School of Mathematical
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A High Frequency Boundary Element Method for Scattering by a Class of Nonconvex Obstacles

by

S. N. Chandler-Wilde, D. P. Hewett, S. Langdon

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A high frequency boundary element method for scattering by a class of nonconvex obstacles

S. N. Chandler-Wilde · D. P. Hewett · S. Langdon · A. Twigger

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Abstract In this paper we propose and analyse a hybrid numerical-asymptotic boundary element method for the solution of problems of high frequency acoustic scattering by a class of sound-soft nonconvex polygons. The approximation space is enriched with carefully chosen oscillatory basis functions; these are selected via a study of the high frequency asymptotic behaviour of the solution. We demonstrate via a rigorous error analysis, supported by numerical examples, that to achieve any desired accuracy it is sufficient for the number of degrees of freedom to grow only in proportion to the logarithm of the frequency as the frequency increases, in contrast to the at least linear growth required by conventional methods. This appears to be the first such numerical analysis result for any problem of scattering by a nonconvex obstacle.

Keywords High frequency scattering · Boundary Element Method · Helmholtz equation

1 Introduction

There has been considerable interest in recent years in the development of numerical methods for time harmonic acoustic and electromagnetic scattering problems that are able to efficiently resolve the scattered field at high frequencies. Standard finite or boundary element methods, with piecewise polynomial approximation spaces, suffer from the restriction that a fixed number of degrees of freedom is required per wavelength in order to represent the oscillatory solution, leading to excessive computational cost when the scatterer is large compared to the wavelength.

Department of Mathematics and Statistics, University of Reading, Whiteknights PO Box 220, Reading RG6 6AX, UK.
Tel: +44-118-3785012
Fax:+44-118-9313423
E-mail: d.p.hewett@reading.ac.uk.
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A methodology that has shown a great deal of promise is the so-called "hybrid numerical-asymptotic" approach, where partial knowledge of the high frequency asymptotic behaviour is incorporated into the approximation space. This approach is particularly attractive when employed within a boundary element method (BEM) framework, since knowledge of the high frequency asymptotics is required only on the boundary of the scatterer. Whereas conventional BEMs for two-dimensional (2D) problems require the number of degrees of freedom to grow at least linearly with respect to frequency in order to maintain a prescribed level of accuracy as the frequency increases, hybrid numerical-asymptotic BEMs have been shown, for a range of problems, to require a significantly milder (often only logarithmic) growth in computational cost. We refer to [14] (and the very

Of course, the same issues must be overcome in fully asymptotic methods. Although the original version of the Geometrical Theory of Diffraction (GTD) [29] was deficient in the sense that it did not include the shadow boundary behaviour, more sophisticated, uniform versions of GTD have been developed which capture this [30, 7]. However, we emphasize that, in the numerical-asymptotic approach developed here, we do not require the computation of a full asymptotic solution in order to design our hybrid approximation space. Rather, we need merely a representation of the form (1), with an explicit (and relatively simple) term V_0 and explicit phases ψ_m, that captures the high frequency oscillations present in the solution. To design hybrid algorithms optimally, and prove their effectiveness by rigorous numerical analysis, we need additionally to understand the regularity of the amplitudes V_m, $m = 1, \ldots, M$, moreover obtaining bounds on these amplitudes that are explicit in their dependence on the wavenumber. This requires high frequency asymptotics of a new kind which aims at coarser information than the full asymptotic solution. The results of this kind that we require in this paper are proved in §3 and §4 below.

As alluded to above, most high frequency algorithms developed to date have been for convex scatterers. The case of multiple smooth convex scatterers, which shares many of the difficulties associated with single nonconvex scatterers, has been considered in [27, 24, 25, 3]. The key theme of that body of work is a decomposition of the multiple scattering problem into a series of problems of scattering by single convex obstacles, with in each case the incident field consisting of a combination of the original incident field with previously scattered waves. This approach does not generalise easily to single nonconvex scatterers, since the number of terms required in the series increases rapidly as the distance between the separated convex scatterers decreases. It may however provide some insight into how to extendhg-353(or6-ae1(es)-1(es)-291(rapie)-1(r38e8(v7 frequency increases. Moreover, it was shown that, when the frequency is fixed, the convergence rate is exponential as a function of the number of degrees of freedom.

