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Equivalence of Weak Formulations of the Steady Water Waves Equations

by

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Abstract. We prove the equivalence of three weak formulations of the steady water waves equations, namely the velocity formulation, the stream function formulation, and the Dubreil-Jacotin formulation, under weak Hölder regularity assumptions on their solutions.

1. Introduction

an open problem by [3]. (A result on the equivalence of the formulations is given in [3], but under different, and not the most natural, regularity assumptions.)

The main result of the present paper, Theorem 1, which is given in Section 4, provides an affirmative answer to the above open problem, albeit only in the case when the Hölder exponent satisfies $\alpha > 1/3$. More precisely, Theorem 1 proves that, under the above regularity assumptions, the weak velocity and the weak stream function formulations are equivalent for any $\alpha \in (1/3, 1]$, while the weak stream function and the weak height formulations are equivalent for any $\alpha \in (0, 1]$. An important consequence of our result is that, at least in the case $\alpha > 1/3$, the solutions constructed in [3] of the Dubreil-Jacotin formulation are relevant (in the sense that they give rise to corresponding solutions) for the velocity formulation of the steady water waves equations. Our result is in the same spirit as, and its proof is inspired by, the Onsager conjecture as proved (partially) in [5]. The Onsager conjecture is, essentially, the statement that solutions of the time-dependent incompressible Euler equations on a fixed domain (in dimension three, with no external forces), of class $C^{0,\alpha}$ in the space variables for each value of the time variable, conserve their energy in time if $\alpha > 1/3$ and may fail to do so if $\alpha \leq 1/3$. The paper [5] proves that $\alpha > 1/3$ implies conservation of energy (and leaves open the reverse statement in the conjecture). As in [5], our proof is based on regularizing the equations and, roughly speaking, the assumption $\alpha > 1/3$ is used in an essential way to show that certain remainder terms converge to 0 as the regularization parameter tends to 0. An important problem left open by the present paper is that of whether the weak velocity formulation and the weak stream function formulation are also equivalent in the case when the Hölder exponent satisfies $\alpha \leq 1/3$.

2. Classical formulations of the steady water waves problem

2.1. The velocity formulation. We consider a wave travelling with constant speed and without change of shape on the free surface of a two-dimensional inviscid, incompressible fluid of unit density, acted on by gravity, over a flat, horizontal, impermeable bed. This means that, in a frame of reference moving at the speed c of the wave, the fluid is in steady flow in a fixed domain. Let the free surface be given by y

the kinematic condition that the same particles always form the free surface, while (2.1f) is the dynamic condition that at the free surface the pressure in the fluid equals the constant atmospheric pressure. This is a free-boundary problem, because the domain D_η is not known a priori. The system (2.1) will be referred to as the *velocity formulation* of the steady water waves equations. Throughout the paper we assume that, in the moving frame, the horizontal velocity of all the particles is in the same direction. For definiteness, we assume that

$$u < c \quad \text{in } \overline{D_\eta}: \tag{2.2}$$

(All the results discussed in the paper have corresponding analogues if instead of (2.2) one assumes that $u > c$ in $\overline{D_\eta}$.)

For the remainder of this section we describe informally, following [2], two other equivalent formulations of (2.1), assuming that the solutions are smooth enough. The equivalence of these formulations under weak regularity assumptions is the main aim of the paper, which will be addressed in the subsequent sections.

2.2. **The stream function formulation.** Suppose that (2.1) and (2.2) hold. Equation (2.1a) implies the existence of a function in $\overline{D_\eta}$, called a (relative) stream function, such that

$$= u - c; \qquad = -v \quad \text{in } D_\eta: \tag{2.3}$$

The boundary conditions (2.1d) and (2.1e) imply that is a constant on each of $y = 0$ and $y = \eta(x)$. Since is only determined up to an additive constant, one can assume that $= 0$ on $y = \eta(x)$, and then we obtain that there exists a constant p_0 such that $= -p_0$ on $y = 0$. The condition (2.2) can be rewritten as

$$< 0 \quad \text{in } \overline{D_\eta}; \tag{2.4}$$

a consequence of which is that $p_0 < 0$. After expressing the left-hand side in (2.1b) and (2.1c) in terms of , di erentiation of the first of these equations with respect to y and of the second with respect to x allows us to eliminate the pressure, leading to

$$(\Delta \ ) \quad = (\Delta \ ) \quad \text{in } D_\eta; \tag{2.5}$$

where Δ denotes the Lapla(Δ)5Tf4.55011.95apla(Δ)(dn7d6anca326(to00)c[()]Tls(pn)]TJ F411.9ere)

