite di erentiable functions with compact support in $\mathbb{R}^d$.

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In this paper, we try to find a characterization via moments of measures concentrated on basic semi-algebraic subsets of $\mathcal{D}'(\mathbb{R}^d)$, i.e. sets that are given by polynomial constraints and so are of the following form

$$S = \bigcap_{i \in Y} \left\{ \xi \in \mathcal{D}'(\mathbb{R}^d) \mid P_i(\xi) \geq 0 \right\},$$

where Y is an arbitrary index set (not necessarily countable) and each P_i is a polynomial in $\mathcal{P}_{C_c^\infty}(\mathcal{D}'(\mathbb{R}^d))$. Equality constraints can be handled using P_i and $-P_i$ simultaneously. As far as we are aware, the infinite dimensional moment problem has only been treated in general on affine subsets [4, 2] and cones [42] of nuclear spaces (these results are stated in Section 2 and Subsection 5.3). Special situations have also been handled; see e.g. [46, 3, 17].

Previous results.

Characterization results via moments are built up out of five completely different types of conditions

I. positivity conditions on the moment sequence
II. conditions on the asymptotic behaviour of the moments as a sequence of their degree
III. properties of the putative support of the realizing measure
IV. regularity properties of the moments as generalized functions
V. growth properties of the moments as generalized functions.

Conditions of type IV and V are only relevant for the infinite dimensional moment problem. The general aim in moment theory is to construct a solution which is as weak as possible w.r.t. some combination of the above different types of conditions, since it seems unfeasible to get one solution which is optimal in all types simultaneously.

Let us give a review of some previous results on which our approach is based and describe the different types of conditions involved in each of them.

Given a sequence m of putative moments, one can introduce on the set of all polynomials the so-called Riesz functional L_m, which associates to each polynomial its putative expectation. If a polynomial P is non-negative on the prescribed support S, then a necessary condition for the realizability of m on S is that $L_m(P)$ is non-negative as well. The question whether this condition alone is also sufficient for the existence of a realizing measure concentrated on $S \subseteq \mathbb{R}^d$ is answered by the Riesz-Haviland theorem [36, 15]; for infinite dimensional versions of this theorem see e.g [24, 25, 28] for point processes and [19, 20] for the truncated case. The disadvantage of this type of positivity condition is that it may be rather difficult and also computationally expensive to identify all non-negative polynomials on S, especially if the latter is geometrically non-trivial.

A classical result shows that all non-negative polynomials on $\mathbb{R}$ can be written as the sum of squares of polynomials (see [32]). Hence, it is already sufficient for realizability on $S = \mathbb{R}$ to require that L_m is non-negative on squares of polynomials, that is, m is positive semidefinite. For the moment problem on $S = \mathbb{R}^d$ with $d \geq 2$, the positive semidefiniteness of m is no longer sufficient, as already pointed out by D. Hilbert in the description of his 17th problem. However, the positive semidefiniteness of m becomes sufficient if one additionally assumes a condition of type II, that is, a bound on a certain norm of the n-th putative moment $m^{(n)}$. For example, one could require that $|m^{(n)}|$ does not grow faster than $BC^n n!$ or than $BC^n (n \ln(n))^n$ for some constants $B, C > 0$. The weakest known growth condition of this kind is that the sequence m is quasi-analytic (see Appendix 6). We will call such a sequence *determining*, because this property guarantees the uniqueness of the realizing measure. The determinacy condition in the infinite dimensional case additionally involves the types IV and V.

Beyond the results for $S = \mathbb{R}^d$, for a long time the moment problem was only studied for specific proper subsets S of $\mathbb{R}^d$ rather than general classes of sets. How-

We assume that each H_k is embedded topologically into H_0. Let be the projective limit of the family $(H_k)_{k 2 K}$ endowed with the associated projective limit topology and let us assume that is nuclear, i.e. for each k_1 2 K there exists k_2 2 K such that the embedding H_{k_2} H_{k_1} is quasi-nuclear.

Let us denote by ^{0} the topological dual space of . We control the classical rigging by identifying H_0 and its dual H_0^0. With this identi cation one can de ne the duality pairing between elements in H_k and in its dual $H_k^0 = H_{\ k}$ using the inner product in H_0. For this reason, in the following we will denote by hf; i the duality pairing between 2 ^{0} and f 2 (see [1, 2] for more details).

Consider the n th (n 2 N_0) tensor power ^{n} of the space which is de ned as the projective limit of $H_k^{\ n}$; for n = 0, $H_k^{\ n}$ = R. Then its dual space is

$$
(1) \qquad {}^{n\ 0} = \bigcup_{k 2 K} H_k^{\ n\ 0} = \bigcup_{k 2 K} (H_k^0)^{\ n} = \bigcup_{k 2 K} H_{\ k}^{\ n};
$$

which we can equip with the weak topology.

A generalized process is a nite measure de ned on the Borel algebra on ^{0}. Moreover, we say that a generalized process is concentrated on a measurable subset S ^{0} if (^{0} n S) = 0.

Let us introduce the main objects involved in the realizability problem.

