# Spectral theory of some non-selfadjoint linear differential operators

by

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Although it should be clear that these are the same question posed in different contexts, very little is explicitly known beyond the classical cases when the spatial operator has a known basis of eigenfunctions. This basis can be used after separation of variables to express the solution of the boundary value problem.

In this paper we give an explicit connection between the two problems in general; we give a link between the solutions of (1) and (2), and we show precisely how the answer to (Q1) and (Q2) are related. In particular, the rigorous answer to one question can be given through answering the other. Our results are true for general $n$, however they are new and interesting in particular for $n$ odd.

Since in general $S$ will not be self-adjoint, we expect that any spectral decomposition involves not only $S$ but also the adjoint $S^*$. In terms of the PDE problem, we will see that this is reflected in the need to consider both the initial time and the final time problems (the evolution with reversed time direction).

## The operator problem

## The PDE problem

In a separate development, a novel transform method for analysing IBVPs was developed by Fokas (see Fokas, 2008, for an overview). The method was applied to IBVPs posed for evolution equations on the half-line by Fokas and Sung (1999) and on the finite interval by Fokas and Pelloni (2001) with simple, uncoupled boundary conditions. In Smith (2012), Fokas' method was applied to IBVPs whose spatial part is given by the operator $S$, namely those of the form

$$\partial_t q(x;t) + a(-i\partial_x)^n q(x;t) = 0; \quad x \in (0;1); \quad t > 0; \quad a = \pm i; \tag{1.6}$$

with prescribed boundary conditions and an initial condition $q_0(x) = q(x;0)$, assumed smooth to avoid technical complications. Usually the initial condition is assumed to be in $C^\infty$. However, the same results hold assuming that $q_0 \in AC^n$. Indeed, in this case, the uniform convergence of the integral representation (see (1.7) below), the poynomial decay rate of the integrand and the explicit exponential $x$ dependence imply that the solution $q$ belongs to the same class. In what follows we assume $q_0 \in AC^n$.

This method yields an integral representation of the solution of the initial-boundary value problem in the form

$$\begin{aligned} q(x;t) = {} & \frac{1}{2\pi}\int_{\Gamma^+} e^{i\lambda x - a\lambda^n t}\frac{\zeta^+(\lambda)}{\Delta_{PDE}(\lambda)}\,d\lambda + \frac{1}{2\pi}\int_{\Gamma^-} e^{i\lambda(x-1) - a\lambda^n t}\frac{\zeta^-(\lambda)}{\Delta_{PDE}(\lambda)}\,d\lambda \\ & + i\sum_{k\in K^+} e^{i\sigma_k x - a\sigma_k^n t}\operatorname*{Res}_{\lambda=\sigma_k}\frac{\zeta^+(\lambda)}{\Delta_{PDE}(\lambda)}\,d\lambda + i\sum_{k\in K^-} e^{i\sigma_k(x-1) - a\sigma_k^n t}\operatorname*{Res}_{\lambda=\sigma_k}\frac{\zeta^-(\lambda)}{\Delta_{PDE}(\lambda)}\,d\lambda; \end{aligned} \tag{1.7}$$

where the quantities $\hat{q}_0$, $\zeta^\pm$, $\Delta_{PDE}$, $\sigma_k$ and $\Gamma^\pm$ are defined below in Definitions 2.1 and 2.4. In many cases, including all problems with $n$ even, the integrals in equation (1.7) both evaluate to zero (Smith, 2012). We study these cases here.

In Pelloni (2004, 2005) and then in greater generality in Smith (2011), this method is used to characterise boundary conditions that determine well-posed problems, and problems whose solutions admit representation by series. To achieve this characterisation, the central objects of interest are the *PDE characteristic matrix* $A$ (see Definition 2.1 below) and its determinant $\Delta_{PDE}$.

Note that in this work, by 'well-posed', we mean existence and uniqueness of a solution

*If the initial- and final-boundary value problems are well posed, then the eigenfunctions of $S$ and $S^\star$ form a complete biorthogonal system in $\overline{D}(S)$.*

This is the content of Theorem 2.6. The conclusion does not imply that the eigenfunctions necessarily form a basis. However the integral representation (1.7) can *always* be deformed to derive a *series representation* for the solution of the IBVP in terms of the eigenfunctions.

