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by

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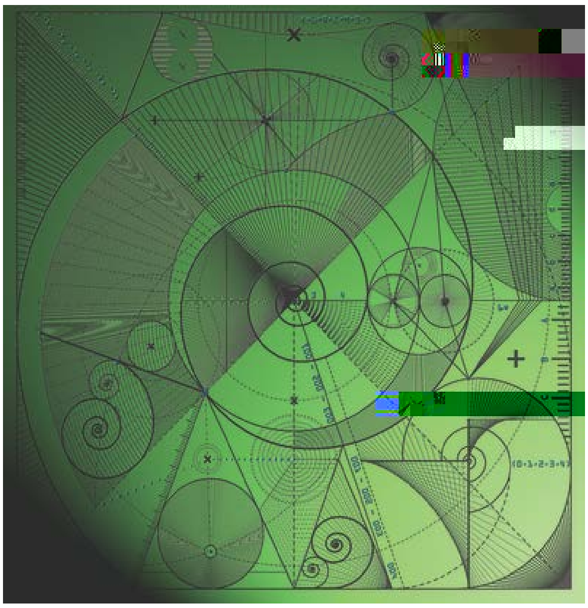

`Quasi'-norm of an arithmetical convolution operator and the order of the Riemann zeta function [1]

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Abstract

In this paper we study Dirichlet convolution with a given arithmetical function f as a linear mapping φ_f that sends a sequence (a_n) to (b_n) where $\|\varphi_f\| = \zeta(\alpha)$.

For the unbounded case, we show that $\varphi_f : \mathcal{M}^2 \to \mathcal{M}^2$ where $\mathcal{M}^2$ is the subset of l^2 of multiplicative sequences, for many $f \in \mathcal{M}^2$. Consequently, we study the `quasi'-norm

$$\sup_{\substack{\|a\| = T \\ a \in \mathcal{M}^2}} \frac{\|\varphi_f a\|}{\|a\|}$$

for large T, which measures the `size' of φ_f on $\mathcal{M}^2$. For the $f(n) = n^{-\alpha}$ case, we show this quasi-norm has a striking resemblance to the conjectured maximal order of $|\zeta(\alpha + iT)|$ for $\alpha > \frac{1}{2}$.

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Introduction

Given an arithmetical function $f(n)$, the mapping φ_f sends $(a_n)_{n\in\mathbb{N}}$ to $(b_n)_{n\in\mathbb{N}}$, where

$$b_n = \sum_{d|n} f(d) a_{n/d}. \tag{0.1}$$

Writing $a = (a_n)$, φ_f maps a to $f * a$ where $*$ is Dirichlet convolution. This is a `matrix' mapping, where the matrix, say $M(f)$, is of `multiplicative Toeplitz' type; that is,

$$M(f) = (a_{ij})_{i,j \geq 1}$$

where $a_{ij} = f(i/j)$ and f is supported on the natural numbers (see, for example, [6], [7]).

Toeplitz matrices (whose ij^{th}-entry is a function of $i - j$) are most usefully studied in terms of a `symbol' (the function whose Fourier coefficients make up the matrix). Analogously, the Multiplicative Toeplitz matrix $M(f)$ has as symbol the Dirichlet series

$$\sum_{n=1}^{\infty} f(n) n^{it}.$$

Our particular interest is naturally the case $f(n) = n^{-\alpha}$ when the symbol is $\zeta(\alpha - it)$. We are especially interested how and to what extent properties of the mapping relate to properties of the symbol for $\alpha \leq 1$.

