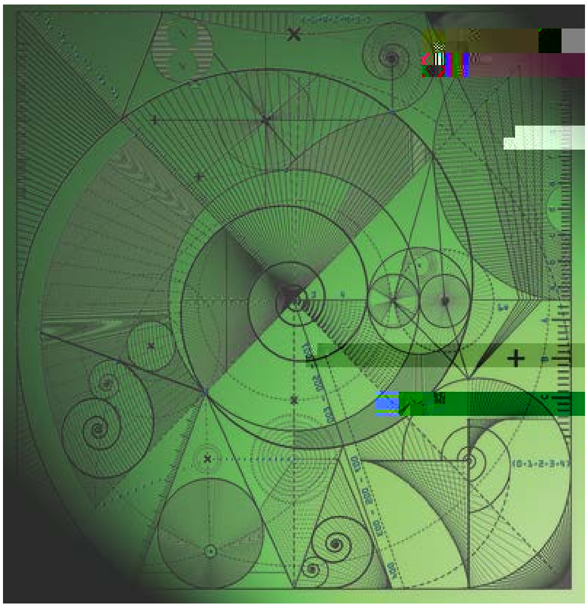
Implementation of an Interior Point Source in the Ultra Weak Variational Formulation through Source Extraction

by

C.J. Howarth, P.N. Childs and A. Moiola



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1 Introduction

The Ultra Weak Variational Formulation (UWVF), originally proposed by Cessenat and Després in [4, 5], is a new-generation finite element method for the accurate simulation of time-harmonic acoustic, elastic, and electromagnetic waves. The area of time-harmonic wave scattering is a subject of much research, with applications in seismology, medical imaging, and radar imaging.

We consider acoustic wave propagation, modelled in two dimensions by the following Helmholtz boundary value problem (BVP):

$$\nabla\cdot\left(\frac{1}{\rho}\nabla u\right) + \frac{\kappa^2}{\rho}u = f \qquad \text{in } \Omega; \tag{1a}$$

$$\left(\frac{1}{\rho}\frac{\partial u}{\partial n} - i\sigma u\right) = Q\left(-\frac{1}{\rho}\frac{\partial u}{\partial n} - i\sigma u\right) + g \qquad \text{on } \Gamma: \tag{1b}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ; the density $\rho(x)$ and the wavenumber $\kappa(x)$ are real positive and may vary throughout the domain. The coupling parameter σ is real and positive, and f and g are the volume and boundary source terms respectively. The parameter $Q \in \mathbb{C}$, $|Q| \le 1$, allows different types of boundary conditions: $Q = 1$, -1 and 0 correspond to Neumann, Dirichlet, and impedance boundary conditions, respectively.

The UWVF is a Trefftz-type method: the exact solution of a Helmholtz boundary value problem is approximated by a linear combination of basis functions that, inside each mesh element, are solutions of the homogeneous Helmholtz equation, i.e. equation (1a) with right-hand side $f = 0$. By incorporating information on the oscillatory behaviour of Helmholtz solutions into the approximation space, the UWVF can produce accurate results requiring significantly fewer degrees of freedom than standard finite element methods, in some cases for mesh sizes encompassing several wavelengths

The solution of the Helmholtz equation is often approximated using a plane wave basis [3, 5, 7, 9, 11]; however, it is also possible to use other solutions of the homogeneous Helmholtz equation, such as a Fourier–Bessel functions as in [13].

As with standard finite element methods (FEM), the domain Ω is partitioned into a polygonal mesh; however the solution variables are impedance traces $\frac{1}{\rho}\frac{\partial u}{\partial n} - i\sigma u$ on the skeleton of the mesh. These traces are approximated by the corresponding traces of a Trefftz trial space and the approximation is automatically achieved also in the element interiors if the discretised BVP is homogeneous ($f = 0$), see [3, Theorem 4.1], [10, Theorem 4.5]. In [3, 6, 7, 9] the UWVF has been shown to be a discontinuous Galerkin (DG) method with Trefftz basis functions, allowing a simpler and more general derivation of the formulation (see e.g. [9, §3.2]) and a more straightforward error analysis.