De nition 1.1 (Finite n

Proposition 1.3. *If is a generalized process on ^{0} with generalized moment functions (in the sense of ^{0}) of any order, then for any $n \in \mathbb{N}$ and for any $f^{(n)} \in {}^{n}$ we have*

$$\int_{{}^{0}} \left| \langle f^{(n)}, {}^{n} \rangle \right| (d) < \infty \quad \text{and} \quad \langle f^{(n)}, m^{(n)} \rangle = \int_{{}^{0}} \langle f^{(n)}, {}^{n} \rangle (d).$$

For a generalized processes the moment functions $m^{(n)}$ are given by an explicit formula. The moment problem, which in an infinite dimensional context is often called the *realizability problem*, addresses exactly the inverse question.

Problem 1.4 (Realizability problem on $S{}^{0}$). Let $N \in \mathbb{N}_0 \cup \{+\infty\}$ and let $m = (m^{(n)})_{n=0}^{N}$ be such that each

one de nes r := r [1]

Corollary 4.2. *The semi-algebraic set S defined as in (8) is measurable w.r.t. the Borel σ-algebra $\mathcal{B}(\tau_w^{ind})$ generated by the weak topology on $\mathcal{D}'_{ind}(\mathbb{R}^d)$.*

Proof. The previous proposition implies that $S \in \mathcal{B}(\tau_s^{ind})$. As $(\mathcal{D}'_{ind}(\mathbb{R}^d), \tau_s^{ind})$ is a Lusin space and so Suslin, $\mathcal{B}(\tau_w^{ind})$ and $\mathcal{B}(\tau_s^{ind})$ coincide (see [40, Corollary 2, p.101]). Hence, $S \in \mathcal{B}(\tau_w^{ind})$. $\square$

In the following, we are going to investigate the full realizability problem (see Problem 1.4) on S of the form (8). Let us introduce the version of the Riesz linear functional for the moment problem on $\mathcal{D}'_{proj}(\mathbb{R}^d)$.

Definition 4.3. Given $m \in \mathcal{F}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right)$, we define its associated Riesz functional L_m as

$$
\begin{aligned}
L_m : \quad & \mathcal{P}_{C_c^\infty}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right) \to \mathbb{R} \\
& P(\eta) = \sum_{n=0}^{N} \langle p^{(n)}, \eta^{\otimes n} \rangle \mapsto L_m(P) := \sum_{n=0}^{N} \langle p^{(n)}, m^{(n)} \rangle.
\end{aligned}
$$

Note that in the case when the sequence m is realized by a non-negative measure $\mu \in \mathcal{M}^*(S)$ on a subset $S \subseteq \mathcal{D}'_{proj}(\mathbb{R}^d)$, then a direct calculation shows that for any polynomial $P \in \mathcal{P}_{C_c^\infty}(\mathcal{D}'_{proj}(\mathbb{R}^d))$

$$
L_m(P) = \int_S P(\eta)\, \mu(d\eta). \tag{9}
$$

The Riesz functional allows us to state our main result in a concise form.

Theorem 4.4. *Let $m \in \mathcal{F}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right)$ be determining and S be a basic semi-algebraic set of the form (8). Then m is realized by a unique non-negative measure $\mu \in \mathcal{M}^*(S)$ if and only if the following inequalities hold*

$$
L_m(h^2) \geq 0, \; L_m(P_i h^2) \geq 0, \; \forall h \in \mathcal{P}_{C_c^\infty}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right), \; \forall i \in Y. \tag{10}
$$

Equivalently, if and only if the functional L_m is non-negative on the quadratic module $Q(\mathcal{P}_S)$.

Despite of the apparently abstract character of the determinacy condition given in Definition 2.2, the latter becomes actually concrete whenever one can explicitly construct the set E. This is possible for the nuclear space $\mathcal{D}$

For such a set E, using (4), we get that

$$m_n \le c^{dn}_{k_1^{(n)}} \left(\sup_{\substack{z\in\mathbb{R}^d \\ \|z\|\le n}} \sup_{x\in[-1,1]^d} \sqrt{k_2^{(n)}(z+x)} \right)^n \|m^{(2n)}\|^{\frac{1}{2}}_{H^{2n}_{k^{(2n)}}}.$$

Remark 4.6.
The more regularity is known on the sequence m the weaker is the restriction on the growth of the $m^{(2n)}$ required in Theorem 4.4. Let us discuss two extremal cases.

- If each $m^{(n)}$ is in H^n_k where $k = (k_1, k_2(r)) \in I$ with both k_1 and k_2 independent of n, then both $c_{k_1^{(n)}}$ and $\sup_{\substack{z\in\mathbb{R}^d \\ \|z\|\le n}} \sup_{x\in[-1,1]^d} \sqrt{k_2^{(n)}(z+x)}$ in Lemma 4.5 are constant

Proposition 4.10. If m is realized by a measure $\mu \in \mathcal{M}^*(\mathcal{D}'_{proj}(\mathbb{R}^d))$ and m is determining, then the sequence $\mathcal{P}m$ is also determining.

Proof. Let us first recall that $\mathcal{D}_{proj}(\mathbb{R}^d) = \text{proj}\lim_{k \in I} H_k$, where I is as in Definition 3.1 and $H_k := W_2^{k_1}(\mathbb{R}^d; k_2(r)dr)$ for any $k = (k_1; k_2(r)) \in I$ (see Section 3.1).