The starting point of the boundary integral equation (BIE) formulation is that, if u satisfies the BVP then a form of Green's representation theorem holds, namely (see [16] and [14, (2.107)])

$$u(\mathrm{x}) = u^i(\mathrm{x}) - \int_\Gamma \Phi_k(\mathrm{x}, \mathrm{y}) \frac{\partial u}{\partial \mathrm{n}}(\mathrm{y})\, \mathrm{d}s(\mathrm{y}), \qquad \mathrm{x} \quad D, \tag{5}$$

where $\Phi_k(\mathrm{x}, \mathrm{y}) := (\mathrm{i}/4) H_0^{(1)}(k|\mathrm{x} - \mathrm{y}|)$ is the fundamental solution for (3), $H_\nu^{(1)}$ the Hankel function of the first kind of order ν, and $\partial u/\partial \mathrm{n}$ is the normal derivative, with n the unit normal directed into D. We note that, as discussed in [16] and [14, Theorem 2.12], it holds that $\partial u/\partial \mathrm{n} \quad L^2(\Gamma)$. It is well known (see, e.g., [14, §2]) that, starting from the representation formula (5), we can derive various BIEs for $\partial u/\partial \mathrm{n} \quad L^2(\Gamma)$, each taking the form

$$A \frac{\partial u}{\partial \mathrm{n}} = f, \tag{6}$$

where $f \quad L^2(\Gamma)$ and $A : L^2(\Gamma) \to L^2(\Gamma)$ is a bounded linear operator.

In the standard combined potential formulation (see [14, (2.114) and (2.69)]),

$$A = A_{k,\eta} := \frac{1}{2} I + D_k - \mathrm{i}\eta S_k, \tag{7}$$

and $f = \partial u^i/\partial \mathrm{n} - \mathrm{i}\eta u^i$, where $\eta \quad \mathbb{R}$ is a coupling parameter, I is the identity operator, and the single-layer potential operator S_k and the adjoint double-layer potential operator D_k are defined by

$$S_k\psi(\mathrm{x}) := \int_\Gamma \Phi_k(\mathrm{x}, \mathrm{y})\psi(\mathrm{y})\, \mathrm{d}s(\mathrm{y}), \qquad \mathrm{x} \quad \Gamma, \quad \psi \quad L^2(\Gamma),$$

$$D_k\psi(\mathrm{x}) := \int_\Gamma \frac{\partial \Phi_k(\mathrm{x}, \mathrm{y})}{\partial \mathrm{n}(\mathrm{x})} \qquad \text{is) the idenx} \int_{\Gamma, x}$$

any Galerkin method of approximation of (6) is invertible, and, via Céa's lemma, implies explicit error estimates for the Galerkin method solution, as discussed below.

For both formulations the following lemma holds provided Ω is Lipschitz and provided $|\eta| \leq Ck$ in the standard formulation (we shall assume henceforth that this condition always holds). Here and for the remainder of this paper $C > 0$ denotes a constant whose value may change from one occurence to the next, but which is always independent of k, although it may (possibly) be dependent on Ω.

Lemma 21 *[13, Theorem 3.6], [36, Theorem 4.2] Assume that Ω is a bounded Lipschitz domain and $k_0 > 0$. In the case $A = A_{k,\eta}$ assume additionally that $|\eta| \leq Ck$. Then for both $A = A_k$ and $A = A_{k,\eta}$ there exists a constant $C_0 > 0$, independent of k, such that*

$$A_{L^2(\Gamma)} \leq C_0 k^{1/2}, \qquad k \geq k_0.$$

Lemma 21 suggests at worst mild growth in $A_{L^2(\Gamma)}$ for both formulations as k increases. For the case $A = A_{k,\eta}$, with η proportional to k, it is shown in [13,5] that $A_{L^2(\Gamma)}$ does grow proportionally to $k^{1/2}$ for a polygonal scatterer, i.e. for this case at least it is known that the bound is sharp.

As alluded to above, in certain cases A also satisfies the following assumptions:

Assumption 22

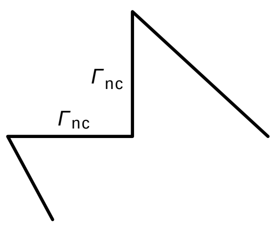
3 Regularity of Solutions

Our goal is to derive a numerical method for the solution of the BIE (6) (and hence of the scattering problem), whose performance does not deteriorate significantly as the wavenumber k (which is proportional to frequency) increases, equivalently as the wavelength $\lambda := 2\pi/k$ decreases. Specifically, we wish to avoid the requirement of conventional schemes for a fixed number of degrees of freedom per wavelength. To achieve this goal, our numerical method for solving (6) uses an approximation space (defined explicitly in §5) which is adapted to the high frequency asymptotic behaviour of the solution $\partial u/\partial \mathrm{n}$ on each of the sides of the polygon. For the case of a sound-soft convex polygon, this behaviour was determined in [28, 16]. A key contribution of this paper is to introduce new methods of argument which enable us to deduce precisely and rigorously this behaviour for a range of cases when the polygon is not convex.

At present our full analysis applies only to a particular class of polygons, defined below. This class includes all convex polygons, but also a large set of nonconvex star-like and non-star-like polygons.

Definition 31 *Let C denote the class of all polygons Ω $\mathbb{R}^2$ for which the following two conditions are satisfied:*

1. *Each external angle is either greater than π or equal to $\pi/2$.*
2. *For each external angle equal to $\pi/2$, if Ω is rotated into the configuration in Figure 2(a), then Ω is contained entirely in the region bounded by the sides Γ_{nc} and Γ_{nc} and the two dotted lines.*



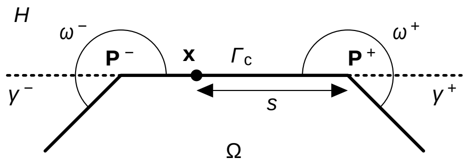


Fig. 3 Geometry of a typical convex side Γ_c.