0^{2p}

for some constant Q. We have therefore obtained the *stream function formulation* of the steady water waves equations, which is to find a domain D_η and a function   in D_η such that

(2.8a) $\Delta \quad = -\gamma(\quad)$ in $D_\eta;$

(2.8b) $= -p_0$ on $y = 0;$

(2.8c) $= 0$ on $y = \eta(x);$

(2.8d) $j\, r \quad j^2 + 2gy = Q$ on $y = \eta(x);$

for some constants $p_0 < 0$ and Q, and some function $\gamma : [0; -p_0] \;!\; \mathbb{R}$.

Conversely, suppose that   satisfies (2.8) and (2.4) in a domain D_η. Then one can define in D_η a velocity field $(u; v)$ by (2.3) and a pressure field P by (2.7) with a suitable choice of the constant in the right-hand side, and easily check that (2.1) and (2.2) hold.

2.3. **The height (or Dubreil-Jacotin) formulation.** An elegant way to overcome the di culty that in (2.8) the fluid domain D_η needs to be found as part of the solution has been first observed by Dubreil-Jacotin: the fact that   is constant on the top and the bottom of D_η canP

Using these identities, one can easily reformulate (2.8) as the following system for the function h defined above:

$$(1 + h_q^2)h_{pp} - 2h_q h_p h_{pq} + h_p^2 h_{qq} = -\gamma(-p)h_p^3 \quad \text{in } R; \tag{2.15a}$$

$$h = 0 \quad \text{on } p = p_0; \tag{2.15b}$$

$$1 + h_q^2 + (2gh - Q)h_p^2 = 0 \quad \text{on } p = 0. \tag{2.15c}$$

This is the *height (or Dubreil-Jacotin) formulation* of the steady water waves equations.

Conversely, suppose that h satisfies (2.15) and (2.12). Let $\eta : \mathbb{R} \to \mathbb{R}$ be given by $\eta(q) = h(q, 0)$ for all $q \in \mathbb{R}$. Then (2.12) implies that $(q, p) \mapsto (x, y) = (q, h(q, p))$ is a bijection between $\overline{R}$ and $\overline{D_\eta}$. Defining ψ by (2.11), the formulae (2.13)–(2.14) are valid, and one can easily deduce from (2.15) and (2.12) that (2.8) and (2.4) hold.

3. Weak formulations of the steady water waves problem

3.1. **Weak velocity formulation.** For sufficiently smooth scef51.910Td[) $(\overline{D_\eta})$]0a[()]TF8117.011917.20 Td[(()]TJ /F6 11.955 Tf 910

Again, (3.2a) may be required to hold in the sense of distributions. We will be interested in solutions of (3.2) with η 2 $C^{1,\alpha}(\mathbb{R})$, 2 $C^{1,\alpha}(\overline{D_\eta})$ and Γ 2 $C^{0,\alpha}([p_0;0])$ for some α 2 $(0;1]$, with (3.2b){(3.2d) being satisfied in the classical sense, and (3.2a) being satisfied in the sense of distributions (with ; being understood in the classical sense).

3.3. **Weak height formulation.** For su ciently smooth functions h and γ, the algebraic identity

$$\left\{-\frac{1+h^2}{2h^2}+\Gamma(p)\right\} + \left\{\frac{h}{h}\right\} = \frac{1}{h^3}\left\{(1+h^2)h - 2h\,h\,h + h^2h + \gamma(-p)h^3\right\}$$

shows that, in the presence of (2.12), (2.15) is equivalent to

$$\left\{-\frac{1+h^2}{2h^2}+\Gamma(p)\right\} + \left\{\frac{h}{h}\right\} = 0 \quad \text{in } R; \tag{3.3a}$$

$$h = 0 \quad \text{on } p = p_0; \tag{3.3b}$$

$$\frac{1+h^2}{2h^2} + gh - \frac{Q}{2} = 0 \quad \text{on } p = 0: \tag{3.3c}$$

We will be interested in solutions of (3.3) with h 2 $C^{1,\alpha}(\overline{R})$ and Γ 2 $C^{0,\alpha}([p_0;0])$ for some α 2 $(0;1]$, with (3.3b){(3.3c) being satisfied in the classical sense, and (3.3a) being satisfied in the sense of distributions (with h ; h understood in the classical sense).