These type of mappings were considered by various authors (for example Wintner [15]) and most notably Toeplitz [13], [14] (although somewhat indirectly, through his investigations of so-called

[1] To appear in Functiones et Approximatio Commentarii Mathematici

"D-forms"). In essence, Toeplitz proved that $\varphi_f : l^2 \to l^2$ is bounded if and only if $\sum_{n=1}^{\infty} f(n) n^{-s}$ is defined and bounded for all $\Re s > 0$. In particular, if $f(n) \ge 0$ then φ_f is bounded on l^2 if and only if $f \in l^1$; furthermore, the operator norm is $\|\varphi_f\| = \|f\|_1$. We prove this in Theorem 1.1 following Toeplitz's original idea. For example, for $f(n) = n^{-\alpha}$, φ_f is bounded on l^2 for $\alpha > 1$ with operator norm $\zeta(\alpha)$. In this special case, the mapping was studied in [7] for $\alpha \le 1$ when it is unbounded on l^2 by estimating the behaviour of the quantity

$$\Phi_f(N) = \sup_{\|a\|_2=1} \left(\sum_{n=1}^{N} |b_n|^2 \right)^{1/2}$$

for large N. Approximate formulas for $\Phi_f(N)$ were obtained and it was shown that, for $\frac{1}{2} < \alpha \le 1$, $\Phi_f(N)$ is a lower bound for $\max_{1 \le t \le T} |\zeta(\alpha + it)|$ with $N = T^{\lambda}$ (some $\lambda > 0$ depending on α only). In this way, it was proven that the measure of the set

$$\left\{ t \in [1, T] : |\zeta(1 + it)| \ge e^{\gamma} \log\log T - A \right\}$$

is at least $T \exp\left(-a \frac{\log T}{\log\log T} \right)$ (some $a > 0$) for A sufficiently large, while for $\frac{1}{2} < \alpha < 1$ one has

$$\max_{t \le T} |\zeta(\alpha + it)| \ge \exp\left\{ \frac{c(\log T)^{1-\alpha}}{\log\log T} \right\}$$

for some $c > 0$ depending on α only, as well providing an estimate for how often $|\zeta(\alpha + iT)|$ is as large as the right-hand side above. The method is akin to Soundararajan's `resonance' method and incidentally shows the limitation of this approach for $\alpha > \frac{1}{2}$ since $|\zeta(\alpha + iT)|$

Similarly one can study the quantity

$$m_f(T) = \inf_{\substack{\|a\| = T \\ a \in M_2}} \frac{\|\varphi_f a\|}{\|a\|}.$$

With $f(n) = n^{-\alpha}$ this is shown to behave like the known and conjectured minimal order of $|\zeta(\alpha + iT)|$ for $\alpha > \frac{1}{2}$. It should be stressed here that, unlike the case of $\Phi_f(N)$ which was shown to be a lower bound for $Z_\alpha(T)$ in [7], we have not proved any connection between $\zeta(\alpha + iT)$ and $M_f(T)$. Even to show $M_f(T)$ is a lower bound would be very interesting.

Our results, though motivated by the special case $f(n) = n^{-\alpha}$, extend naturally to completely multiplicative f for which $f|_P$ is regularly varying (see section 2 for the definition).

Addendum. I would like to thank the anonymous referee for some useful comments and for pointing out a recent paper by Aistleitner and Seip [1]. They deal with an optimization problem which is different yet curiously similar. The function $\exp\{c_\alpha(\log T)^{1-\alpha}(\log\log T)^{-\alpha}\}$ appears in the same way, although their c_α is expected to remain bounded as $\alpha \to \frac{1}{2}$. It would be interesting to investigate any links further.

1. Bounded operators

Notation: Let l^1 and l^2

$\varphi_f : l^2 \to l^2$ is bounded if and only if $f \in l^1$, in which case $\|\varphi_f\| = \|f\|_1$.