Theorem 32 *On a convex side Γ_c the representation*

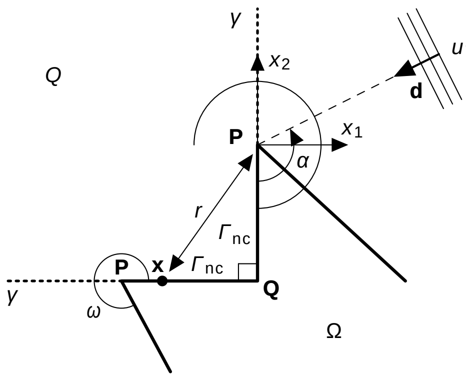
$$\frac{\partial u}{\partial \mathrm{n}}(\mathrm{x}(s)) = \Psi(\mathrm{x}(s)) + v^+(s)e^{iks} + v^-(L_c - s)e^{-iks}, \quad s \quad [0, L_c], \tag{9}$$

holds, where

(i) $\Psi := 2\partial u^i/\partial \mathrm{n}$ *if* Γ_c *is illuminated and* $\Psi := 0$ *otherwise;*
(ii) the functions $v^\pm(s)$ *are analytic in the right half-plane* $\mathrm{Re}[s] > 0$*; further, for every* $k_0 > 0$ *we have*

$$|v^\pm(s)| \le \begin{cases} CM(u)k|ks|^{-\delta^\pm}, & 0 < |s| \le 1/k, \\ CM(u)k|ks|^{-1/2}, & |s| > 1/k, \end{cases} \quad \mathrm{Re}[s] > 0, \tag{10}$$

for $k \ge k_0$*, where* $\delta^\pm := 1 - \pi/\omega^\pm$ (0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(iii) the function $v(s)$ *is analytic in the* k*-independent complex neighbourhood* $D_\varepsilon := \{s \in \mathbb{C} : \mathrm{dist}(s, [0, L_{nc}]) < \varepsilon\}$ *of* $[0, L_{nc}]$*, where*

$$\varepsilon := L_{nc}/(32\sqrt{2}); \qquad (13)$$

further

$$|v(s)| \leq CC_1 k^{1+\beta_1} \log^{1/2}(2 + D$$

Dirichlet Green's function for this domain is known explicitly (see (15)) by the method of images. (This simple representation for the Green's function simplifies the calculations throughout this section; it is this which motivates the requirement in Definition 31 that the exterior angles less than π are exactly $\pi/2$.) This gives $\partial u/\partial \mathbf{n}$ on Γ_{nc}

As a consequence of Lemma 41(iii) we have that

$$\frac{\partial u}{\partial \mathrm{n}}(\mathrm{x}) = \frac{\partial \Psi_1}{\partial \mathrm{n}}(\mathrm{x}) + \int_{\gamma \cup \gamma'} \frac{\partial^2 G_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} u(\mathrm{y})\,\mathrm{d}s(\mathrm{y}), \qquad \mathrm{x} \in \Gamma_{\mathrm{nc}}. \tag{21}$$

Theorem 36 follows from a careful analysis of the integral in (21). The terms $v^+(L_{\mathrm{nc}}+s)\mathrm{e}^{\mathrm{i}ks}$ and $v^-(L_{\mathrm{nc}}-s)\mathrm{e}^{-\mathrm{i}ks}$ in the representation (12) arise from the integral over γ. Indeed, noting that

$$\frac{\partial^2 G_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} = 2\frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} - 2\frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y}^*)}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})}, \quad \mathrm{x} \in \Gamma_{\mathrm{nc}},\ \mathrm{y} \in \gamma,$$

where $\mathrm{y}^* := (-y_1, y_2)$, we find that, for $\mathrm{x} \in \Gamma_{\mathrm{nc}}$,

$$\int_\gamma \frac{\partial^2 G_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} u(\mathrm{y})\,\mathrm{d}s(\mathrm{y}) = 2\int_\gamma \frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} u(\mathrm{y})\,\mathrm{d}s(\mathrm{y}) - 2\int_{\tilde\gamma} \frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} u(\mathrm{y}^*)\,\mathrm{d}s(\mathrm{y}), \tag{22}$$

with $\tilde\gamma := \{(x_1, -L_{\mathrm{nc}}) : x_1 > L_{\mathrm{nc}}\}$. This expression is very similar to that encountered in the derivation of the regularity results on a convex side. Indeed, arguing almost exactly as in [28, §3] (and see also [16, §3]), it can be shown from (22) that

$$\int_\gamma \frac{\partial^2 G_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} u(\mathrm{y})\,\mathrm{d}s(\mathrm{y}) = v^-(L_{\mathrm{nc}}-s)\mathrm{e}^{-\mathrm{i}ks} + v^+(L_{\mathrm{nc}}+s)\mathrm{e}^{\mathrm{i}ks}, \quad \mathrm{x}(s) \in \Gamma_{\mathrm{nc}},$$

where $v^\pm(s)$ are analytic in $\mathrm{Re}[s] > 0$, where they satisfy the bounds (10) with $\delta^\pm = 1 - \pi/\omega$. This is the assertion in paragraph (ii) of Theorem 36.