Since m is determining in the sense of Definition 2.2, there exists a subset E total in $\mathcal{D}_{proj}(\mathbb{R}^d)$ such that for any $n \in \mathbb{N}_0$, $m_n < \infty$ and the class $C\{m_n\}$ is quasi-analytic, where

$$m_n := \sqrt{\sup_{f_1,\ldots,f_{2n} \in E} \left|\langle f_1 \otimes \cdots \otimes f_{2n}, m^{(2n)}\rangle\right|}.$$

It is easy to see that, since m is realized by a measure $\mu \in \mathcal{M}^*(\mathcal{D}'_{proj}(\mathbb{R}^d))$, the sequence $(m_n)_{n \in \mathbb{N}_0}$ is also log-convex.

We will show that there exists a finite positive constant $c_{\mathcal{P}}$ such that

$$\bar{m}_n := \sqrt{\sup_{f_1,\ldots,f_{2n} \in E} \left|\langle f_1 \otimes \cdots \otimes f_{2n}, (\mathcal{P}m)^{(2n)}\rangle\right|} \leq \sqrt{c_{\mathcal{P}}\, m_{2n}}. \tag{12}$$

The latter bound is sufficient to prove that the sequence $\mathcal{P}m$ is determining. In fact, the log-convexity of $(m_n)_{n \in \mathbb{N}_0}$ and the quasi-analiticity of $C\{m_n\}$ imply that the class $C\{\sqrt{c_{\mathcal{P}}\, m_{2n}}\}$ is also quasi-analytic (see Lemma 6.8 and Proposition 6.5). Hence, (12) gives that $C\{\bar{m}_n\}$ is also quasi-analytic.

It remains to show the bound in (12).
Let us fix $n \in \mathbb{N}$. Using Definition 4.7 and the assumption that m is realized by μ on $\mathcal{D}'_{proj}(\mathbb{R}^d)$, we get that for any $f_1, \ldots, f_{2n} \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\left|\langle f_1 \otimes \cdots \otimes f_{2n}, (\mathcal{P}m)^{(2n)}\rangle\right| &\leq \sum_{j=0}^{N} \int_{\mathcal{D}'_{proj}(\mathbb{R}^d)} \left|\langle p^{(j)}, \eta^{\otimes j}\rangle \langle f_1 \otimes \cdots \otimes f_{2n}, \eta^{\otimes(2n)}\rangle\right| \mu(d\eta) \\
&\leq c_{\mathcal{P}} \left(\int_{\mathcal{D}'_{proj}(\mathbb{R}^d)} \left|\langle f_1 \otimes \cdots \otimes f_{2n}, \eta^{\otimes 2n}\rangle\right|^2 \mu(d\eta) \right)^{\frac{1}{2}} \\
&= c_{\mathcal{P}} \langle (f_1 \otimes \cdots \otimes f_{2n})^{\otimes 2}, m^{(4n)}\rangle^{\frac{1}{2}};
\end{aligned}$$

where

$$c_{\mathcal{P}} := \sum_{j=0}^{N} \int_{\mathcal{D}'_{proj}(\mathbb{R}} \ldots$$

Since integrals of non-negative functions w.r.t. a non-negative measure are non-negative, the inequalities in (10) hold.

Su ciency

As already observed in Remark 4.8, the assumptions in (10) mean that the sequence m and $_P$ m are positive semide nite. Since m is assumed to be determining, Theorem 2.3 guarantees the existence of a unique non-negative measure 2 M $(D^0_{proj}(R^d))$ realizing m. On the one hand, according to Lemma 4.9 the sequence $_{P_i}$ m is realized by the signed measure P_i, i.e. for any $f^{(n)}$ 2 $C^1_c(R^{nd})$

$$(13) \qquad hf^{(n)};(_{P_i} m)^{(n)}i = \int_{D^0_{proj}(R^d)} hf^{(n)}; \quad {}^n i P_i(\ ) \ (d\ ):$$

On the other hand, by Proposition 4.10, the sequence $_{P_i}$ m is also determining. Hence, applying again Theorem 2.3, the sequence $_{P_i}$ m is realized by a unique non-negative measure 2 M $(D^0_{proj}(R^d))$, namely for any $f^{(n)}$ 2 $C^1_c(R^{nd})$

$$(14) \qquad hf^{(n)};(_{P_i} m)^{(n)}i = \int_{D^0_{proj}(R^d)} hf^{(n)}; \quad {}^n i \ (d\ ):$$

Let A_i := 2 $D^0_{proj}(R^d)$: $P_i(\ )$ 0 and let us de ne $^+_i(B)$:= $(B \setminus A_i)$ and $_i(B)$:= $(B \setminus (D^0_{proj}(R^d) \, n \, A_i))$, for all B 2 $B(D^0_{proj}(R^d))$. Moreover, let us consider the non-negative measures ^+_i and $_i$ given by $^+_i(B)$:= $\int_B P_i(\ ) \ ^+_i(d\ )$ and $_i(B)$:= $\int_B P_i(\ ) \ _i(d\ )$, for all B 2 $B(D^0_{proj}(R^d))$. Hence, we have that = ^+_i + $_i$ and P_i = ^+_i $_i$. According to this notation, (13) and (14) can be rewritten as

$$(15) \qquad \int_{D^0_{proj}(R^d)} hf^{(n)}; \quad {}^n i \ ^+_i(d\ ) = \int_{D^0_{proj}(R^d)} hf^{(n)}; \quad {}^n i \ _i(d\ ) + \int_{D^0_{proj}(R^d)} hf^{(n)}; \quad {}^n i \ (d\ ):$$

Since m is determining and since $^+$, the sequence m^+ consisting of all moment functions of $^+$ is also determining. By Proposition 4.10, the sequence $_{P_i} m^+$ is determining, too.