Proof. After (1.1), and since $\Phi_f(N)$ increases with N, we need only provide a lower bound for an infinite sequence of Ns. Let $a_n = d(N)^{-\frac{1}{2}}$ for $n|N$ and zero otherwise (N to be chosen later), where $d(\cdot)$ is the divisor function. Thus $a_1^2 + \cdots + a_N^2 = 1$ and

$$\Phi_f(N) \geq \sum_{n \leq N} a_n b_n = \frac{1}{d(N)} \sum_{n|N} \sum_{d|n} f(d) = \frac{1}{d(N)} \sum_{d|N} f(d) d\left(\frac{N}{d}\right); \qquad (1.2)$$

say. We choose N

(see [2] for a detailed treatise on the subject). For example $x^{\rho}(\log x)^{\tau}$ is regularly-varying of index ρ for any τ. The Uniform Convergence Theorem says that the above asymptotic formula is automatically uniform for λ in compact subsets of $(0,\infty)$. Note that every regularly varying function of non-zero index is asymptotic to one which is strictly monotonic and continuous. We shall make use of Karamata's Theorem: for ℓ regularly varying of index ρ,

$$\int^{x} \ell \sim \frac{x\ell(x)}{\rho+1} \quad \text{if } \rho > -1, \qquad \int_{x}^{\infty} \ell \sim -\frac{x\ell(x)}{\rho+1} \quad \text{if } \rho < -1,$$

while if $\rho = -1$, $\int^{x} \ell$ is slowly varying (regularly varying with index 0) and $\int^{x} \ell \gg x\ell(x)$.

Notation. Let M^{2}

On the other hand, the RHS of (2.2) is greater than

$$\sum_{k=1}^{\infty}\sum_{s=1}^{k}\sum_{r=1}^{s} f(p^r)f(p^s)g(p^{k-r+1})g(p^{k-s+1}):$$

Hence $h \in M^2$ if and only if

$$\sum_{p}\sum_{m,n \ge 1}\sum_{k=0}^{\infty} f(p^m)g(p^n)f(p$$

Thus, in particular, $\mathcal{M}^2_c \subset \mathcal{M}^2_0$. For $f \in \mathcal{M}^2_c$ if and only if $|f(p)| < 1$ for all primes p and $\sum_p |f(p)|^2 < \infty$. Thus

$$\sum_{k=1}^{\infty} |f(p^k)| = \frac{|f(p)|}{1 - |f(p)|} \le A,$$

independent of p (since $f(p) \to 0$).

The "quasi-norm" $M_f(T)$

Let $f \in \mathcal{M}^2_0$. From above we see that $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$ but, typically, φ_f is not 'bounded' on $\mathcal{M}^2$ (if $f \notin \ell^1$) in the sense that $\|\varphi_f a\| / \|a\|$ is not bounded by a constant for all $a \in \mathcal{M}^2$. It therefore makes sense to define, for $T \ge 1$,

$$M_f(T) = \sup_{\substack{a \in \mathcal{M}^2 \\ \|a\| = T}} \frac{\|\varphi_f a\|}{\|a\|}.$$

We aim to find the behaviour of $M_f(T)$ for large T.

We shall consider f completely multiplicative and such that $f|_P$ is regularly varying of index $-\alpha$ with $\alpha > 1/2$ in the sense that there exists a regularly varying function $\tilde{f}$ (of index $-\alpha$) with $\tilde{f}(p) = f(p)$ for every prime p.

Our main result here is the following:

Theorem 2.3
Let $f \in \mathcal{M}^2_c$, such that $f \ge 0$ and $f|_P$ is regularly varying of index $-\alpha$ where $\alpha \in (\frac{1}{2}, 1)$. Then

$$\log M_f(T) \sim c(\alpha) \tilde{f}(\log T \log\log T) \log T$$

where $\tilde{f}$ is any regularly varying extension of $f|_P$ and

$$c(\alpha) = B(\tfrac{1}{\alpha}$$

Collecting those terms for which $(c; d) = k$, writing $c = km$, $d = kn$

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By Cauchy-Schwarz,

$$\sum_{k=1}^{\infty} a_{p^k} b_{p^{k-1}} \le \left(\sum_{k=1}^{\infty} a_{p^k}^2 \sum_{k=1}^{\infty} b_{p^{k-1}}^2 \right)^{1/2} = \sqrt{\alpha_p (1+\beta_p)};$$

so, on rearranging

$$(1+\beta_p) - \frac{2f(p)\sqrt{\alpha_p(1+\beta_p)}}{1-f(p)^2} \le \frac{1+\alpha_p}{1-f(p)^2}.$$