*The departure of the family of eigenfunctions of $S$ and $S^\star$ from being a biorthogonal basis can be estimated in terms of the integrand in the representation of the solution of the associated IBVP.*

This is the content of Theorem 2.12. This departure is quantified in the notion of 'wildness' (see Davies, 2007). Indeed, if the eigenfunction of $S$ and $S^\star$ form a wild system in $L^2[0,1]$, then we provide an estimate of the wildness of the system in terms of the quantities used to determine whether the initial- and final-boundary value problems are well posed.

### Outline of paper

In section 2, we review the necessary definitions and notation. Following this, we precisely state and prove the results described above.

Each of sections 3 and 4 is devoted to the analysis of an example which illustrates the above general results. We compare and contrast the results obtained through the new theorems with those yielded by Davies' wildness method.

# 2 Complete and basic systems of eigenfunctions

## 2.1 Notation, definitions and preliminary results

In this paper, we make extensive use of the notation developed in Smith (2012). We refer to that paper for details, but we list here some of the notation used throughout the rest of this work.

**The initial-boundary value problem** $\Pi(n,A,a,q_0)$: Find $q \in AC^n([0,1]\times[0,T])$ which satisfies the linear, evolution, constant-coefficient partial differential equation

$$\partial_t q(x,t) + a(-i\partial_x)^n q(x,t) = 0 \tag{2.1}$$

subject to the initial condition

$$q(x,0) = q_0(x) \tag{2.2}$$

and the boundary conditions

$$\left(\partial_x^{n-1} q(0,t), \partial_x^{n-1} q(1,t), \partial_x^{n-2} q(0,t), \partial_x^{n-2} q(1,t), \ldots, q(0,t), q(1,t)\right)^T = h(t), \tag{2.3}$$

**Definition 2.1.** *Let* $^{?}_{kj}$, $^{?}_{kj}$ *be the boundary coefficients of the operator* $S^{?}$, *adjoint to* $S$. *We define*

$$A^{+}_{kj}(\ ) = \sum_{r=0}^{n-1} (\ i!^{k\ 1}\ )^{r}\ ^{?}_{kj}; \tag{2.5}$$

$$A_{kj}(\ ) = \sum_{r=0}^{n-1} (\ i!^{k\ 1}\ )^{r}\ ^{?}_{kj}; \tag{2.6}$$

$$\text{then } A_{kj}(\ ) = A^{+}_{kj}(\ ) + A_{kj}(\ )e^{\ i!^{k\ 1}} \tag{2.7}$$

*is called the* PDE characteristic matrix. *The determinant* $_{\mathrm{PDE}}$ *of* $A$ *is called the* PDE characteristic determinant.

**Remark 2.2.** The PDE characteristic matrix is a realisation of Birkhoff's characteristic matrix for $S^{?}$ and also represents the Dirichlet-to-Neumann map for the problem . Indeed, it is through this matrix that the unknown (Neumann) boundary values are obtained from the (Dirichlet) boundary data of the problem. Smith (2012) uses a different but equivalent definition of $A$ which generalises the construction via determinants and Cramer's rule originally found in Fokas and Sung (1999). The validity of the new definition is established in Fokas and enclosed by the contours thus, by Jordan's Lemma, well-conditioning of the problem with the opposite direction coe cient is equivalent to the two integrals in (1.7) vanishing (Smith, 2012).

The reader will recall that a system $( \ _n)_{n \in \mathbb{N}}$ in a Banach space is said to be *complete* if its linear span is dense in the space and such a system is a *basis* if for each $f$ in the space there exists a unique sequence of scalars $( \ _n)_{n \in \mathbb{N}}$ such that

$$f = \lim_{r \to \infty} \left( \sum_{n=1} \ {}_n \ {}_n \right):$$

## 2.2 Well-posed PDE systems and bases of eigenfunctions

It is well known (see Coddington and Levinson, 1955, Section 12.5) that if the zeros of the characteristic determinant of $S$ are all simple then the eigenfunctions of $S$ form a complete system in $\overline{D}(S)$. This theorem is proven using an analysis of the Green's functions of both the operator $S$ and its adjoint $S^{?}$. We prove the following result without directly analysing the adjoint operator.