As the two non-negative measures ^+_i and $_i$ + both realize the determining sequence $_{P_i} m^+$, they coincide because Theorem 2.3 also guarantees the uniqueness of the realizing measure. This implies that the signed measure P_i is actually a non-negative implies that there exists a finite open subcover of K_ε, i.e. there exists a finite subset $J \subseteq Y$ such that $K_\varepsilon \subseteq \bigcup_{i\in J}\left(\mathcal{D}'_{ind}(\mathbb{R}^d)\setminus A_i\right)$. Therefore, we have that

$$0 \le \nu_0(K_\varepsilon) \le \nu_0\left(\bigcup_{i\in J}\mathcal{D}'_{ind}(\mathbb{R}^d)\setminus A_i\right) \le \sum_{i\in J}\nu\left(\left(\mathcal{D}'_{ind}(\mathbb{R}^d)\setminus A_i\right)\cap\mathcal{D}'_{proj}(\mathbb{R}^d)\right) = 0;$$

where in the last equality we used (16). Moreover, by (17), we have that

$$\nu_0\left(\mathcal{D}'_{ind}(\mathbb{R}^d)\setminus S\right) \le \nu_0(K_\varepsilon) + \varepsilon = \varepsilon.$$

Since this holds for any $\varepsilon > 0$, we get $\nu_0\left(\mathcal{D}'_{ind}(\mathbb{R}^d)\setminus S\right) = 0$ and hence, $0 = \nu_0\left(\mathcal{D}'_{ind}(\mathbb{R}^d)\setminus S\right) = \nu\left((\mathcal{D}'_{ind}(\mathbb{R}^d)\setminus S)\cap\mathcal{D}'_{proj}(\mathbb{R}^d)\right) = \nu\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\setminus S\right)$.

Theorem 4.4 does still hold for any basic semi-algebraic set S which is subset of $\mathcal{D}'_{ind}(\mathbb{R}^d)$ (instead of $\mathcal{D}'_{proj}(\mathbb{R}^d)$) and gives a realizing measure actually concentrated on $S\cap\mathcal{D}'_{proj}(\mathbb{R}^d)$. If $S\cap\mathcal{D}'_{proj}(\mathbb{R}^d) = \emptyset$, then there is no contradiction because Theorem 4.4 shows that the only realizing measure is identically equal to zero, and so we know *a posteriori* that all the moment functions were zeros. However, the case $S\cap\mathcal{D}'_{proj}(\mathbb{R}^d) \neq \emptyset$ is very common, since $\mathcal{D}'_{proj}(\mathbb{R}^d)$ contains all tempered distributions, Radon measures and all locally integrable functions. Hence, if at least a single one of such generalized functions is contained in S then $S\cap\mathcal{D}'_{proj}(\mathbb{R}^d)\neq\emptyset$ and Theorem 4.4 can be applied to get a non-zero realizing measure supported on S, indeed on $S\cap\mathcal{D}'_{proj}(\mathbb{R}^d)$. Note that in Theorem 4.4 it is not sufficient to just assume that $m \in \mathcal{F}\left(\mathcal{D}'_{ind}(\mathbb{R}^d)\right)$. However, the assumption $m \in \mathcal{F}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right)$ is not a restrictive requirement in any application.

5. Applications

In this section we give some concrete applications of Theorem 4.4.
In Subsection 5.1, we present Theorem 4.4 in the finite dimensional case. This theorem generalizes the results already know in literature about the classical moment problem on a basic semi-algebraic set of $\mathbb{R}^d$.
In Subsection 5.2, we study the case when we assume more regularity of type IV on the putative moment functions, that is, we require that they are non-negative symmetric Radon measures. The advantage of this additional assumption is that it allows us to simplify the condition of determinacy and hence, to give an adapted version of Theorem 4.4. In Subsection 5.3, we derive conditions on the putative moment functions to be realized by a random measure, that is, we assume S to be the set of all Radon measures on $\mathbb{R}^d$. In this case, the fact that all the moment functions are themselves Radon measures is a necessary condition and so the results of Subsection 5.2 can be exploited. In Subsection 5.4, we consider the case when S is the set of Radon measures with Radon-Nikodym densities w.r.t. the Lebesgue measure fulfilling an *a priori* L^1 bound.

From now on let us denote by $\mathcal{R}(\mathbb{R}^d)$ the space of all Radon measures on $\mathbb{R}^d$, namely the space of all non-negative Borel measures that are finite on compact sets in $\mathbb{R}^d$.

5.1. Finite dimensional case.

The d-dimensional moment problem on a closed basic semi-algebraic set S of $\mathbb{R}^d$ is a special case of Problem 1.4 for $\Omega = \mathcal{H}_0 = \mathbb{R}^d$. Hence, Theorem 4.4 can be applied also in the finite dimensional case, where the condition $m := (m^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}\left(\mathbb{R}^d\right)$ holds for any multi-sequence of real numbers. In fact, if we denote by $\{e_1,\dots,e_d\}$

the canonical basis of $\mathbb{R}^d$ then we have that for each $n \in \mathbb{N}_0$,

m

As suggested by the name, the condition (18) is an infinite-dimensional weighted version of the classical Carleman condition, which ensures the uniqueness of the solution to the d-dimensional moment problem (for $d = 1$ see [8], for $d \geq 2$ see e.g. [41, 31, 5, 11]).