Completing the square we find

$$\left(\sqrt{1+\beta_p} - \frac{f(p)\sqrt{\alpha_p}}{1-f(p)^2} \right)^2 \le \frac{1+\alpha_p}{(1-f(p)^2)^2}.$$

The term on the left inside the square is non-negative for p sufficiently large since $f(p) \to 0$; in fact from (2.4), $1+\beta_p \ge \frac{1+\alpha_p}{1-f(p)^2}$ which is greater than $\frac{f(p)^2\alpha_p}{(1-f(p)^2)^2}$ if $f(p) \le 1/\sqrt{2}$. Rearranging gives

$$\sqrt{\frac{1+\beta_p}{1+\alpha_p}} \le \frac{1}{1-f(p)^2} \left(1 + f(p) \sqrt{\frac{\alpha_p}{1+\alpha_p}} \right).$$

Let $\gamma_p = \sqrt{\frac{\alpha_p}{1+\alpha_p}}$. Taking the product over all primes p gives

$$\frac{\|\varphi_f a\|}{\|a\|} \le A\|f\|^2 \prod_p (1+f(p)\gamma_p) \le A' \exp\left\{ \sum_p f(p)\gamma_p \right\} \qquad (2.5)$$

fo955059ome(95505const)]T(est]TJ/F16 9.96TJ/F2 19856 TJ/F11 5.9[4(A)]TJ[(−4457(2.V)83(e(95505can(95505tak7(e)-3J/F16 subject to $0 \le \gamma_p < 1$ and $\prod_p \frac{1}{1-\gamma_p^2} = T^2$. The maximum clearly occurs for γ_p decreasing (if $\gamma_{p'} > \gamma_p$ for primes $p < p'$, then the sum increases in value if we swap γ_p and $\gamma_{p'}$). Thus we may assume that γ_p is decreasing.

By interpolation we may write $\gamma_p = g(\frac{p}{P})$ where $g : (0,\infty) \to (0,1)$ is continuously differentiable and decreasing. Of course g will depend on P. Let $h = \log \frac{1}{1-g^2}$, which is also decreasing. Note that

$$2\log T = \sum_p h\left(\frac{p}{P}\right) \ge \sum_{p \le aP} h\left(\frac{p}{P}\right) \ge h(a)\pi(aP) \ge cah(a)\log T;$$

for P sufficiently large, for some constant $c > 0$. Thus $h(a) \le C_a$ (independent of T).

Now, for $F : (0,\infty) \to [0,\infty)$ decreasing,

$$\sum_{ax<p\le bx} F\left(\frac{p}{x}\right) = \frac{x}{\log x}\int_a^b F + O\left(\frac{xF(a)}{(\log x)^2}\right); \qquad (2.9)$$

where the implied constant is independent of F (and x). For, on writing $\pi(x) = \mathrm{li}(x) + E(x)$, the LHS is

$$\begin{aligned}\int_{ax}^{bx} F\left(\frac{t}{x}\right) d\pi(t) &= x\int_a^b \frac{F(t)}{\log xt}\,dt + \int_a^b F(t)\,dE(xt)\\ &= \frac{x}{\log \theta x}\int_a^b F + \Big[F(t)E(xt)\Big]_a^b - \int_a^b E(xt)\,dF(t) \qquad (\text{some } \theta \in [a,b])\\ &= \frac{x}{\log x} \;\Big])\end{aligned}$$

me $\theta \in [a,b]$)

approximate arbitrarily closely with such functions. On writing $g = s \circ h$ where $s(x) = \sqrt{1 - e^{-x}}$, we have