**Theorem 2.6.** *Let $S$ be such that the zeros of $\ _{\mathrm{PDE}}$ are all simple. Let $\ = \ (n; a; A; q_0; 0)$ be an IBVP associated with $A$ and $\ ^0$ be the corresponding problem with the opposite direction coe cient, $(n; \ a; A; q_0; 0)$. If is well-posed and $\ ^0$ is well-conditioned in the sense of De nition 2.5 then the eigenfunctions of $S$ form a complete system in $\overline{D}(S)$.*

Rather than analysing both the original operator $S$ and the adjoint operator $S^{?}$, one needs information on both the initial- and nal-boundary value problems associated with the operator $S$.

A stronger, but essentially straightforward, result in the reverse direction is:

**Proposition 2.7.** *If the eigenfunctions of $S$ form a basis in $\overline{D}(S)$ and, for some $a$, the associated IBVP is well-posed, then $\ ^0$ is well-conditioned.*

*Further, if $( \ _k)_{k \in \mathbb{N}}$ are the eigenfunctions of $S$, with corresponding eigenvalues $( \ ^n_k)_{k \in \mathbb{N}}$ then there exists a sequence $( \ _k)_{k \in \mathbb{N}}$ biorthogonal to $( \ _k)_{k \in \mathbb{N}}$ such that the Fourier expaJ/F21 5.9776 TfunctioTd [(then)-*

**Proposition 2.8.** *For each $k \in \mathbb{N}$ and for each $j \in \{1, 2, \ldots, n\}$, the function*

$$\psi_k^j(x) = \sum_{r=1}^{n} e^{-i\omega^{r-1}\lambda_k(1-x)} \det X^{rj}(\lambda_k) \tag{2.15}$$

*is an eigenfunction of $S$ with eigenvalue $\lambda_k^n$. Further,*

$$\hat{q}_j(\lambda_k; q_0) = \frac{1}{C_j}\langle q_0, \psi_k^j \rangle; \qquad j = 1, \ldots, n; \quad k \in \mathbb{N} \tag{2.16}$$

$$\hat{q}_j(\lambda_k; q_0) =$$

- $\lambda_k \to \infty$ as $k \to \infty$.
- $|\lambda_k| < |\lambda_{k+1}|$
- $(\lambda_k)$ *is bounded away from the set of zeros of* $\Delta_{\mathrm{PDE}}$*, uniformly in* $k$*:*

$$\exists \epsilon > 0 : \forall k, j \in \mathbb{N}, \quad ||\lambda_k| - |\sigma_j|| > \epsilon$$

*Then*

$$\|Q_k\| = O\left(\sup_{\rho(\lambda_k)} \left[\frac{\eta_j(\lambda_k; \rho_k^j)}{\Delta_{\mathrm{PDE}}(\lambda_k)} - \frac{\Delta_{\mathrm{PDE}}(\lambda_k)}{\eta_j(\lambda_k; \rho_k^j)}\right]\right), \quad \text{as } k \to \infty.$$

## 2.3 Sketch of proofs

*Proof of Theorem 2.6.* As $\Pi$ is well-posed and $\Pi'$ is well-conditioned, by Smith (2012, 2013a) the solution $q$ of the problem $\Pi$ can be expressed using a series as

$$q(x,t) = i \sum_{k \in K^+} \operatorname*{Res}_{\lambda=\lambda_k} \frac{e^{i\lambda x - a\lambda^n t}}{\Delta_{\mathrm{PDE}}(\lambda)} \zeta^+(\lambda) + i \sum_{k \in K^-} \operatorname*{Res}_{\lambda=\lambda_k} \frac{e^{i\lambda(x-1) - a\lambda^n t}}{\Delta_{\mathrm{PDE}}(\lambda)} \zeta^-(\lambda).$$