We now consider the integral over γ' in (21). Noting that

$$\frac{\partial^2 G_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} = -4\frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y})}{\partial x_2 \partial y_1}, \qquad \mathrm{x} \in \Gamma_{\mathrm{nc}},\ \mathrm{y} \in \gamma',$$

and using the decomposition $u = u^i + u^s$, we have, for $\mathrm{x} \in \Gamma_{\mathrm{nc}}$, that

$$\int_{\gamma'} \frac{\partial^2 G_k(\mathrm{x},\mathrm{y})}{\partial \mathrm{n}(\mathrm{x})\partial \mathrm{n}(\mathrm{y})} u(\mathrm{y})\,\mathrm{d}s(\mathrm{y}) = -4\int_{\gamma'} \frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y})}{\partial x_2 \partial y_1} u^i(\mathrm{y})\,\mathrm{d}s(\mathrm{y}) - 4\int_{\gamma'} \frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y})}{\partial x_2 \partial y_1} u^s(\mathrm{y})\,\mathrm{d}s(\mathrm{y}). \tag{23}$$

The assertions in paragraphs (i) and (iii) of Theorem 36 then follow from (21), (23) and Lemmas 42 and 43 below.

Lemma 42 *For* $\mathrm{x} = (-s, -L_{\mathrm{nc}}) \in \Gamma_{\mathrm{nc}}$,

$$-4\int_{\gamma'} \frac{\partial^2 \Phi_k(\mathrm{x},\mathrm{y})}{\partial x_2 \partial y_1} u^i(\mathrm{y})\,\mathrm{d}s(\mathrm{y}) = \Psi(\mathrm{x}) - \frac{\partial \Psi_1}{\partial \mathrm{n}}(\mathrm{x}) + \mathrm{e}^{\mathrm{i}kr} W^i(s),$$

where $W^i(s)$ is analytic in D_ε, with ε given by (13)*; further, for every $k_0 > 0$,*

Lemma 43 *If Assumption 22 holds, then, for* $\mathbf{x} = (-s, -L_{\mathrm{nc}}) \in \Gamma_{\mathrm{nc}}$,

$$-4\int_{\gamma'} \frac{\partial^2 \Phi_k(\mathbf{x}, \mathbf{y})}{\partial x_2 \partial y_1} u^s(\mathbf{y})\, \mathrm{d}s(\mathbf{y}) = \mathrm{e}^{\mathrm{i}kr} W^s(s),$$

where $W^s(s)$ *is analytic in* D_ε, *with* ε *given by* (13)*; further, where* β_1, k_1 *and* C_1 *are the constants in Assumption 22,*

$$|W^s(s)| \le C C_1 k^{1+\beta_1} \log^{1/2}(2 + k), \qquad s \in D_\varepsilon,\ k \ge k_1, \tag{24}$$

where $C > 0$ *depends only on* Ω *and* k_1.

We begin by proving Lemma 43. Some of the results needed for this proof will be used again in the proof of Lemma 42.

4.1 Proof of Lemma 43

For $\mathbf{x} = (-s, -L_{\mathrm{nc}}) \in \Gamma_{\mathrm{nc}}$ we have $r = r(s) = \sqrt{s^2 + L_{\mathrm{nc}}^2}$. Thus, to prove the lemma we have to show that

$$W^s(s) := -4 \exp\left(-\mathrm{i}k\sqrt{s^2 + L_{\mathrm{nc}}^2}\right) \int_{\gamma'} \frac{\partial^2 \Phi_k(\mathbf{x}, \mathbf{y})}{\partial x_2 \partial y_1} u^s(\mathbf{y})\, \mathrm{d}s(\mathbf{y})$$

is analytic in D_ε, satisfying the bound (24). Substituting for u^s using (5), and switching the order of integration, justified by Fubini's theorem, gives

$$W^s(s) = \int_\Gamma K(s, \mathbf{z}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{z})\, \mathrm{d}s(\mathbf{z}), \tag{25}$$

where, for $s \in \mathbb{R}$ and $\mathbf{z} \in \Gamma$,

$$K(s, \mathbf{z}) := 4 \exp\left(-\mathrm{i}k\sqrt{s^2 + L_{\mathrm{nc}}^2}\right) \int_0^\infty \frac{\partial^2 \Phi_k\big((-s, -L_{\mathrm{nc}}), (0, y_2)\big)}{\partial x_2 \partial y_1} \Phi_k((0, y_2), \mathbf{z})\, \mathrm{d}y_2, \tag{26}$$

and, by the recurrence and differentiation formulae for Hankel functions [1, §10.6],

$$\Phi_k((0, y_2), \mathbf{z}) = \frac{\mathrm{i}}{4} H_0^{(1)}\left(k\sqrt{z_1^2 + (y_2 - z_2)^2}\right),$$

$$\frac{\partial^2 \Phi_k((-s, -L_{\mathrm{nc}}), (0, y_2))}{\partial x_2 \partial y_1} = -\frac{\mathrm{i}k^2 s(L_{\mathrm{nc}} + y_2)}{4\,(s^2 + (L_{\mathrm{nc}} + y_2)^2)} H_2^{(1)}\left(k\sqrt{s^2 + (L_{\mathrm{nc}} + y_2)^2}\right).$$