Corollary 5.3. Let $m \in \mathcal{F}\left(\mathcal{R}(\mathbb{R}^d)\right)$ fulfill the weighted Carleman type condition in Definition 5.2 and let $S \subseteq \mathcal{D}'_{proj}(\mathbb{R}^d)$ be a basic semi-algebraic of the form (8). Then m is realized by a unique non-negative measure $\mu \in \mathcal{M}(S)$ with

$$\int_S \left\langle \frac{1}{k_2^{(n)}}, \eta^{n} \right\rangle \mu(d\eta) < \infty, \quad \forall n \in \mathbb{N}_0; \tag{19}$$

if and only if the following inequalities hold

$$L_m(h^2) \geq 0; \; L_m(P_i h^2) \geq 0; \; \forall h \in \mathcal{P}_{C_c^\infty}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right); \forall i \in Y; \tag{20}$$

and for any $n \in \mathbb{N}_0$ we have

$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(dr_1, \ldots, dr_{2n})}{\prod_{l=1}^{2n} k_2^{(2n)}(r_l)} < \infty. \tag{21}$$

Remark 5.4. If m is realized by a non-negative measure $\mu \in \mathcal{M}(\mathcal{D}'_{proj}(\mathbb{R}^d))$ and m satisfies (18) then (21) holds also for the odd orders.

Corollary 5.3 is essentially a consequence of the following proposition.

Proposition 5.5. If m satisfies (18) and (21), then m is a determining sequence in the sense of Definition 2.2.

Proof. Let us preliminarily recall that $\mathcal{R}(\mathbb{R}^d) \subseteq \mathcal{D}'_{proj}(\mathbb{R}^d)$ and so m is automatically in $\mathcal{F}(\mathcal{D}'_{proj}(\mathbb{R}^d))$ as required by Definition 2.2.
For any $f_1, \ldots, f_n \in C_c^\infty$ m

Hence, by choosing E as in Lemma 4.5, we have that

$$m_n := \sqrt{\sup_{f_1,\ldots,f_{2n}\in E} \langle f_1 \otimes \cdots \otimes f_{2n}, m^{(2n)}\rangle}$$
$$\leq \sqrt{C^{2n}\left(\sup_{f\in E}\|f\|_{H_{k^{(n)}}}\right)^{2n} \int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(dr_1,\ldots,dr_n)}{\prod_{l=1}^{2n} k_2^{(2n)}(r_l)}}$$

$$\text{(23)} \qquad \leq \left(C c_d^{\lceil\frac{d+1}{2}\rceil} \sup_{\substack{z\in\mathbb{R}^d\\ \|k\|\leq n}} \sup_{x\in[-1,1]^d} \sqrt{k_2^{(2n)}(z+x)} \right)^n \sqrt{\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(dr_1,\ldots,dr_{2n})}{\prod_{l=1}^{2n} k_2^{(2n)}(r_l)}}.$$

Then the condition (21) guarantees that the m_n's are finite and (18) implies that the class $C\{m_n\}$ is quasi-analytic.

Proof. (Corollary 5.3).
Since the necessity part follows straightforwardly, let us focus on the sufficiency. Since m is determining by Proposition 5.5 and (20) holds by assumption, we can apply Theorem 4.4 to get that m is realized by $\mu \in \mathcal{M}^*(S)$.

It remains to show (19). For any positive real number R let us define a function χ_R such that

$$\text{(24)} \qquad \chi_R \in C_c^{\infty}(\mathbb{R}^d) \text{ and } \chi_R(r) := \begin{cases} 1 & \text{if } |r| \leq R \\ 0 & \text{if } |r| \geq R+1. \end{cases}$$

Since m is realized by $\mu \in \mathcal{M}^*(S)$, for any $n \in \mathbb{N}_0$ and for any positive real number R we have that

$$\int_S \left\langle \frac{\chi_R}{k_2^{(n)}}, \cdot \right\rangle^n \mu(d\cdot) = \int_{\mathbb{R}^{nd}} \prod_{l=1}^{n} \frac{\chi_R(r_l)}{k_2^{(n)}(r_l)} m^{(n)}(dr_1,\ldots,dr_n).$$

Hence, the monotone convergence theorem for $R \to \infty$ and Remark 5.4 give (19).

Remark 5.6.
The proof of Proposition 5.5 is a particular instance of what we were pointing out in Remark 4.6. In fact, the regularity assumed on the sequence m, that is m consisting of Radon measures, allowed us to get the bound (23) from (18) and (21) for some index $k^{(n)} = (k_1^{(n)}, k_2^{(n)})$ with $k_1^{(n)} = \lceil\frac{d+1}{2}\rceil$ and so independent of n.
Note that to obtain this result it was important to use our definition of determining sequence (see Definition 2.2). In fact, if we used the one given in [2] involving the norms $\|m^{(2n)}\|_{H_{k^{(2n)}}^{2n}}$ (see Remark 2.4), we would have got $k_1^{(n)} > n(d$ (see [...]