$$\int_0^\infty \frac{g(u)}{u^\alpha}\,du = \left[\frac{g(u)u^{1-\alpha}}{1-\alpha}\right]_0^\infty - \frac{1}{1-\alpha}\int_0^\infty g'(u)u^{1-\alpha}\,du$$
$$= -\frac{1}{1-\alpha}\int_0^\infty s'(h(u))h'(u)u^{1-\alpha}\,du = \frac{1}{1-\alpha}\int_0^{h(0^+)} s'(x)l(x)^{1-\alpha}\,dx;$$

where $l = h^{-1}$, since $\sqrt{u}\,g(u) \to 0$ as $u \to \infty$. The final integral is, by Hölder's inequality at most

$$\left(\int_0^{h(0^+)} s'^{1/\alpha}\right)^\alpha \left(\int_0^{h(0^+)} l\right)^{1-\alpha}. \qquad (2.10)$$

But $\int_0^{h(0^+)} l = -\int_0^\infty uh'(u)\,du = \int_0^\infty h \le 2$, so

$$\int_0^\infty \frac{g(u)}{u^\alpha}\,du \le \frac{2^{1-\alpha}}{1-\alpha}\left(\int_0^\infty s'^{1/\alpha}\right)^\alpha.$$

A direct calculation shows that[3] $\int_0^\infty (s')^{1/\alpha} = 2^{-1/\alpha}B(\frac{1}{\alpha}; 1 - \frac{1}{2\alpha})$. This gives the upper bound.

The proof of the upper bound leads to the optimum choice for g and the lower bound. We note that we have equality in (2.10) if $l/(s')^{1/\alpha}$ is constant; i.e. $l(x) = cs'(x)^{1/\alpha}$ for some constant $c > 0$ — chosen so that $\int_0^\infty l = 2$. This means we take

$$h(x) = (s')^{-1}\left(\left(\frac{x}{c}\right)^\alpha\right) = \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \left(\frac{c}{x}\right)^{2\alpha}}\right).$$

from which we can calculate g. In fact, we show that we get the required lower bound by just considering a_n completely multiplicative. To this end we use (2.3), and define a_p by:

$$a_p = g_0\left(\frac{p}{P}\right);$$

where $P = \log T \log\log T$ and g_0 is the function

$$g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + (\frac{c}{x})^{2\alpha}}}};$$

with $c = 2^{1+1/\alpha}/B(\frac{1}{\alpha}; 1 - \frac{1}{2\alpha})$. As such, by the same methods as before, we have $\|a\| = T^{1+o(1)}$ and

$$\log\frac{\|\varphi_\alpha a\|}{\|a\|} = \sum_p f(p)g_0\left(\frac{p}{P}\right) + O(1) \sim \frac{P\tilde{f}(P)}{\log P}\int_0^\infty \frac{g_0(u)}{u^\alpha}\,du.$$

By the choice of g_0, the integral on the right is $\frac{B(\frac{1}{\alpha}; 1-\frac{1}{2\alpha})^\alpha}{(1-\alpha)2^\alpha}$, as required.

$\square$

Remark. From the above proof, we see that the supremum of $\|\varphi_f a\|/\|a\|$ over $\mathcal{M}_c^2$ is roughly the same size as the supremum over $\mathcal{M}^2$; i.e. they are log-asymptotic to each other. Is it true that these respective suprema are closer still; eg. are they asymptotic to each other for $\frac{1}{2} < \alpha < 1$?

3. The special case $f(n) = n^{-\alpha}$.
In this case we can take $\tilde{f}(x) = x^{-\alpha}$ which is regularly varying of index $-\alpha$. Here we shall write φ_α for φ_f and $\mathcal{M}_\alpha$ for $\mathcal{M}_f$.

[3] The integral is $2^{-1/\alpha}\int_0^\infty e^{-x/\alpha}(1-e^{-x})^{-1/2\alpha}\,dx = 2^{-1/\alpha}\int_0^1 t^{1/\alpha-1}(1-t)^{-1/2\alpha}\,dt$.