As each $\lambda_k$ is a simple zero of $\Delta_{\mathrm{PDE}}$, the series is separable into $x$-dependent and $t$-dependent parts

$$\phi_k(x) = \begin{cases} \frac{i}{2} e^{i\lambda_k x} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\zeta^+(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} & \text{if } k \in K^+, \\ \frac{i}{2} e^{i\lambda_k (x-1)} \operatorname*{Res}_{\lambda=\lambda_k} \frac{\zeta^-(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} & \text{if } k \in K^-, \end{cases} \tag{2.24}$$

$$\psi_k(t) = e^{-a\lambda_k^n t}, \tag{2.25}$$

so that

$$q(x,t) = \sum_{k \in \mathbb{N}} \phi_k(x)\psi_k(t). \tag{2.26}$$

Further, Smith (2012, Lemma 6.1) guarantees the existence of a nonzero complex constant $C$ such that $\lambda_k = Ck + O(1)$ as $k \to \infty$, which, by Sedletskii (2005, Theorems 3.3.3 & 4.1.1), guarantees that $(\psi_k)_{k\in\mathbb{N}}$ is a minimal system in $L^2[0,T]$.

As $q$ is the solution of $\Pi$, $q$ satisfies

$$A \begin{pmatrix} \partial_x^{n-1} q(0,t) \\ \partial_x^{n-1} q(1,t) \\ \vdots \\ q(0,t) \\ q(1,t) \end{pmatrix} = 0, \quad \forall t \in [0,T].$$

The minimality of the $t$-dependent system means that this implies each $\phi_k$ satisfies the boundary conditions of $S$, so $\phi_k \in D(S)$.

As $q$ satisfies the PDE,

$$0 = a \sum_{k \in \mathbb{N}} [\lambda_k^n I + S](\phi_k)(x)\psi_k(t)$$

so, by minimality of $(\psi_k)_{k\in\mathbb{N}}$, each $\phi_k$ is an eigenfunction of $S$ with eigenvalue $\lambda_k^n$.

Evaluating equation (2.26) at $t = 0$ yields an expansion of $q_0$ in the system $(\phi_k)_{k\in\mathbb{N}}$. □

**Remark 2.13.** We have to require the zeros of $\Delta_{\mathrm{PDE}}$ are all simple. It would be desirable to be able to say that the zeros of $\Delta$ and $\Delta_{\mathrm{PDE}}$ are all the same and of the same order. It has been shown that this holds under certain symmetry restrictions on the boundary conditions (Smith, *Proof of Proposition 2.7.* As $(\ _k)_{k \in \mathbb{N}}$ is a basis, the Fourier expansion

$$q_0(x) = \sum_{k \in \mathbb{N}} \ _k(x) \langle q_0, \ _k \rangle$$

$\sum_{k \in q}^{2}$

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2. For all $m$, multiply the $m^{\text{th}}$ column by $\;_m$.
3. Apply the permutation $r \mapsto \hat{r}$ to the row index.
4. Take the complex conjugate of each entry.

we nd that the characteristic determinant (1.5) is given by

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De ne $_{k}(x)$ =

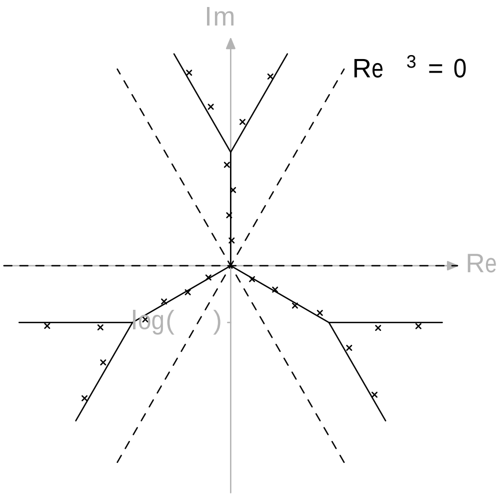


Figure 1: The asymptotic position of $_k$ for $2$ ( $1;0$).