If $S \subseteq \mathcal{D}'_{proj}(\mathbb{R}^d)$ is a basic semi-algebraic of the form (8), then m is realized by a unique non-negative measure $\mu \in \mathcal{M}^*(S)$ with

$$\int_S \left\langle \frac{1}{k_2}, \eta \right\rangle^n \mu(d\eta) < \infty, \quad \forall n \in \mathbb{N}_0;$$

if and only if the following inequalities hold

$$L_m(h^2) \geq 0; \; L_m(P_i h^2) \geq 0; \; \forall h \in \mathcal{P}_{C_c^\infty}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right), \; \forall i \in Y;$$

and for any $n \in \mathbb{N}_0$ we have

$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(dr_1, \ldots, dr_{2n})}{\prod_{l=1}^{2n} k_2(r_l)} < \infty.$$

5.3. Realizability on the space of Radon measures $\mathcal{R}(\mathbb{R}^d)$.

Example 5.8.
The set $\mathcal{R}(\mathbb{R}^d)$ of all Radon measures on $\mathbb{R}^d$ is a basic semi-algebraic subset of $\mathcal{D}'_{proj}(\mathbb{R}^d)$, i.e.

$$\mathcal{R}(\mathbb{R}^d) = \bigcap_{\varphi \in C_c^{+,\infty}(\mathbb{R}^d)} \left\{ \eta \in \mathcal{D}'_{proj}(\mathbb{R}^d) : P_\varphi(\eta) \geq 0 \right\} \tag{25}$$

where $P_\varphi(\eta) := \langle \varphi, \eta \rangle$.

Proof.
The representation (25) follows from the fact that there exists a one-to-one correspondence between the Radon measures on $\mathbb{R}^d$ and the continuous non-negative linear functionals on the space $\mathcal{D}_{proj}(\mathbb{R}^d)$. In fact, for any $\nu \in \mathcal{R}(\mathbb{R}^d)$ the functional

$$\begin{aligned} C_c^\infty(\mathbb{R}^d) &\to \mathbb{R} \\ \varphi &\mapsto \langle \varphi, \nu \rangle = \int_{\mathbb{R}^d} \varphi(r)\, \nu(dr) \end{aligned}$$

is non-negative and it is an element of $\mathcal{D}'_{proj}(\mathbb{R}^d)$. Conversely, by a theorem due to L. Schwartz (c.f. [39, Theorem V]), every non-negative linear functional on $C_c^\infty(\mathbb{R}^d)$ can be represented as integral w.r.t. a Radon measure on $\mathbb{R}^d$.

Using the representation (25), we obtain a realizability theorem for $S = \mathcal{R}(\mathbb{R}^d)$, namely Corollary 5.3 becomes

Theorem 5.9.
Let $m \in \mathcal{F}\left(\mathcal{R}(\mathbb{R}^d)\right)$ fulfill the weighted Carleman type condition (18). Then m is realized by a unique non-negative measure $\mu \in \mathcal{M}^*(\mathcal{R}(\mathbb{R}^d))$ with

$$\int_S \left\langle \frac{1}{k_2^{(n)}}, \eta \right\rangle^n \mu(d\eta) < \infty, \quad \forall n \in \mathbb{N}_0;$$

if and only if the following inequalities hold

$$L_m(h^2) \geq 0; \; \forall h \in \mathcal{P}_{C_c^\infty}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right); \tag{26}$$

$$L_m(P_\varphi h^2) \geq 0; \; \forall h \in \mathcal{P}_{C_c^\infty}\left(\mathcal{D}'_{proj}(\mathbb{R}^d)\right), \; \forall \varphi \in C_c^{+,\infty}(\mathbb{R}^d); \tag{27}$$

$$\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(dr_1, \ldots, dr_{2n})}{\prod_{l=1}^{2n} k_2^{(2n)}(r_l)} < \infty; \; \forall n \in \mathbb{N}_0. \tag{28}$$

Note that if μ is concentrated on $\mathcal{R}(\mathbb{R}^d)$ then $m^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ for all $n \in \mathbb{N}_0$.

The previous theorem still holds even when m does not consist of Radon measures. In this case, instead of (18) and (28), one has to assume that m is determining in the sense of Definition 2.2.

The assumption (18) can be actually weakened by taking into account a result due to S.N. Šifrin about the infinite dimensional moment problem on dual cones in nuclear spaces (see [42]). Indeed, applying Šifrin's results to the cone $C_c^{+,\infty}(\mathbb{R}^d)$, it is possible to obtain a particular instance of our Theorem 4.4 for the case $S = \mathcal{R}(\mathbb{R}^d)$ (the latter is in fact the dual cone of $C_c^{+,\infty}(\mathbb{R}^d)$) but with the difference that in the determinacy condition the quasi-analyticity of the m_n's is replaced by the so-called Stieltjes condition $\sum_{n=1}^{\infty} m_n^{-\frac{1}{2n}} = \infty$. As a consequence, the condition (18) in Theorem 5.9 can be replaced by the following weaker one

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sup\limits_{\substack{z\in\mathbb{R}^d\\ \|z\|\le n}} \sup\limits_{x\in[-1,1]^d} \sqrt{k_2^{(n)}(z+x)}\ \sqrt[4n]{\int_{\mathbb{R}^{2nd}} \frac{m^{(2n)}(dr_1,\ldots,dr_{2n})}{\prod_{l=1}^{2n} k_2^{(2n)}(r_l)}}}} = \infty;$$

which we call *weighted generalized Stieltjes condition*.

