School of Mathematical
and Physical Sciences

Department of Mathematics and Statistics

Preprint MPS-2011-13

21 October 2011

Resolution of sharp fronts in the presence of model error in variational data assimilation

by

M.A. Freitag, N.K. Nichols and C.J. Budd

solution, fronts are resolved more accurately than with the standard $_2$-norm regularisation of 4DVar.

The aim of this paper is to examime the potential benefits of using -norm regularisation in variational data assimilation. It presents a preliminary study showing that the method has potential to give improvement over existing approaches. Further investigation remains to be done in order to evaluate the technique in an operational setting.

Section 2 gives an introduction to 4DVar and shows its relation to Tikhonov regularisation. In Section 3 we introduce the new algorithm and in Section 4 we explain how we solve the -norm regularisation problem and the mixed TV - $_2$-norm regularisation problem. In Section 5 we state the model equations. Section 6 presents numerical examples, where the new -norm regularisation is compared to standard 4DVar. In our examples we introduce several kinds of model error. Under these conditions it can be seen that -norm regularisation outperforms 4DVar when sharp fronts are present (see Sections 6). We conclude with a section on future work.

2 4DVar and its relation to Tikhonov regularisation

In nonlinear 4DVar we aim to minimise the objective function

$$
\begin{aligned}
J(x\ ) &= \frac{1}{2}(x\ - x^b)^T B^{-}\ (x\ - x^b) \\
&\quad + \frac{1}{2}\ {}^{N}
\end{aligned}
$$

are vectors whereas $\square_i$, $i = 0$ N are square matrices of the dimension of the system state.

The approximate Hessian $\widetilde{\square\square} \mathbf{J}(x)$ and $\square \mathbf{J}(x)$ are

Firstly, in this paper we consider the effects of -

subject to

$$w = \quad \text{and} \quad w \geq 0 \tag{25}$$

where

$$w = \begin{bmatrix} \\ \\ - \end{bmatrix} \qquad = \begin{bmatrix} 2(\ ^T\ + \mu^2 I) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2\ ^T \\ 1 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} C_B^{\ I2} & -I & I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix} \qquad = - \quad x^b$$

and the block matrices I and **0** as well as the vectors **1** of all ones in the matrices , , and are of appropriate size. The objective function in (24) is convex as is symmetric positive semi-definite. In order to solve the quadratic programming problem (24) with constraints (25) we use the MATLAB in-built function `quadprog.m`.

In the following section we consider a square wave advected using the linear advection equation as an example. We use a 'true' model (from which we take the observations) and another model, which is different from the truth and hence introduces a model error. The different models we use are introduced in the next section. In all examples we observe that the new edge-preserving mixed TV - $_2$-norm regularisation indeed gives better results than the standard $_2$-norm approach and the simple - norm regularisation.

In all the examples we keep the regularisation parameter μ fixed, as we are only investigating the influence of the norm in the regularisation term, but not the size of the regularisation parameter μ.