The case $\neq 0$

It is already well known (Fokas and Pelloni, 2005; Smith, 2011) that in this case we have the following result.

**Theorem 3.2.** *The initial-boundary value problem associated with* ($S$ ; $i$) *is well-posed and its solution admits a series representation.*

The case $= 0$

**Theorem 3.3.** *The initial-boundary value problem associated with* ($S^0; i$) *is well-posed but the problem* ($S^0;$ $i$) *is ill-conditioned.*

*Proof.* The proof of the well-posedness claim in this statement can be found in Smith (2011). However, for this example we now show that the statement ` ( )= $_{PDE}$( ) $!$ 0 as $!$ $1$ from within the sets enclosed by ' does not hold, implying that ($S^0;$ $i$) is ill-conditioned.

The reduced global relation matrix in this case is given by

$$A( ) = \begin{pmatrix} c_2( ) & c_2( )e^{\ i} & c_1( )e^{\ i} \\ c_2( ) & c_2( )e^{\ i!} & c_1( )!e^{\ i!} \\ c_2( ) & c_2( )e^{\ i!^2} & c_1( )!^2e^{\ i!^2} \end{pmatrix};$$

hence its determinant $_{PDE}( ) = {}^0( )$ given by (3.4), and the functions

$$_1( ) = i\ (!^2\ \ !) \sum_{r=0} !^r \hat{q}_0(!^r\ )e^{i!^r}\ ;$$

$$_2( ) = i \sum_{r=0} \hat{q}_0(!^r\ )\ \ !^{r+1}e^{\ i!^{r+1}}\ \ !^{r+2}e^{\ i!^{r+2}}\ ;$$

$$_3( ) = i \sum_{r=0} \hat{q}_0(!^r\ )\ \ e^{\ i!^{r+2}}\ \ e^{\ i!^{r+1}}\ ;$$

$$_4( ) = {}_5( ) = {}_6( ) = 0:$$

As $+2$

We consider the particular ratio

$$\frac{\zeta_3(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}, \qquad \lambda \in \hat{E}_2. \tag{3.13}$$

For $\lambda \in \hat{E}_2$, $\operatorname{Re}(i\omega^r\lambda) < 0$ if and only if $r = 2$ so we approximate ratio (3.13) by its dominant terms as $\lambda \to \infty$ from within $\hat{E}_2$,

$$\frac{(\hat{q}_0(\lambda) - \hat{q}_0(\omega\lambda))e^{-i\omega^2\lambda} + \hat{q}_0(\omega^2\lambda)(e^{-i\omega\lambda} - e^{-i\lambda}) + o(1)}{(\omega^2 - \omega)e^{i\lambda} + (1 - \omega^2)e^{i\omega\lambda} + o(1)}.$$

We expand the integrals from $\hat{q}_0$ in the numerator and multiply the numerator and denominator by $e^{-i\omega\lambda}$ to obtain

$$\frac{i\int_0^1 \left(e^{i\lambda(1-x)} - e^{i\lambda(1-\omega x)} - e^{i\lambda\omega^2(1-x)} + e^{-i\lambda(2\omega-\omega^2 x)}\right) q_0(x)\,\mathrm{d}x + o\left(e^{\operatorname{Im}(\omega\lambda)}\right)}{\sqrt{3}(e^{i\lambda(1-\omega)} + \omega) + o\left(e^{\operatorname{Im}(\omega\lambda)}\right)}. \tag{3.14}$$