We recall that $H_n^{(1)}(z)$ is analytic in $|z| > 0$, $|\arg(z)| < \pi$. To derive bounds on $K(s, \mathbf{z})$ we need bounds on $H_n(z) := \mathrm{e}^{-\mathrm{i}z} H_n^{(1)}(z)$. From [1, §10.2(ii), §10.17.5] it follows that, for some constant $C > 0$,

$$|H_0(z)| \le C|z|^{-1/2}, \quad |z| > 0,\ |\arg(z)| \le \pi/2 \tag{27}$$

and that, for every $c > 0$ there exists $C > 0$ such that

$$|H_2(z)| \le C|z|^{-1/2}, \quad |z| > c,\ |\arg(z)| \le \pi/2. \tag{28}$$

Note that, for s R and $\mathbf{z}$ Γ, [redacted]

$$K(s, \mathbf{z}) = \int_0^\infty e^{\mathrm{i}k\varphi(s,y_2,\mathbf{z})} S_k(s, y_2, \mathbf{z})\,\mathrm{d}y_2,$$ [redacted]

where $\varphi(s, y_2, \mathbf{z}) := \chi(s, L_{\mathrm{nc}}, y_2) - \chi(s, L_{\mathrm{nc}}, 0) + \chi(z_1, -z_2, y_2)$ and [redacted]

$S_k(s, y_2$ nc ([redacted]

it follows that $\mu_1 \geq 0$, and hence (30) holds.

For (ii), note that, when $c = te^{i\pi/4}$ with $t \geq 0$, we have $\xi = a^2 + b^2 + \sqrt{2}bt \geq a^2 + b^2$ and $\eta = t(\sqrt{2}b + t) \geq t^2$, and (32) follows from (34). Also, by (35),

$$2(a^2 + b^2)\,\mathrm{Im}\left[\chi(a, b, te^{i\pi/4})\right]^2 = b^2t^2 + \mu_2,$$

where $\mu_2 := -(b^2t^2 + (a^2+b^2)\xi) + (a^2+b^2)\sqrt{\xi^2 + \eta^2}$. A little algebraic manipulation reveals that

$$(a^2+b^2)^2(\xi^2+\eta^2) - (b^2t^2 + (a^2+b^2)\xi)^2 = t^3a^2\left(2\sqrt{2}b(a^2+b^2) + (a^2+2b^2)t\right) \geq 0.$$

Hence $\mu_2 \geq 0$ and (33) follows.

In order to prove Lemma 43 we must consider the analytic continuation of $K(s, \mathbf{z})$ into the complex s-plane. But before complexifying s it is helpful to modify the representation (29) by deforming the contour of integration off the real line. From (29) it follows from Cauchy's theorem that, for $s \in \mathbb{R}$ and $\mathbf{z} \in \Gamma$, where $f(w) := e^{ik\varphi(s,w,\mathbf{z})}S_k(s, w, \mathbf{z})$,

$$K(s, \mathbf{z}) = \int_{\gamma^*} f(w)\,dw = e^{i\pi/4}\int_0^\infty e^{ik\varphi(s,te^{i\pi/4},\mathbf{z})}S_k(s, te^{i\pi/4}, \mathbf{z})\,dt, \qquad (36)$$

where $\gamma^* = \{w = te^{i\pi/4} : t \geq 0\}$. This application of Cauchy's theorem is valid since, by Lemma 44(i), $f(w)$ is analytic in $\mathrm{Re}[w] > 0$; further, $\mathrm{Im}[\varphi(s, w, \mathbf{z})] \geq 0$, so that $|e^{ik\varphi(s,w,\mathbf{z})}| \leq 1$, if $\mathrm{Re}[w] > 0$ and $\mathrm{Im}[w] \geq 0$; moreover, the bounds (30), (27), and (28) imply that

$$S_k(s, w, \mathbf{z}) = O\left(|w|^{-1/2}\right), \text{ as } |w| \to 0, \quad S_k(s, w, \mathbf{z}) = O\left(|w|^{-2}\right), \text{ as } |w| \to \infty,$$

uniformly in $\arg(w)$, for $0 \leq \arg(w) \leq \pi/4$.

Having established the validity of the representation (36) for $s \in \mathbb{R}$, we now show that this same formula represents the analytic continuation of $K(s, \mathbf{z})$.