Theorem 3.1 *We have*

$$M_1(T) = e^{\gamma}(\log\log T + \log\log\log T + 2\log 2 - 1 + o(1)); \tag{3.1}$$

while for $\frac{1}{2} < \alpha < 1$,

$$\log M_\alpha(T) = \left(\frac{B(\frac{1}{\alpha}, 1-\frac{1}{2\alpha})^\alpha}{(1-\alpha)2^\alpha} + o(1)\right)\frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha}. \tag{3.2}$$

Remark. As noted in the introduction, these asymptotic formulae bear a strong resemblance to the (conjectured) maximal order of $|\zeta(\alpha + iT)|$. It is interesting to note that the bounds found here are just larger than what is known about the lower bounds for $Z_\alpha(T) = \max_{1\le t\le T}|\zeta(\alpha + it)|$. In a recent paper (see [8]), Lamzouri suggests $\log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}(\log\log T)^{-\alpha}$ with some specific function[4] $C(\alpha)$ (for $\frac{1}{2} < \alpha < 1$). We note that the constant appearing in (3.2) is not $C(\alpha)$ since, for α near $\frac{1}{2}$, the former is roughly $\frac{1}{\sqrt{\alpha-\frac{1}{2}}}$, while $C(\alpha) \sim \frac{1}{\sqrt{2\alpha-1}}$. For $\alpha = 1$, see the comment in the introduction.

It would be very interesting to be able to extend these ideas (and results) to the $\alpha = \frac{1}{2}$ case. As we show in the appendix, we cannot do this by restricting $\varphi_{\frac{1}{2}}$ to smaller domains in l^2. Somehow the analogy — if such exists — between M_α and Z_α breaks down just here.

Proof of Theorem 3.1. For $\frac{1}{2} < \alpha < 1$ the result follows from Theorem 2.3, so we only concern ourselves with $\alpha = 1$.

For an upper bound we use (2.5) with $f(p) = 1/p$ (and $A = 1$). Thus

$$\frac{\|\varphi_1 a\|}{\|a\|} \le \zeta(2)\prod_p\left(1 + \frac{\gamma_p}{p}\right).$$

by (2.9). Thus

$$\frac{\|\varphi_1 a\|}{\|a\|} \le e^{\gamma}\left(\log_2 T + \log_3 T + \int_a^A \frac{g(u)}{u}\,du - \int_a^1 \frac{1}{u}\,du + \frac{2}{\sqrt{A}} + o(1)\right)$$

for all $A > 1 > a > 0$. We need to minimise the constant term. Since $g(u) < 1$, the minimum occurs for a arbitrarily small. On the other hand $\int_A^\infty \frac{g(u)}{u}du \le \left(\frac{1}{A}\int_A^\infty g^2\right)^{1/2} = o(1$ du $\le$ A after some calculation.

The proof of the upper bound leads to the optimum choice for g and the lower bound. We note that we have equality in (3.3) if l/s' is constant; i.e. $l(x) = cs'(x)$ for some constant $c > 0$ | chosen so that $\int_0^1 l = 2$ (i.e. we take $c = 2$). Thus, actually $\cdot = 2\log 2 - 1$ and the supremum is achieved for the function g_0, where

$$g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + (\frac{2}{x})^2}}}.$$

In fact, we show that we get the required lower bound by just considering a_n completely multiplicative. To this end we use Corollary 2.5, and define a_p by:

$$a_p = g_0\left(\frac{p}{P}\right);$$

where $P = \log T \log\log T$. As such, by the same methods as before, we have $\|a\| = T^{1+o(1)}$. Let $a > 0$ and $P = \log T \log\log T$. By Corollary 2.5

$$\frac{\|\varphi_1 a\|}{\|a\|} \geq \prod_p \frac{1}{1 - \frac{a_p}{p}} = \prod_{p \leq aP} \frac{1}{1 - \frac{1}{p}} \prod_{p \leq aP} \frac{1}{1 + \frac{1 - a_p}{p - 1}} \prod_{p > aP} \frac{1}{1 - \frac{a_p}{p}}. \tag{3.4}$$