In seismic imaging applications, point sources (monopoles or dipoles) are used in the interior of the domain, for example to represent an explosive sound source. Modelling this situation requires solving the inhomogeneous Helmholtz equation for a non-zero and singular source term f, for example a Dirac delta function. To date, the use of the UWVF to solve the inhomogeneous form of the Helmholtz equation has not received a great deal of attention in the literature: typically, sources in the exterior of the domain have been simulated by imposing non-zero boundary conditions in BVPs for the homogeneous Helmholtz equation, in order to demonstrate superior approximation properties of Trefftz methods.

In [4–7] the UWVF with non-zero source term f has been investigated, and both a priori analysis and numerical experiments have been presented. Loeser and Witzigman [12] use UWVF to solve the Helmholtz equation (1a) with a source term $f = 1$ in Ω_S and $f = 0$ elsewhere, for an active region $\Omega_S \subset \Omega$. The UWVF solution is found in the source-free region $\Omega \setminus \Omega_S$ only, after which, in an additional post-processing step, a standard finite element method (FEM) is used in the active region where f is non-zero. In practice, [12] suggests that the FEM mesh size in the active region should be no larger than $\lambda/30$, where λ is the problem wavelength, leading to a potentially computationally expensive scheme.

Here, we investigate the applicability of the UWVF to seismic imaging by considering the typical situation of an interior point source. We first consider a domain of constant wave speed, and then extend our investigations to the simulation of wave propagation through a layered velocity profile. We present a simple yet accurate method to augment the UWVF in the case of a localised non-zero source term f, which we call the Source Extraction UWVF. In this approach, the domain Ω is split into two regions: an inner source region containing the source, and an outer region comprising the remainder of the domain. In the inner region, a particular radiating solution of the inhomogeneous Helmholtz equation with source f is subtracted from the field, so that the remainder of the wave field is amenable to a Trefftz approximation in the interior (this remainder is the wave field which is back-scattered from the outer region into the inner region). In the outer region we solve for the total field. The solutions in the two regions are matched by prescribing the jumps of the impedance and the conjugate-impedance traces across element boundaries. If we consider a point source (a Dirac delta), then we subtract the fundamental solution in the source region. However the method can be easily generalised to other forms of sources, such as for a dipole source. A related approach based on splitting of outgoing and back-scattered fields is used in [2, 17] for finite difference methods in time domain. A similar approach for the UWVF has been derived separately by Gabard in [6, Section 5.1] for a system of linear hyperbolic equations, applied with accurate results to the linearised Euler equations.

Details of the UWVF are given in Section 2, with explanation given as to and we represent any $v \in H$ as a vector $\{v_k\}_{k=1}^{K}$ with $v_k := v|_{k}$. To avoid technical difficulties with the regularity of f and the solution u of the BVP (

$$= 2i \int_{\partial \Omega_k} \frac{1}{\gamma_k} \left(u_k \overline{\frac{\partial v_k}{\partial n_k}} - \frac{\partial u_k}{\partial n_k} \overline{v_k} \right) dS = 2i \int_{\Omega_k} f \, \overline{v_k} \, dV; \qquad (4)$$

which holds for all $v \in H$ and for $u \in H^e$ solution of (1a), and substituting the term denoted by A_k with the corresponding trace from the neighbouring element or from the boundary condition. Note that complex wavenumbers (i.e. absorbing media) can be considered as in [3, Section 5].

The usual UWVF discretisation consists in restricting the variational problem (3) to the discrete space $H_h = \prod_{k=1}^{K} \operatorname{span}\{\varphi_{k;l}\}_{l=1}^{p_k} \subset H$ defined by the basis functions $\varphi_{k;l} \in H_k$, $1 \leq k \leq K$, $1 \leq l \leq p_k$, where p_k is the number of degrees of freedom located in Ω_k and may vary in different elements.