Lemma 45 *For $\mathbf{z} \in \Gamma$, $K(s, \mathbf{z})$, defined by* (36)*, is analytic as a function of s in D_ε, with ε given by* (13)*. Further, for every $k_0 > 0$,*

$$|K(s, \mathbf{z})| \leq Ck^{1/2}\zeta(\mathbf{z}), \qquad s \in D_\varepsilon,\ k \geq k_0,\ \mathbf{z} \in \Gamma, \qquad (37)$$

where $C > 0$ depends only on Ω and k_0, and

$$\zeta(\mathbf{z}) := \begin{cases} 1, & 0 < k|\mathbf{z}| < 1, \\ (k|\mathbf{z}|)^{-1/2}, & k|\mathbf{z}| \geq 1. \end{cases}$$

The proof of Lemma 45 is based on the following two intermediate results.

Lemma 46 *For $t \geq 0$ and $\mathbf{z} \in \Gamma$, $\varphi(s, te^{i\pi/4}, \mathbf{z})$ is analytic as a function of s in D_ε, with ε given by* (13)*. Further,*

$$\mathrm{Im}\left[\varphi(s, te^{i\pi/4}, \mathbf{z})\right] \geq \frac{L_{nc}t}{2\sqrt{2}\sqrt{L_{nc}^2 + L_{nc}^2}}, \qquad s \in D_\varepsilon,\ t \geq 0,\ \mathbf{z} \in \Gamma. \qquad (38)$$

Proof Suppose $t \geq 0$ and $z \in \Gamma$. For $s_0 \in [0, L_{nc}]$,

$$\mathrm{Im}\left[\varphi(s_0, te^{i\pi/4}, z)\right] = \mathrm{Im}\left[\chi(s_0, L_{nc}, te^{i\pi/4})\right] + \mathrm{Im}\left[\chi(z_1, -z_2, te^{i\pi/4})\right] \geq \frac{L_{nc}t}{\sqrt{2}\sqrt{s_0^2 + L_{nc}^2}}, \tag{39}$$

by (33) and (31), applied to $\chi(s_0, L_{nc}, te^{i\pi/4})$ and $\chi(z_1, -z_2, te^{i\pi/4})$, respectively. We next note that, for $s \in \mathbb{C}$,

$$\varphi(s, te^{i\pi/4}, z) = \frac{A}{B(s)} + \chi(z_1, -z_2, te^{i\pi/4}),$$

where $A := te^{i\pi/4}(2L_{nc} + te^{i\pi/4})$ and $B(s) := \chi(s, L_{nc}, te^{i\pi/4}) + \chi(s, L_{nc}, 0)$. Thus, for $s_0 \in [0, L_{nc}]$ and $|s - s_0| < \varepsilon$,

$$|\varphi(s, te^{i\pi/4}, z) - \varphi(s_0, te^{i\pi/4}, z)| = \frac{|A|\,|B(s) - B(s_0)|}{|B(s_0)|\,|B(s)|} \leq \frac{|A|\,|B(s) - B(s_0)|}{|B(s_0)|\,||B(s_0)| - |B(s) - B(s_0)||}. \tag{40}$$

Now $|A| \leq t(2L_{nc} + t)$, and, by (30),

$$|B(s_0)| \geq \mathrm{Re}\,[B(s_0)] \geq L_{nc} + \frac{t}{\sqrt{2}} + \sqrt{s_0^2 + L_{nc}^2}.$$

Also, $\mathrm{Re}\left[\chi(s, L_{nc}, te^{i\pi/4})\right] > 0$ for $s \in D_\varepsilon$, since $\varepsilon < L_{nc}$ so that $\mathrm{Re}\left[s^2 + (L_{nc} + te^{i\pi/4})^2\right] \geq -\mathrm{Im}\,[s]^2 + L_{nc}^2 > 0$. Thus, using (32),

$$\begin{aligned}
\left|\chi(s, L_{nc}, te^{i\pi/4}) - \chi(s_0, L_{nc}, te^{i\pi/4})\right| &= \frac{|s - s_0|\,|s + s_0|}{\left|\chi(s, L_{nc}, te^{i\pi/4}) + \chi(s_0, L_{nc}, te^{i\pi/4})\right|} \\
&\leq \frac{\varepsilon(2s_0 + \varepsilon)}{\mathrm{Re}\left[\chi(s_0, L_{nc}, te^{i\pi/4})\right]} \\
&\leq \frac{4\varepsilon(s_0 + L_{nc})}{\sqrt{s_0^2 + L_{nc}^2}} \leq 4\sqrt{2}\varepsilon.
\end{aligned} \tag{41}$$

This implies that $|B(s) - B(s_0)| \leq 8\sqrt{2}\varepsilon$. Inserting these bounds into (40) gives

$$\begin{aligned}
|\varphi(s, te^{i\pi/4}, z) - \varphi(s_0, te^{i\pi/4}, z)| &\leq \frac{8\sqrt{2}\,t\varepsilon(2L_{nc} + t)}{\left(2L_{nc} + t/\sqrt{2}\right)\left(L_{nc} + \sqrt{s_0^2 + L_{nc}^2} - 8\sqrt{2}\varepsilon\right)} \\
&\leq \frac{L_{nc}t}{2\sqrt{2}\sqrt{s_0^2 + L_{nc}^2}},
\end{aligned} \tag{42}$$

on using (13). The result (38) follows by combining (39) and (42).