4. The equivalence of the weak formulations

Weak solutions (in the sense described in the previous section) of the steady water waves problem have been studied only very recently in [3]. That paper deals with waves which are periodic in the horizontal direction, the subscript *per* being used in what follows to indicate this periodicity requirement. In [3] the authors develop a global bifurcation theory for weak solutions of (3.3) with h 2 $C^{1,\alpha}_{per}(\overline{R})$, under the assumption Γ 2 $C^{0,\alpha}([p_0;0])$, for some α 2 $(0;1)$. These would formally correspond to solutions of the weak velocity formulation with η 2 $C^{1,\alpha}_{per}(\mathbb{R})$ and $u;v;P$ 2 $C^{0,\alpha}_{per}(\overline{D_\eta})$. However, no rigorous proof of this equivalence is given in [3]. The only result there on the equivalence of the weak formulations, see [3, Theorem 2], is the following:

Let $0 < \alpha < 1$ *and* $r = \frac{2}{1-\alpha}$. *Then the following are equivalent:*

(i) *the weak velocity formulation (3.1) together with (2.2), for* η 2 $C^{1,\alpha}_{per}(\mathbb{R})$ *and* $u;v;P$ 2 $W^{1,r}_{per}(D_\eta)$ $C^{0,\alpha}_{per}(\overline{D_\eta})$;
(ii) *the stream function formulation (2.8) together with (2.4), for* γ 2 $L^r[0;-p_0]$, η 2 $C^{1,\alpha}_{per}(\mathbb{R})$ *and* 2 $W^{2,r}_{per}(D_\eta)$ $C^{1,\alpha}_{per}(\overline{D_\eta})$;
(iii) *the weak height formulation (3.3) together with (2.12), for* Γ 2 $W^{1,r}[p_0;0]$ *and* h 2 $W^{2,r}_{per}(R)$ $C^{1,\alpha}_{per}(\overline{R})$.

As one can see, in the above result the velocity field $(u;v)$, the pressure P, the stream function , the height h, and the function Γ, are assumed to have more regularity, namely an additional weak (Sobolev space) derivative, than one would really like.

Our main result, given below, proves the equivalence of the weak formulations under the `right' regularity assumptions, albeit only for the case when the Hölder exponent satisfies $\alpha \in (1/3, 1]$. (In particular, under our assumptions, the function Γ need not have a (weak) derivative.) While the weak stream function and the weak height formulations will be seen to be, in fact, equivalent for any $\alpha \in (0, 1]$, it remains an open problem whether the weak

Then, for any $" > 0$ *such that* V^{ε} *is non-empty, where* $V^{\varepsilon} \overset{\text{de}}{=} f$

of (3.3a), which we need to prove: for any $\zeta \in C_0^1(R)$,

$$\text{(4.7)} \qquad \int_R \left(-\frac{1 + h^2}{2h^2} + \Gamma(p) \right) \zeta \ + \frac{h}{h} \zeta \ dqdp = 0.$$

For any such ζ, let $\varphi \in C_0^1(D_\eta)$ be given by $\varphi(x,y) = \zeta(x, -\ (x,y))$ for all $(x,y) \in D_\eta$. By changing variables in the integral, using (2.13)–(2.14), one can rewrite (4.7) as

$$\text{(4.8)} \qquad \int_D \Gamma(-\ )\varphi \ - (\ \ )\varphi \ + \frac{1}{2}(\ ^2 - \ ^2)\varphi \ dxdy = 0.$$

But (4.8) is valid, as a consequence of (3.2a). This shows that (3.3a) holds. We have thus proved that (iii) holds.