Using Merten's Theorem, the first product on the right is $e^{\gamma}(\log aP + o(1))$, while the second product is greater than

$$\exp\left\{-\sum_{p \leq aP} \frac{1 - a_p}{p - 1}\right\} \geq 1 - 2\sum_{p \leq aP} \frac{1 - g_0(p/P)}{p}.$$

The sum is asymptotic to $\frac{a}{\log P}\int_0^a \frac{1 - g_0(u)}{u}du < \frac{\varepsilon}{\log P}$, for any given $\varepsilon > 0$, for sufficiently small a. The third product in (3.4) is greater than

$$\exp\left\{\sum_{p > aP} \frac{a_p}{p}\right\} = \exp\left\{\frac{(1 + o(1))}{\log P}\int_a^1 \frac{g_0(u)}{u}du\right\}$$

by (2.9). Thus

$$\frac{\|\varphi_1 a\|}{\|a\|} \geq e^{\gamma}\left(\log P + \int_a^1 \frac{g_0(u)}{u}du + \log a - \varepsilon\right) \geq e^{\gamma}\left(\log P + L(g_0) - \varepsilon\right)$$

for a sufficiently small. As $L(g_0) = 2\log 2 - 1$ and ε arbitrary, this gives the required lower bound. □

Lower bounds for φ_α

We see that $m_\alpha(T)$ corresponds closely to the conjectured minimal order of $|\zeta(\alpha + iT)|$ (see [3] and [9]). We omit the proofs, but just point out that for an upper bound (for $1/m_\alpha(T)$) we use

$$\frac{\|a\|}{\|\varphi_\alpha a\|} \le \prod_p \left(1 + \frac{\gamma_p}{p^\alpha}\right);$$

which can be obtained in much the same way as (2.5). For the lower bound, we choose a_p as -1 times the choice in Theorem 3.1 and use Corollary 2.5.

The above formulae suggest that the supremum (respectively infimum) of $\|\varphi_\alpha a\|/\|a\|$ with $a \in \mathcal{M}^2$ and $\|a\| = T$ are close to the supremum (resp. infimum) of $|\zeta_\alpha|$ on $[1, T]$. One could therefore speculate further that there is a close connection between $\|\varphi_\alpha a\|/\|a\|$ (for such a) and $|\zeta(\alpha + iT)|$, and hence between $Z_\alpha(T)$ and $M_\alpha(T)$. Recent papers by Gonek [4] and Gonek and Keating [5] suggest this may be possible, or at least that M_α is a lower bound for Z_α. On the Riemann Hypothesis, it was shown in [4] (Theorem 3.5) that $\zeta(s)$ may be approximated for $\sigma > \frac{1}{2}$ up to height T by the truncated Euler product

$$\prod_{p \le P} \frac{1}{1 - p^{-s}} \quad \text{for } P \ll T.$$

Thus one might expect that, with $a \in \mathcal{M}^2_{c+}$ maximizing $\frac{\|\varphi_\alpha a\|}{\|a\|}$ subject to $\|a\| = T$, and $A(s) = \prod_{p \le P} \frac{1}{1 - a_p p^{-s}}$ (with $P \ll T$),

$$\int_{-T}^{T} |\zeta(\alpha - it)|^2 |A(it)|^2 \, dt \sim \int_{-T}^{T} \prod_{p \le P} \left| \left(1 - \frac{p^{it}}{p^\alpha}\right)(1 - a_p p^{it})^{-1} \right|^2 dt = \int_{-T}^{T} \prod_{p \le P} |B_p(it)|^2 \, dt, \quad B_p(s) =$$