When solving the homogeneous Helmholtz equation, all of the integrals in (3) are defined on the element boundaries (as $f \equiv 0$ the only volume integral in (3) vanishes). On the other hand, in the general case the right-hand side of (3) includes an integral over all the elements where the source term f is non zero (or point evaluations if f is a linear combination of point sources).

A standard choice of the Trefftz basis functions $\varphi_{k;l}$, i.e. equispaced plane waves or circular waves (Fourier–Bessel functions), allows high orders of approximation in the elements where $f = 0$; see [16]. On the contrary, when $f \neq 0$ inside Ω_k, Trefftz functions lose their approximation properties. The use of plane waves in the inhomogeneous case can provide the same approximation of u as piecewise-linear polynomials only; this is supported by numerical experiments that found moderately high orders of convergence for the approximation of u on the skeleton of the mesh but only linear order in the meshsize h for the volume error measured in the $L^2(\Omega)$-norm, see [5, Tables 3.3 and 3.4] and [7, Section 5].

These two reasons, the integration on the mesh skeleton only and the higher orders of approximation, motivated the invest where δ is the Dirac delta function and $x_0 \in \Omega$. In this case, the right-hand side of the UWVF formulation (3) becomes $\int_{\Omega_k} f\,\overline{v_k}\,dV = \overline{v_k}(x_0)$; $f \notin L^2(\Omega)$ and $u \notin H^1(\Omega)$. As it might be expected, numerical tests using the formulation (3) proved extremely inaccurate at representing the source, with high errors in the element containing x_0; numerical experiments for this case are provided in Section 4.1.

In order to introduce a modified formulation, we now fix some notation. We split the domain in two open regions Ω^S and Ω^E, $\Omega = \Omega^S \cup \Omega^E \cup \Gamma_S$ where $\Gamma_S = \partial\Omega^S$ (as illustrated in Figure 1) such that the two regions correspond to a partition of the mesh: $\mathcal{T} = \mathcal{T}^S \cup \mathcal{T}^E$ with $\Omega_k \in \mathcal{T}^S$ if $\Omega_k \subset \Omega^S$ and $\Omega_{k'} \in \mathcal{T}^E$ if $\Omega_{k'} \subset \Omega^E$. On Γ_S, we denote by n_S the unit normal vector outward pointing from Ω^S, and set $n_E = -n_S$.

where H_0^1 is the Hankel function of the first kind and order zero. Then u^I is the fundamental solution of the Helmholtz equation (with constant pa-

In the space X we define the impedance and the "adjoint impedance" trace operators

$$I : H \to$$

p	L^2( ) relative error, classical UWVF	L^2( ) relative error, Source Extraction UWVF	N
9	4:6148 10^{1}	9:8941 10^{3}	6:7672
10	4:6138 10^{1}	5:2901 10^{3}	7:1332
11	4:6087 10^{1}	1:5578 10^{3}	7:4814
12	4:6159 10^{1}	8:2696 10^{4}	7:8140
13	4:6154 10^{1}	3:3(1)-2.813857]TJ /R2960 Td [(73857]TJ /R26 11.9552	