Remark 5.10.
The condition (26) can be rewritten as

$$\sum_{i,j} \langle h^{(i)} \otimes h^{(j)}, m^{(i+j)} \rangle \ge 0, \quad \forall\, h^{(i)} \in C_c^{\infty}(\mathbb{R}^{id});$$

and (27) as

$$\sum_{i,j} \langle h^{(i)} \otimes h^{(j)} \otimes \varphi, m^{(i+j+1)} \rangle \ge 0, \quad \forall\, h^{(i)} \in C_c^{\infty}(\mathbb{R}^{id});\ \forall\, \varphi \in C_c^{+,\infty}(\mathbb{R}^d).$$

Recalling Definition 4.7, we can restate these conditions as follows: the sequence $(m^{(n)})_{n\in\mathbb{N}_0}$ and its shifted version $((\varphi . m)^{(n)})_{n\in\mathbb{N}_0}$ are positive semidefinite in the sense of Definition 2.1.
In particular, if for each $n \in \mathbb{N}_0$, $m^{(n)}$ has a Radon-Nikodym density, that is there exists $\rho^{(n)} \in L^1(\mathbb{R}^n, \lambda)$ s.t. $m^{(n)}(dr_1,\ldots,dr_n) = \rho^{(n)}(r_1,\ldots,r_n)dr_1 \cdots dr_n$, then (26) and (27) can be rewritten as

$$\sum_{i,j} \int_{\mathbb{R}^{d(i+j)}} h \ldots$$

where N is a positive integer and $\cdot$ is a smooth characteristic function of the support of a function $\varphi \in C_c(\mathbb{R}^d)$ (see (24)).

As a consequence of the equivalence of the two topologies, the associated Borel algebras also coincide and they are equal to $(\ _w^{proj}) \cap R\ (\mathbb{R}^d)$.

5.4.

Remark 5.13.
Proceeding as in Remark 5.10, we can work out the analogy between the realizability problem on S_c and the moment problem on $[0; c]$. Indeed, if each $m^{(n)}$ has density $^{(n)}$ w.r.t. the Lebesgue measure, then (33)

Theorem 6.4 (The Denjoy-Carleman Theorem).
Let $(M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers. Then the following conditions are equivalent

(1) $C\{M_n\}$ *is quasi-analytic,*
(2) $\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty$ *with* $\beta_n := \inf_{k\geq n} \sqrt[n]{M_n}$;
(3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n^c}} = \infty$,
(4) $\sum_{n=1}^{\infty} \frac{M_{n-1}^c}{M_n^c} = \infty$,

where $(M_n^c)_{n\in\mathbb{N}_0}$ is the convex regularization of $(M_n)_{n\in\mathbb{N}_0}$ by means of the logarithm.

Let us now state a simple result which has been repeatedly used throughout this paper.

Proposition 6.5.
Let $(M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers. Then $C\{M_n\}$ is quasi-analytic if and only if for any positive constant γ the class $C\{\gamma M_n\}$ is quasi-analytic.

In conclusion, let us introduce some interesting properties of log-convex sequences.

Remark 6.6.
For a sequence of positive real numbers $(M_n)_{n\in\mathbb{N}_0}$ the following properties are equivalent

(a): $(M_n)_{n=0}^{\infty}$ is log-convex.
(b): $\left(\frac{M_n}{M_{n-1}}\right)_{n=1}^{\infty}$ is monotone increasing.
(c): $(\ln(M_n))_{n=1}^{\infty}$ is convex.

where the last inequality is due to Proposition 6.7. Hence, if $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}}$ diverges then $\sum_{n=1}^{\infty} \frac{1}{\sqrt[jn]{M_{jn}}}$ diverges as well. On the other hand, if the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[jn]{M_{jn}}}$ diverges for some $j \in \mathbb{N}$, then also $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}}$ diverges since the latter contains more summands.

6.2. Complements about the space $C_c^{\infty}(\mathbb{R}^d)$.

Let us recall the definition of the inductive topology on $C_c^{\infty}(\mathbb{R}^d)$ (see [35, Section V.4, vol. I]) for a more detailed account on this topic).

Definition 6.9. Let $(\Omega_n)_{n\in\mathbb{N}}$ be an increasing family of relatively compact open subsets of $\mathbb{R}^d$ such that $\mathbb{R}^d = \bigcup_{n\in\mathbb{N}} \Omega_n$. Let us consider the space $C_c^{\infty}(\overline{\Omega_n})$ of all infinitely differentiable functions on $\mathbb{R}^d$ with compact support contained in $\overline{\Omega_n}$ and let us endow $C_c^{\infty}(\overline{\Omega_n})$ with the Fréchet topology generated by the directed family of seminorms given by

$$\|\varphi\|_a := \sum_{|\beta| \leq a} \max_{r\in\overline{\Omega_n}} |D^{\beta}\varphi(r)|. \tag{36}$$

Then as sets

$$C_c^{\infty}(\mathbb{R}^d) = \bigcup_{n\in\mathbb{N}} C_c^{\infty}(\overline{\Omega_n}).$$

We denote by $\mathcal{D}_{ind}(\mathbb{R}^d)$ the space $C_c^{\infty}(\mathbb{R}^d)$ endowed with the inductive limit topology τ_{ind} induced by this construction.