Suppose now that (iii) holds. Let $h \in C^{1,\alpha}_{\text{per}}(\overline{R})$ be such that (3.3) and (2.12) hold, where $\Gamma \in C^{0,\alpha}([p_0, 0])$. Defining η and as in Section 2, we then have that $\eta \in C^{1,\alpha}_{\text{per}}(\mathbb{R})$ and $\in C^{1,\alpha}_{\text{per}}(\overline{D_\eta})$, and the formulae (2.13)–(2.14) are still valid. Clearly (3.3b)–(3.3c) imply (3.2b)–(3.2d), and (2.12) implies (2.4). The weak form of (3.2a), which we need to prove, is written explicitly as (4.8), for any $\varphi \in C_0^1(D_\eta)$. For any such φ, let $\zeta \in C_0^1(R)$ be given by $\zeta(q,p) = \varphi(q, h(q,p))$ for all $(q,p) \in R$. By changing variables in the integral, using (2.13)–(2.14), one can rewrite (4.8) as (4.7). But (4.7) is valid, as a consequence of (3.3a). This shows that (3.2a) holds. We have thus proved that (ii) holds.

We now prove the equivalence of (i) and (ii), making essential use of the assumption $\alpha > 1/3$.

Suppose that (i) holds. Since $\eta \in C^{1,\alpha}_{\text{per}}(\mathbb{R})$ and $u, v \in C^{0,\alpha}_{\text{per}}(\overline{D_\eta})$, it follows from (3.1a), by arguments similar to those in [1, Lemma 3], in which our Lemma 2 plays a key role, that there exists $\in C^{1,\alpha}_{\text{per}}(\overline{D_\eta})$, uniquely determined up to an additive constant, such that (2.3) holds. Clearly, (2.2) implies (2.4). Also, it follows from (3.1d) and (3.1e) that is constant on each of $y = 0$ and $y = \eta(x)$. The additive constant in the definition of may be chosen so that (3.2c) holds, and then (3.2b) also holds for some constant $p_0 < 0$. Using the definition of we rewrite (3.1b)–(3.1c) in the weak distributional form (with $\ ,\ $ in the classical sense):

$$\text{(4.9a)} \qquad (\ ^2) \ - (\ \ ) \ = -P \qquad \text{in } D_\eta,$$

$$\text{(4.9b)} \qquad -(\ \ ) \ + (\ ^2) \ = -P \ - g \qquad \text{in } D_\eta.$$

Let us denote

$$\text{(4.10)} \qquad F \stackrel{\text{def}}{=} P + \frac{1}{2}|\nabla \ |^2 + gy \quad \text{in } D_\eta.$$

It follows from (4.9) that we have, in the sense of distributions (with $\ ,\ $ in the classical sense):

$$\text{(4.11a)} \qquad F \ = \frac{1}{2}(\ ^2 - \ ^2) \ + (\ \ ) \quad \text{in } D_\eta,$$

$$\text{(4.11b)} \qquad F \ = (\ \ ) \ - \frac{1}{2}(\ ^2 - \ ^2) \quad \text{in } D_\eta.$$

We now show that there exists a function $\Gamma \in C^{0,\alpha}([p_0, 0])$ such that

$$\text{(4.12)} \qquad F(x,y) = \Gamma(-\ (x,y)) \quad \text{for all } (x,y) \in D_\eta.$$

to prove (4.15) for $'$, we write, for any $'' \ 2 \ (0; ''_0)$,

$$\begin{aligned}
&\int_D F(\quad ' \ - \quad ')\, dxdy \\
&= \int_K (F \quad - F^{\varepsilon} \ {}^{\varepsilon})\,' \ - (F \quad - F^{\varepsilon} \ {}^{\varepsilon})\,' \ dxdy + \int_K F^{\varepsilon} \ {}^{\varepsilon}\,' \ - F^{\varepsilon} \ {}^{\varepsilon}\,' \ dxdy \\
&\stackrel{\mathrm{de}}{=} I_\varepsilon + J_\varepsilon :
\end{aligned}$$