Let $(R_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that $\lambda_j = R_j e^{i\frac{7\pi}{6}} \in \hat{E}_2$, $R_j$ is bounded (uniformly in $j$ and $k$) away from $\{\frac{2\pi}{\sqrt{3}}(k+\frac{1}{6}) : k \in \mathbb{N}\}$ and $R_j \to \infty$ as $j \to \infty$. Then $\lambda_j \to \infty$ from within $\hat{E}_2$. We evaluate ratio (3.14) at $\lambda = \lambda_j$,

$$\frac{i\int_0^1 \left(2ie^{\frac{R_j}{2}(1-x) - \frac{\sqrt{3}R_j}{2}i}\sin\left(\frac{\sqrt{3}R_j x}{2}\right) - e^{-R_j(1-x)}\left(1 - e^{-\sqrt{3}R_j i}\right)\right) q_0(x)\,\mathrm{d}x + o\left(e^{-\frac{R_j}{2}}\right)}{\sqrt{3}(e^{-\sqrt{3}R_j i} + \omega) + o\left(e^{-\frac{R_j}{2}}\right)}. \tag{3.15}$$

The denominator of ratio (3.15) is bounded away from 0 by the definition of $R_j$ and the numerator tends to $\infty$ for any nonzero initial datum. □

**Remark 3.4.** In the proof of Theorem 3.3 we use the example of the ratio $\frac{\zeta_3(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$ being unbounded as $\lambda \to \infty$ from within $\hat{E}_2$. It may be shown using the same argument that $\frac{\zeta_2(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$ is unbounded in the same region and that both these ratios are unbounded for $\lambda \in \hat{E}_3$ using $\lambda_j = R_j e^{i\frac{11\pi}{6}}$ for appropriate choice of $(R_j)_{j\in\mathbb{N}}$. However the ratio

$$\frac{\zeta_1(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)} = \frac{\zeta^+(\lambda)}{\Delta_{\mathrm{PDE}}(\lambda)}$$

is bounded in $\hat{E}_1 = \hat{E}^+$ hence it is possible to deform the contours of integration in the upper half-plane. This permits a partial series representation of the solution to the initial-boundary value problem.

**Remark 3.5.** For all $\beta \in (-1, 1)$ the final time boundary value problem is ill-posed. The asymptotic location of the zeros of $\Delta_{\mathrm{PDE}}$, along rays wholly contained within $\{\lambda \in \mathbb{C} : \operatorname{Re}(-i\lambda^3) < 0\}$ means that for nozero initial data the solution exhibits instantaneous blow-up. Nevertheless, for all $\beta \in (-1, 0) \cup (0, 1)$ the final time problem is well-conditioned. In the case $\beta = 0$, the final-time problem becomes ill-conditioned and $S$ becomes degenerate irregular under Locker's classification.

When $\beta = \pm 1$, $S$ is self-adjoint and the initial- and final-boundary value problems are both well-posed. For $|\beta| > 1$, the final-boundary value problem remains well-posed but the initial-boundary value problem becomes ill-posed. Thus the self-adjoint cases represent the transitions between well-posedness of the initial- and final-boundary value problems. Analogous to the $\beta = 0$ case, in the limit $\beta = \infty$, the initial-boundary value problem becomes ill-conditioned, the solution to the final-boundary value problem may not be represented as a series and $S$ becomes degenerate irregular.

## 3.3 Comparison

The explicit computation of the operator norms in section 3.2 requires the evaluation of the biorthogonal family of eigenfunctions and the precise asymptotics for the corresponding eigenvalues.

On the other hand, the integral representation of the solution of the boundary value problem can be constructed algorithmically from the given data, without the need for any precise asymptotic information about the eigenvalues, except their asymptotic location (always along a ray for odd-order problems; see Smith, 2012, Theorem 6.3). This is sufficient for a direct analysis of the terms that blow up and prevent deformation of the contour of integration and a residue computation around the eigenvalues, thereby precluding a series representation of the solution.

In the example above, the particular term in the integral representation exhibiting this blow-up is the term

$$\int_0^1 2i \sin \left( \frac{\sqrt{3} R_j x}{2} \right) e^{\frac{R_j}{2}(1 - x)} \quad \sqrt{3} R_j$$

## The associated differential operator

For the real parameter $\in (2 \; ; 2]$, we investigate the differential operator $S$ with pseudoperiodic boundary coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix};$$

and the associated initial- and final-boundary value problems and $'$.