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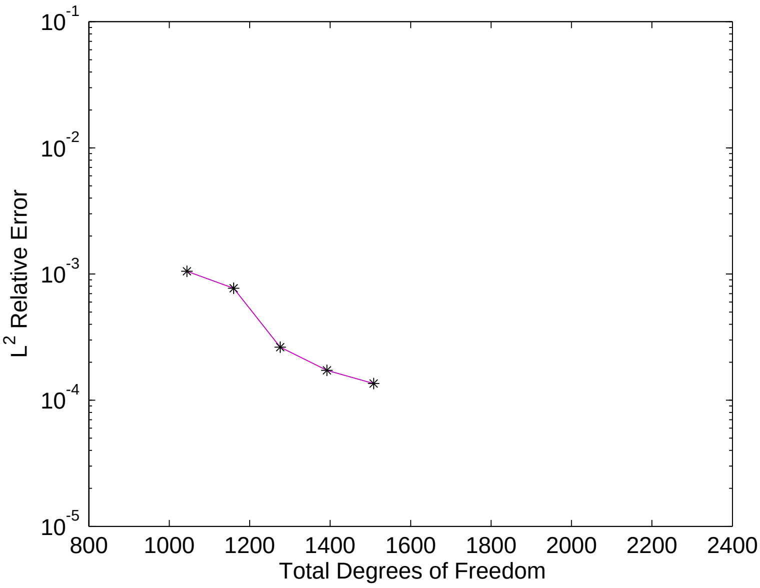
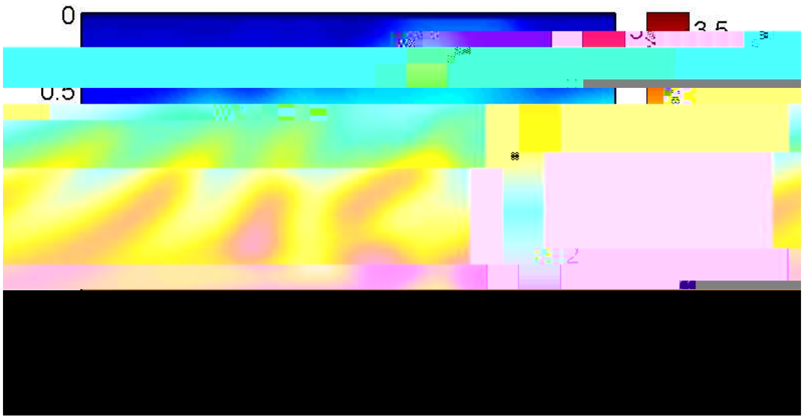


10^-1
10^-2
10^-3
10^-4
10^-5
L^2 Relative Error
800
1000
1200
1400
1600
1800
2000
2200
2400
Total Degrees of Freedom


0
0.5

(piecewise constant) wavenumber for a frequency of 5 Hz and the position of the point source. The same discretisations are used for the frequency 10 Hz, resulting in the wavenumber in each element being doubled.

The angularly equispaced basis (11) is used, with $R = 100$ to replicate the conventional plane wave basis. An initial maximum number $p = 15$ of basis functions per element is set, and then p_k reduced if the condition number of the submatrix D_k is above the tolerance level of 10^{10}. The range of values taken by p_k across the mesh and the total number of degrees of freedom obtained for the frequencies 5 and 10 Hz and for the two meshes is summarised in Table 3.

Frequency	K	Range of p_K	Total number of degrees of freedom
5 Hz	485	$[8, \dots, 15]$	5,162
5 Hz	771	$[8, \dots, 13]$	6,636
10 Hz	485	$[11, \dots, 15]$	7,417
10 Hz	771	$[10, \dots, 15]$	9,749

Table 3: The range of the values taken by the local number of degrees of freedom p_k and the total number of degrees of freedom $\sum_{k=1}^{K} p_k$ obtained with the adaptive procedure for the frequencies 5 and 10 Hz and for the two meshes with 485 and 771 triangles shown in Figure 4.

The upper and centre plots of Figure 5 show the real part of the Source Extraction UWVF solution for the frequency 5 Hz and for the discretisations with $K = 485$ and $K = 771$ elements, respectively. The lower plot shows the real part of a reference solution computed with a finite difference scheme for comparison. (This was obtained on a regular structured grid with 180 points per wavelength and using the method described in [8].) Figure 6 shows