It is a consequence of Lemma 1(i) that $I_\varepsilon \ ! \ 0$ as $'' \ ! \ 0$. To estimate J_ε, we first integrate by parts, then use (4.18) to cancel some terms, and then integrate by parts again, to get

$$\begin{aligned}
J_\varepsilon &= \int_K (F^{\varepsilon} \ {}^{\varepsilon} - F^{\varepsilon} \ {}^{\varepsilon})\,' \, dxdy \\
&= -\int_K \left[\frac{1}{2}R^\varepsilon(\ ;\ )\ \frac{1}{2}R^\varepsilon(\ ;\ ) + R^\varepsilon(\ ;\ )\right](\ {}^{\varepsilon}\,')\, dxdy \\
&\quad + \int_K \left[R^\varepsilon(\ ;\ ) - \frac{1}{2}R^\varepsilon(\ ;\ ) + \frac{1}{2}R^\varepsilon(\ ;\ )\right](\ {}^{\varepsilon}\,')\, dxdy \\
&= \int_K \left[\frac{1}{2}R^\varepsilon(\ ;\ ) - \frac{1}{2}R^\varepsilon(\ ;\ )\right][(\ {}^{\varepsilon}\,') + (\ {}^{\varepsilon}\,')\ ]\, dxdy \\
&\quad + \int_K R^\varepsilon(\ ;\ )[(\ {}^{\varepsilon}\,') - (\ {}^{\varepsilon}\,')\ ]\, dxdy :
\end{aligned}$$

Expanding the square brackets, we write J_ε as a sum of six terms, all of which can be estimated in a similar way, by using Lemma 1, to the one shown below:

$$\begin{aligned}
j\int_K R^\varepsilon(\ ;\ )(\ {}^{\varepsilon}\,')\ dxj &= j\int_K R^\varepsilon(\ ;\ )\big(\ {}^{\varepsilon}\,' \ + \ {}^{\varepsilon}\,'\big)\, dxj \\
&\le C(''^{2\alpha}k \quad k^2_{C^0\ (K_0)}k \quad k_{C^0\ (K_0)} + ''^{3\alpha-1}k \quad k^3_{C^0\ (K_0)});
\end{aligned}$$

where C is a constant which depends on $'$, but is independent of $'' \ 2 \ (0; ''_0)$. The assumption $\alpha > 1{=}3$ now implies that $J_\varepsilon \ ! \ 0$ as $'' \ ! \ 0$. We have thus proved that (4.15) holds for any $' \ 2 \ C^1_0(D_\eta)$. As discussed earlier, this implies the existence of $\Gamma \ 2 \ C^{0,\alpha}([p_0; 0])$ such that (4.12) holds. It therefore follows from (4.11) that, in the sense of distributions (with $\ ;$ in the classical sense),

$$\Gamma(-\ ) = \frac{1}{2}(\ ^2 - \ ^2) + (\qquad) \quad \text{in } D_\eta; \tag{4.19a}$$

$$\Gamma(-\ ) = (\qquad) - \frac{1}{2}(\ ^2 - \ ^2) \quad \text{in } D_\eta: \tag{4.19b}$$

Rearranging (4.19b) gives exactly (3.2a). Also, recalling (4.10), we obtain from (3.1f) the validity of (3.2d) for some constant Q. We have thus proved that (ii) holds.

Suppose now that (ii) holds. We define in D_η the velocity $(u; v)$ by (2.3) and, up to an additive constant, the pressure P, by

$$P \stackrel{\mathrm{de}}{=} -\frac{1}{2}j\,r \quad j^2 - gy + \Gamma(-\ ) \quad \text{in } D_\eta: \tag{4.20}$$