5 Models

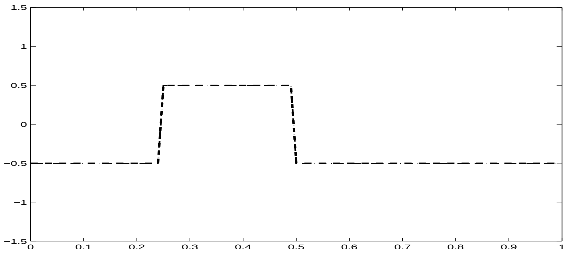
In this section we consider the problem

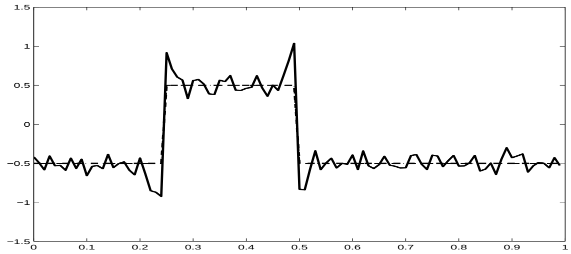
$$_t + [\ (\ )]_x = 0 \tag{26}$$

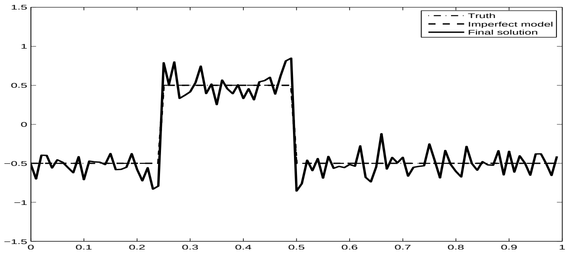
where $(\ )$ is given by

$$(\ ) = \tag{27}$$

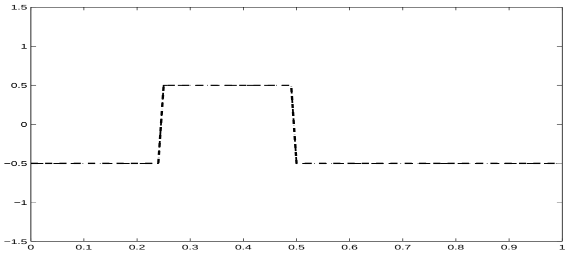
lnt e d o-2.8(Gi0)4.49(e)76.5(n)3(e)3.15(i)-2.1(5)-2.12(...)-2835639(e)11.29(t)-2.65993(i)-2.657g , bnfos rio


1.5
1
0.5
0
−0.5
−1
−1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


1.5
1
0.5
0
−0.5
−1
−1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Truth
Imperfect model
Final solution

is obtained if we use the correct initial conditions and the (imperfect) model. It represents the best solution that we


1.5
1
0.5
0
−0.5
−1
−1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

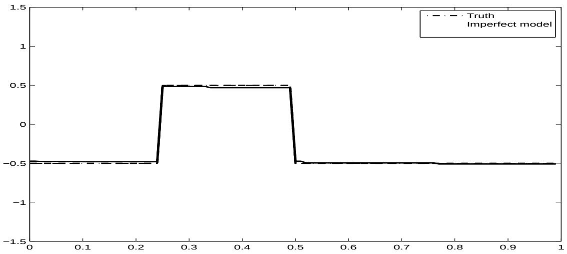
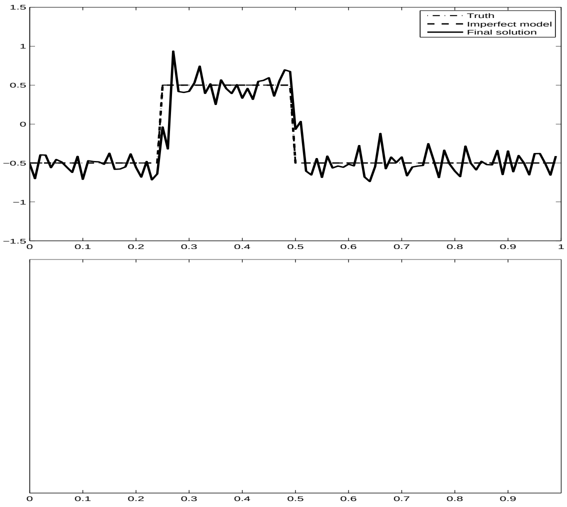
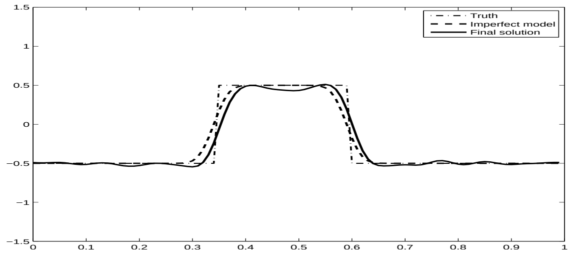

Truth
Imperfect model

Table I. Comparison between errors in the analysis in standard 4DVar, L_1-norm regularisation and mixed TV L_1-L_2-norm regularisation measured in the L_2-norm


Truth
Imperfect model
Final solution


Truth
Imperfect model
Final solution

the second author is supported by the National Centre for Earth Observation (NCEO).

References

Agarwal V, Gribok AV, Abidi MA. 2007. Image restoration using norm penalty function. *Inverse Probl. Sci. Eng.* **15**(8): 785–809.

Bennett A. F. 2002. *Inverse Modelling of the Ocean and Atmosphere (234 pp.)*. Cambridge: Cambridge University Press.

Daley R. 1991. *Atmospheric Data Analysis (457 pp.)*. Cambridge: Cambridge University Press.

Dennis Jr JE, Schnabel RB. 1983. *Numerical methods for*