**Remark 4.1.** The restriction from $\in \mathbb{R} \setminus \{ 1; 0; 1/2$

Let

$$_k(x) = \frac{_k(x)}{\langle \ _k, \ _k\rangle}. \tag{4.6}$$

Then there exists a minimal $Y \in \mathbb{N}$ such that $((\ _k)_{k=Y}^{\infty}, (\ _k)_{k=Y}^{\infty})$ is a biorthogonal sequence in $AC^n[0,1]$. Moreover

$$\langle \ _k, \ _k\rangle = \frac{\sqrt{3}}{\ k \ \frac{1}{2}} e^{\frac{4}{\sqrt{3}}(k \ \frac{1}{2})} + O(e^{\sqrt{3}\ k} k^{\ 1}) \text{ as } k \to \infty. \tag{4.7}$$

The eigenfunctions have the same norm and it grows *at the same rate* as their inner product.

$$\| \ _k\|^2 = \| \ _k\|^2 = \frac{3\sqrt{3}}{2 \ k \ \frac{1}{2}} e^{\frac{4}{\sqrt{3}}(k \ \frac{1}{2})} + O(e^{\sqrt{3}\ k} k^{\ 1}{}_{k}^{\ 1} + \ \| \ \|$$

Also

$$
\begin{aligned}
\Delta_2^{(2)}(\lambda_k) &= -i\omega^2\sqrt{3}R_k^2\left[q_T(\lambda_k)e^{-i\omega\lambda_k} - \omega^2 q_T(\omega\lambda_k)e^{-i\lambda_k} + \omega q_T(\omega^2\lambda_k)e^{i\omega^2\lambda_k}\right] \\
&\qquad + O(e^{R_k/2}R_k^2) \\
&= -\omega^2 i\sqrt{3}R_k^2\left[\int_0^1 q_T(x)e^{-\frac{R_k}{2}[x+1+i\sqrt{3}(x-1)]}\,\mathrm{d}x \right.\\
&\qquad \left. - \omega^2\int_0^1 q_T(x)e^{-\frac{R_k}{2}[x+1-i\sqrt{3}(x-1)]}\,\mathrm{d}x + \omega\int_0^1 q_T(x)e^{R_k[1-x]}\,\mathrm{d}x\right] \\
&\qquad + O(e^{R_k/2}R_k^2) \\
&= -\omega^2 i\sqrt{3}R_k^2\left[-\frac{2q_T(1)e^{R_k}}{R_k(1+i\sqrt{3})} - \omega^2\frac{2q_T(1)e^{R_k}}{R_k(1-i\sqrt{3})} + \omega\frac{q_T(0)e^{R_k}}{R_k} \right.\\
&\qquad \left. + O(e^{R_k}R_k^{-2})\right] + O(e^{R_k/2}R_k^2) \\
&= -i\sqrt{3}R_k\left(2q_T(1) - q_T(0)\right)e^{R_k} + O(e^{R_k}).
\end{aligned}
$$

Note that $q_T(1) = q(1;T) = -q(0;T) = -q_T(0)$, by the first boundary condition. Hence, provided we can be sure that $q_T(0) \neq 0$, $2q_T(1) - q_T(0) \neq 0$. □

As $0 < \arg(\lambda_k) < \pi/3$, and $R_k$ was chosen to ensure that $\Delta_{\mathrm{PDE}}(\lambda_k)$ is bounded away from $0$, $\lambda_k \in D_1$. Hence, by Smith (2012, Theorem 1.1), $\Pi$ is ill-posed.

The rate of blowup exhibited in Proposition 4.2 is maximal in the sense that for any sequence $(\mu_k)_{k\in\mathbb{N}}$ such that $|\lambda_{k-1}| < |\mu_k| < |\lambda_k|$ and for any $j \in \{1,2,3\}$,

$$\frac{\Delta_j^{(2)}(\mu_k)}{\Delta_{\mathrm{PDE}}^{(2)}(\mu_k)} = O(e^{R_k/2}R_k^{-2}).$$

The problem $\Pi^0$ is well-conditioned for all $\beta \in (2-\ ,2]$. Indeed,

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