Then $u; v; P \ 2 \ C^{0,\alpha}_{\mathrm{pe}}(\overline{D_\eta})$. Moreover, the definition of u and v implies (3.1a), while (3.2b){(3.2d) imply (3.1d){(3.1f), provided the additive constant in the definition of P is chosen in a suitable way. Also, (2.4) implies (2.2). It therefore remains to prove the validity of (3.1b){(3.1c). Using our definition of $u; v$ and P, (3.1b){(3.1c) can be equivalently rewritten as (4.19). However, (4.19b) is exactly (3.2a), which we are assuming to hold, and therefore it only remains to prove (4.19a). We now show that (4.19b) implies (4.19a). For notational convenience, we denote $F \stackrel{\text{de}}{=} \Gamma(- \ )$. We claim that, with this definition of F, (4.16) necessarily holds. Indeed, (4.16) can be written explicitly as (4.15) for any $' \ 2 \ C_0^1(D_\eta)$, which, using the same notation as earlier in the proof, is equivalent to (4.14) for any $\ 2 \ C_0^1(R)$, which is clearly true with our definition of F. Let $V \stackrel{\text{de}}{=} D_\eta$ and let, for any $'' > 0$, let $V^\varepsilon \stackrel{\text{de}}{=} f(x;y) \ 2 \ V : \mathrm{dist}((x;y);\mathbb{R}^2 \ n \ V) > ''g$. Using Lemma 2, (4.16) implies that, for any $'' > 0$ such that V^ε is not empty, in the notation (4.1),

$$(F^\varepsilon \ \ ^\varepsilon)_y - (F^\varepsilon \ \ ^\varepsilon)_y + R^\varepsilon(F; \ \ ) - R^\varepsilon(F; \ \ )_y = 0 \quad \text{in} \tag{4.21}$$

It is a consequence of Lemma 1(i) that K_ε ! 0 as ″ ! 0. Now note that (2.4) implies that there exists ꝃ 2 (0; ″$_0$) and δ > 0 such that, for all ″ 2 (0; ꝃ),

$$\quad ^{\varepsilon} \geq \delta \quad \text{in } K: \tag{4.26}$$

To estimate L_ε, we first integrate by parts using (4.17a) with θ := $^{\varepsilon}$, then use (4.24) to cancel some terms, and then integrate by parts again, to obtain, for any ″ 2 (0; ꝃ),

$$\begin{aligned} L_\varepsilon &= -\int_K (F^\varepsilon - \ ^\varepsilon \Delta \ ^\varepsilon)\ ' \\ &= -\int_K \frac{^\varepsilon}{^\varepsilon}\left[R^\varepsilon(\ ;\ ) - \frac{1}{2}R^\varepsilon(\ ;\ ) + \frac{1}{2}R^\varepsilon(\ ;\ )\right]\ '\ dxdy \\ &\quad + \int_K \frac{1}{^\varepsilon}\left[R^\varepsilon(F;\ ) - R^\varepsilon(F;\ )\right]\ '\ dxdy \\ &= \int_K R^\varepsilon(\ ;\ )\left(\frac{^\varepsilon}{^\varepsilon}\ '\right) - \frac{1}{2}R^\varepsilon(\ ;\ )\left(\frac{^\varepsilon}{^\varepsilon}\ '\right) - \frac{1}{2}R^\varepsilon(\ ;\ )\left(\frac{^\varepsilon}{^\varepsilon}\ '\right)\ dxdy \\ &\quad - \int_K R^\varepsilon(F;\ )\left(\frac{1}{^\varepsilon}\ '\right) - R^\varepsilon(F;\ )\left(\frac{1}{^\varepsilon}\ '\right)\ dxdy: \end{aligned}$$

Thus we have written L_ε as a sum of five terms, all of which can be estimated in a similar way, by using Lemma 1, to the one shown below:

$$\begin{aligned} j\int_K R^\varepsilon(\ ;\ )\left(\frac{^\varepsilon}{^\varepsilon}\ '\right)\ dxdyj &= j\int_K R^\varepsilon(\ ;\ )\left(\frac{^\varepsilon\ ^\varepsilon}{(\ ^\varepsilon)^2}\ ' + \frac{^\varepsilon\ ^\varepsilon - \ ^\varepsilon\ ^\varepsilon}{(\ ^\varepsilon)^2}\ '\right)\ dxdyj \\ &\leq C(″^{2\alpha} jj\ \ jj^2_{C^0\ (K_0)} jj\ \ jj^2_{C^0\ (K_0)} + ″^{3\alpha-1} jj\ \ jj^2_{C^0\ (K_0)} jj\ \ jj^2_{C^0\ (K_0)}); \end{aligned}$$

where C is a constant which C34.770Td[(C)]TJ/F411.955Tf14.090Td[(is)-405(a)-406(constan)26(t)

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