Lemma 47 *For $t \geq 0$ and $z \in \Gamma$, $S_k(s, te^{i\pi/4}, z)$ is analytic as a function of s in D_ε, with ε given by* (13)*. Further, for every $k_0 > 0$,* *Proof* Suppose $t \geq 0$ and $z \in \Gamma$. By (30) and (32) we have, for $s_0 \in [0, L_{\mathrm{nc}}]$,

$$\mathrm{Re}\left[\chi(s_0, L_{\mathrm{nc}}, t\mathrm{e}^{\mathrm{i}\pi/4})\right] \geq L_{\mathrm{nc}}, \quad \mathrm{Re}\left[\chi(z_1, -z_2, t\mathrm{e}^{\mathrm{i}\pi/4})\right] \geq \frac{|z| + t}{2}. \tag{44}$$

Combining (44) with (41) and recalling (13) gives

$$\mathrm{Re}\left[\chi(s, L_{\mathrm{nc}}, t\mathrm{e}^{\mathrm{i}\pi/4})\right] \geq \frac{7L_{\mathrm{nc}}}{8}, \qquad s \in D_\varepsilon. \tag{45}$$

Therefore, $S_k($

of Γ. Then it is clear that, if Γ^* is not one of the sides of Γ adjacent to P, then $\int_{\Gamma^*}(\zeta(z))^2\,ds \le Ck^{-1}$, for $k \ge k_0$. On the other hand, if Γ^* has length L^* and is adjacent to P, then

$$\int_{\Gamma^*}(\zeta(z))^2\,ds \le C\int_0^{1/k} ds + Ck^{-1}\int_{1/k}^{L^*} t^{-1}\,dt \le Ck^{-1}\log(2+k).$$

Thus $\|\zeta\|_{L^2(\Gamma)} \le Ck^{-1/2}\log^{1/2}(2+k)$, so that, by (37),

$$\|K(s,\cdot)\|_{L^2(\Gamma)} \le C\log^{1/2}(2+k), \qquad (49)$$

where $C > 0$ depends only on Ω. Finallyon

If α $(\pi/2, 3\pi/2)$, however, (51) no longer holds (since $\cos\alpha < 0$). In this case we write $u^i = u^d + (u^i - u^d)$, and note that, by Lemma 35, for y γ ,

$$u^i(\mathrm{y}) - u^d(\mathrm{y}) = 2\mathrm{e}^{\mathrm{i}ky_2} h \left(\sqrt{2ky_2}\sin(\alpha/2) \right),$$

where $h(w) := \mathrm{e}^{-\mathrm{i}w^2}\mathrm{Fr}(w)$. The function $h(w$

For α $(\pi/2, 3\pi/2)$ we have $2\partial\Psi$@

Lemma 52 *If the function g is analytic and bounded in $E_{a,b,r}$, for some $a, b, r \in \mathbb{R}$ with $a < b$ and $r > b - a$, then*

$$\inf_{v \in \mathcal{P}_p(a,b)} \|g - v\|_{L^\infty(a,b)} \leq \frac{2}{\rho - 1}\rho^{-p} \|g\|_{L^\infty(E_{a,b,r})},$$

where $\rho = (r + \sqrt{r^2 - (b-a)^2})/(b-a) > 1$.

Lemma 52 implies the following best approximation results for the two nonsingular terms in the representation on a nonconvex side.

Theorem 55 (cf. [28, Theorem 5.4]) *Suppose that Assumption 22 and* (56) *hold. Then, for every $k_0 > 0$, for the approximation of $v^+(s)$ and $v^-(L_c - s)$ on a convex side Γ_c we have*

$$\inf_{v' \in \mathcal{P}_{p,n}(0,L_c)} \left\| v^{\pm} - v' \right\|_{L^2(0,L_c)} \leq C M(u) k^{1-\delta_*} e^{-p\tau}, \quad k \geq k_0,$$

where $\tau > 0$ depends only on σ, the corner angles at the ends of Γ_c, and c (the constant in (56)*), and $C > 0$ only on Ω and k_0. The same estimate holds for the approximation of $v^-(L_{nc} - s)$ on a nonconvex side, except that L_c is replaced by L_{nc} in the above formula, and τ depends now on σ, c, and the exterior angle ω in Figure 4(a).*