It is easy to see that the previous definition is indep TdT q 5(that)-355(the)-3(TdT qs)-357ous dece

Definition 6.11 (Condition (D)).
We say that the set K_0 I satisfies Condition (D) if:
"For any pair $k = (k_1; k_2(r)) \in K_0$ there exists $k' = (k'_1; k'_2(r)) \in K_0$ such that

k'_1 $k_1 + l$ (where l is the smallest integer greater than $\frac{d}{2}$)

$k'_2(r)$ $\max_{|\,|\ l} |(D\ q)(r)|^2$, $\forall r \in \mathbb{R}^d$, for some function $q(r) \in C^l(\mathbb{R}^d)$ chosen such that

$$q^2(r)\quad k_2(r);\ \forall r \in \mathbb{R}^d \quad \text{and} \quad \int_{\mathbb{R}^d}$$

, for some function

q

6.3. Construction of a total subset of test functions.
In this subsection, we provide an outline of the proof of Lemma 4.5 about the explicit construction of a set E of the kind required in De nition 2.2. For convenience, we give here the proofs only in the case when E $D^0_{proj}(R)$. The higher dimensional case follows straightforwardly.

For any n 2 N_0, let $k^{(n)}$:= (

where the last equality is due to (38). Since φ is in some space $H_{k^{(n)}}$ and as (37), holds, we get that

$$|\langle f_{y;p}, \varphi_\varepsilon \rangle| \leq \| f_{y;p} \|_{H_{-k^{(n)}}} \| \varphi_\varepsilon \|_{H_{k^{(n)}}} \leq c(1+|p|)^{k_1^{(n)}} \| \varphi_\varepsilon \|_{H_{k^{(n)}}}; \tag{40}$$

where $c := d_{k_1^{(n)}} (\sqrt{2})^{k_1^{(n)}+1} \sup_{x \in [-1,1]} \sqrt{k_2^{(n)}(x+y)}$ and so it depends only on $k_1^{(n)}; k_2^{(n)}; y$.

Since φ_ε is an approximating identity we get that

$$\lim_{\varepsilon \downarrow 0} \| \varphi_\varepsilon \|_{H_{k^{(n)}}} = \| \varphi \|_{H_{k^{(n)}}}$$

The latter together with (40) imply that the function $\langle f_{y;p}, \varphi_\varepsilon \rangle$ is uniformly bounded in p and ε

It remains to construct an increasing sequence $(c_n)_n$ of positive numbers not quasi-analytic and such that (41) holds. First note that our requirement is equivalent to define an increasing sequence $(d_n)_n$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{d_n}} < \infty$ and $\lim_{n\to\infty} \frac{\sqrt[n]{d_n}}{\sqrt[n]{c_n}} = 0$: Indeed, for each C and for each $\varepsilon > 0$ there exist N such that for all $n \geq N$ holds $d_n \leq \left(\frac{\varepsilon}{C}\right)^n c_n$ and hence also $C^n d_n \leq \varepsilon^n c_n$.

Our problem reduces to find, given a decreasing sequence $(a_n)_n$ of positive numbers with $\sum_{n=1}^{\infty} a_n < \infty$, a decreasing sequence $(b_n)_n$ of positive numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

For any $k \in \mathbb{N}$ let us define $N_k := \min\{m \mid \sum_{n=m}^{\infty} a_n \leq \frac{1}{k^2}\}$ and also

$$b_n := \min\left\{ a_n \left(1 + \sum_{k\in\mathbb{N}\,:\,N_k \leq n} \sqrt{k}\right); b_{n-1} \right\};$$

with $b_0 := a_0 \left(1 + \sum_{k\in\mathbb{N}\,:\,N_k = 0} \sqrt{k}\right)$. Then

$$\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n \left(1 + \sum_{k\in\mathbb{N}\,:\,N_k \leq n} \sqrt{k}\right) \leq \sum_{n=1}^{\infty} a_n + \sum_{k=1}^{\infty} k^{-3/2} < \infty;$$

It follows that $\lim_{n\to\infty} b_n = 0$. Then latter together with the definition $(b_n)_n$ implies that there exists an infinite subsequence $(b_{n_j})_j \subseteq (b_n)_n$ such that

$$\forall j \in \mathbb{N} : b_{n_j} = a_{n_j} \left(1 + \sum_{k\in\mathbb{N}\,:\,N_k \leq n_j} \sqrt{k}\right):$$

For such a subsequence we have that

$$\lim_{j\to\infty} \frac{b_{n_j}}{a_{n_j}} = \lim_{j\to\infty} \left(1 + \sum_{k\in\mathbb{N}\,:\,N_k \leq n_j} \sqrt{k}\right) = \left(1 + \sum_{k=1}^{\infty} \sqrt{k}\right) = \infty: \tag{42}$$

Now let us note that for any $n \in \mathbb{N}$ we have either that $\frac{b_n}{a_n} = \frac{b_{n-1}}{a_n} \geq \frac{b_{n-1}}{a_{n-1}}$ or that

$$\frac{b_n}{a_n} = \frac{a_n \left(1 + \sum_{k\in\mathbb{N}\,:\,N_k \leq n} \sqrt{k}\right)}{a_n}$$

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$\sum^{\infty}$

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