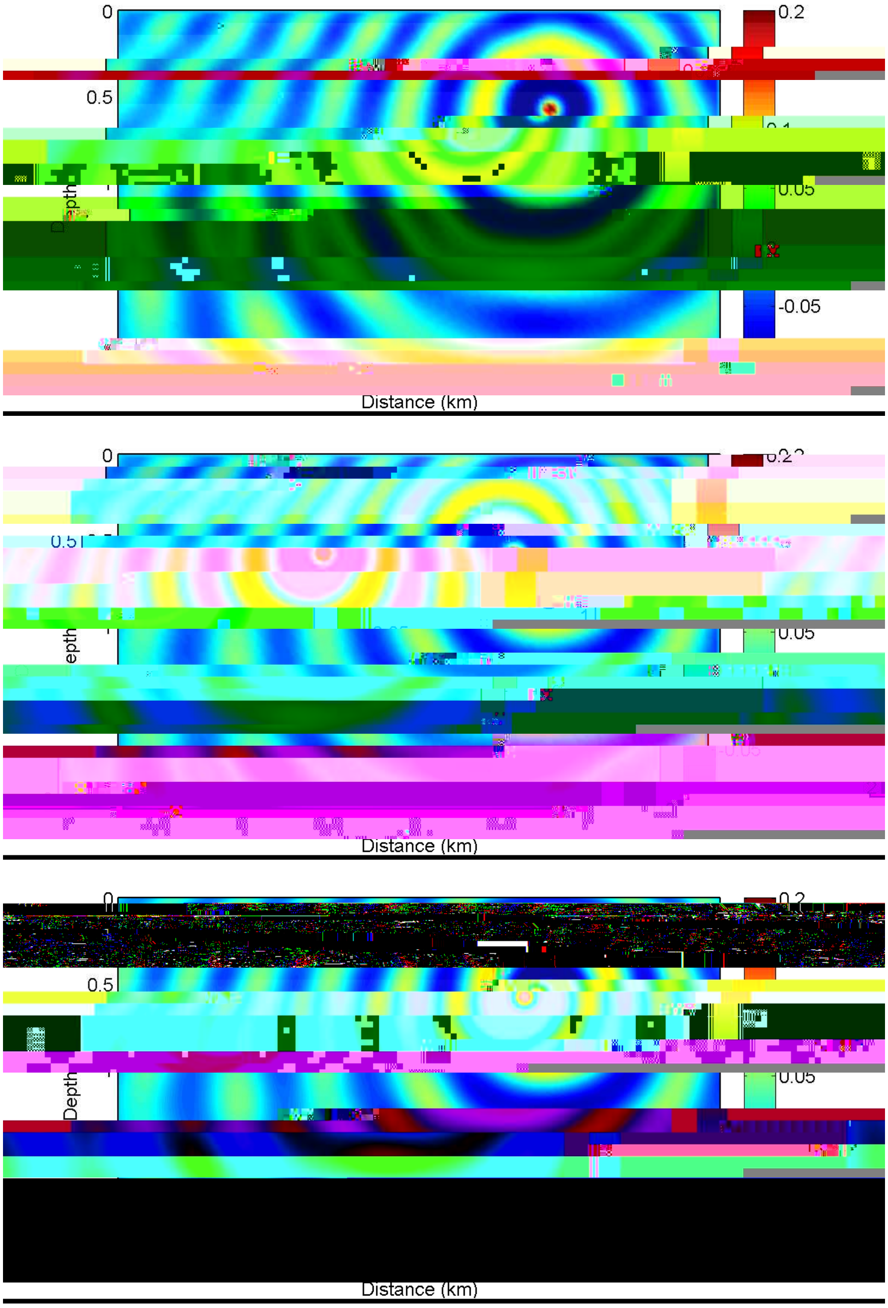


Figure 5: Real part of the total eld approximation in the smoothedMar-

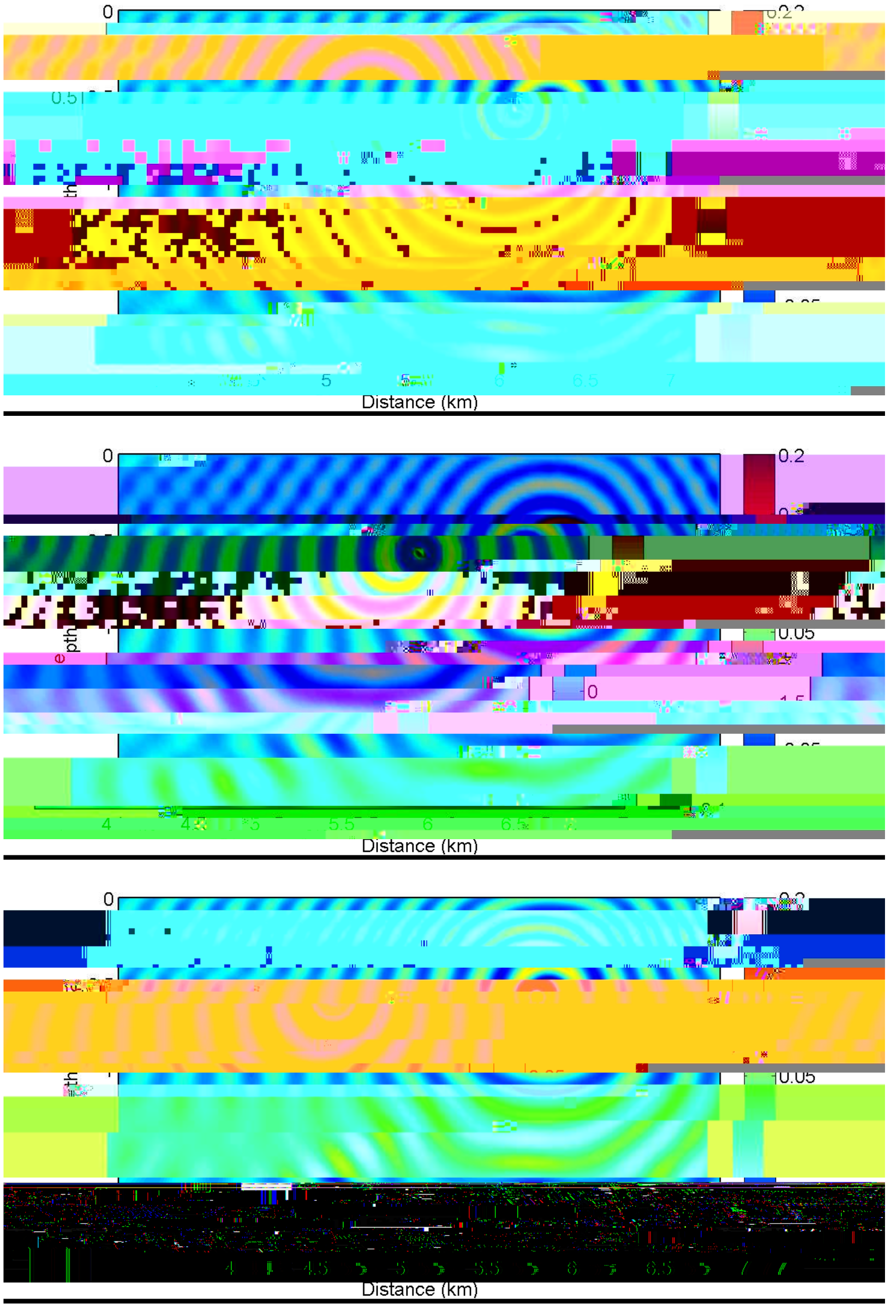


Figure 6: Real part of the total eld approximation in the smoothedMar- mousi section with frequency 10 Hz: UWVF solution withK

of the discrete space. For a point source, we approximate the unknown back-scattered field in a region surrounding the source, and match this to the total field approximated in the remainder of the domain. In the considered examples we use a Dirac delta point source; however, the augmentation of the method can be easily generalised to other forms of source function, such as dipoles and multipoles. Following on from work in [3], we show that the Source Extraction UWVF is well-posed and satisfies the error bound (10) on the mesh skeleton in the case of impedance boundary conditions and sound-

[7] C. J. Gittelson, R. Hiptmair, and I. Perugia. Plane wave discontinuous Galerkin methods: Analysis of the h