We now combine these results into a single estimate for the best approximation

Corollary 64 *Suppose that* Ω *is a star-like member of the class* C

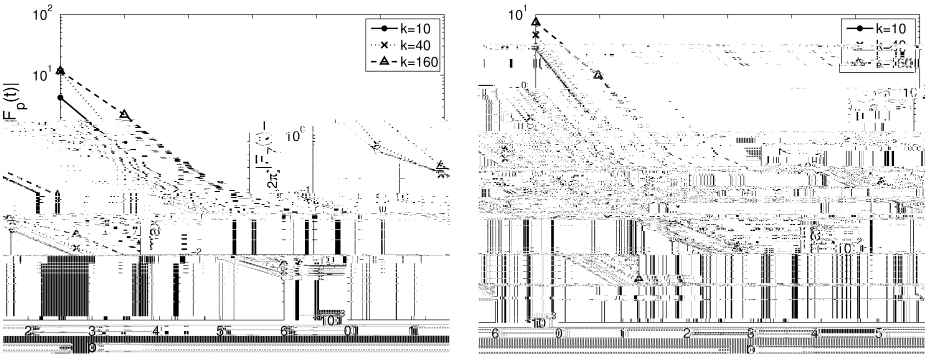
1. $\alpha = 5\pi/4$, as shown in Figure 1(a); in this case, multiply-reflected rays are present in the asymptotic solution.
2. $\alpha = 5\pi/3$, as shown in Figure 1(b); in this case, one of the nonconvex sides is partially illuminated.

Table 1 L^2 and L^1 errors for each example, fixed $p = 4$ (and hence $N = 320$), various k, with $N/(L/\lambda)$ the average number of degrees of freedom per wavelength along the boundary.

α	k	$\frac{N}{L/\lambda}$	$\|\psi_7 - \psi_4\|_{L^2(\Gamma)}$	μ	$\frac{\|\psi_7-\psi_4\|_{L^2(\Gamma)}}{\|\psi_7\|_{L^2(\Gamma)}}$	$\frac{\|\psi_7-\psi_4\|_{L^1(\Gamma)}}{\|\psi_7\|_{L^1(\Gamma)}}$
$5\pi/4$	5	10.67	8.37×10^{-1}	-0.35	3.90×10^{-1}	1.03×10^{-2}
	10	5.33	6.55×10^{-1}	-0.19	4.04×10^{-1}	1.43×10^{-2}
	20	2.67	5.72×10^{-1}	-0.29	4.24×10^{-1}	1.69×10^{-2}
	40	1.33	4.68×10^{-1}	-0.91	4.47×10^{-1}	1.85×10^{-2}
	80	0.67	2.48×10^{-1}	-0.20	4.39×10^{-1}	1.91×10^{-2}
	160	0.33	2.16×10^{-1}		4.62×10^{-1}	2.09×10^{-2}
$5\pi/3$	5	10.67	8.64×10^{-1}	-0.46	4.05×10^{-1}	1.17×10^{-2}
	10	5.33	6.30×10^{-1}	-0.54	4.18×10^{-1}	1.60×10^{-2}
	20	2.67	4.32×10^{-1}	-0.46	4.27×10^{-1}	1.80×10^{-2}
	40	1.33	3.15×10^{-1}	-0.46	4.40×10^{-1}	1.80×10^{-2}
	80	0.67	2.30×10^{-1}	-0.45	4.54×10^{-1}	1.88×10^{-2}
	160	0.33	1.69×10^{-1}		4.69×10^{-1}	1.92×10^{-2}

nitude are seen in the corresponding convex case [28]. There it is noted that the

approximations to $\|F_7 - F_p\|_{L^\infty(\mathbb{S}^1)}$ for $k = 10$, 40, and 160, for the two incident directions. To approximate the L^∞ norm, we compute F_7 and F_p at 30,000 evenly



(a) $\alpha = 5\pi/4$ (b) $\alpha = 5\pi/3$

Fig. 10 Absolute maximum errors $\|F_7 - F_p\|_{L^\infty(0,2\pi)}$ in the far field pattern.

spaced points on the unit circle. The exponential decay as p increases predicted by Theorem 63 is clearly seen. For fixed p, the error does not grow significantly as k increases, indicating that the mild k-dependence of the bound (72) may not be optimal. The errors are comparable in magnitude for each incidence angle, suggesting that our algorithm copes equally well with cases of multiple reflection and partial illumination.

In summary, our numerical examples demonstrate that the predicted exponential convergence of our *hp* scheme is achieved in practice. Moreover, for a fixed number of degrees of freedom, the accuracy of our numerical solution appears to deteriorate only very slowly (or not at all) as the wavenumber k increases. The p- and k-dependence of our results appears to mimic closely that of the comparable results for the convex polygon in [28]. The k-explicit error bounds in Corollary 64 predict at worst a mild growth in errors as k increases, which can be controlled by a logarithmic growth in the degrees of freedom N, as discussed in Remark 66. The numerical results support the conjecture that this mild growth is pessimistic; the estimates in Corollary 64 are not quite sharp in their k-dependence. We suspect that this is due to lack of sharpness in the dependence on k of the estimate (69) for $M(u)$, of our best approximation estimate (60), and of the quasi-optimality estimate (62).

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