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THE UNSTEADY FLOW OF A WEAKLY COMPRESSIBLE FLUID IN A THIN POROUS LAYER II: THREE-DIMENSIONAL THEORY

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Abstract. We consider the problem of determining the pressure and velocity elds for a weakly compressible uid owing in a three-dimensional layer, composed of an inhomogeneous, anisotropic porous medium, with vertical side walls and variable upper and lower boundaries, in the presence of vertical wells injecting and/or extracting uid. Numerical solution of this three-dimensional evolution problem may be expensive, particularly in the case that the depth scale of the layer h is small compared to the horizontal length scale l, a situation which occurs frequently in the application to oil reservoir recovery and which leads to signi cant sti ness in the numerical problem. Under the assumption that $\epsilon \propto h/l \ll 1$, we show that, to leading order in ϵ

studies. Such schemes have thus received considerable attention in the literature over the years, both for well testing applications [11, 6, 17, 2, 10] and also for full eld simulation problems, for porous media with homogeneous and anisotropic permeabilreduces to a linear, inhomogeneous, strongly elliptic two-dimensional boundary value problem [BVP], on the layer cross-sectional projection, that must in general be solved numerically. This can be achieved via standard finite or boundary element methods, and a detailed consideration of the numerical solution of [BVP] is described in [12]. In the inner regions, determination of the leading order terms reduces to the solution of a strongly elliptic problem whose solution can be written analytically in terms of the eigenvalues and corresponding eigenfunctions of a regular Sturm-Liouville eigenvalue problem, identical to that considered in [13]. The asymptotic solution of [EVP] in §4 reduces to a regular two-dimensional strongly elliptic problem, whose numerical solution can also be achieved via standard finite element methods in a very similar manner to the solution of [BVP], and this is also considered in [12]. Finally in §5 we draw some conclusions.

We remark further that full implementation details for an efficient numerical scheme for the computation of the dynamic fluid pressure and the fluid velocity field throughout the layer are provided in [12], where we also apply the theory to some simple model examples, demonstrating the exceptional computational efficiency of our approach via matched asymptotic expansions.

2. Equations of motion. As in Needham et. al [13] we again consider the flow of a weakly compressible fluid in the presence of sources and sinks, in a reservoir of porous medium with variable upper and lower boundary. The reservoir has permeability which is both inhomogeneous and anisotropic. Whilst in [13] we restricted attention to two-dimensional flow in a two-dimensional reservoir, we now extend the theory to fully three-dimensional flow in a three-dimensional reservoir. We adopt the same notation and the same physical model as in [13], and so omit a detailed description of the modelling here. Thus, following [13], the equations of motion of the fluid in the porous reservoir may be written as,

$$c_t\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right)\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{q} = \sum_{i=1}^{N} s_i\left(\frac{z}{h}\right)\frac{1}{l^2}\,\delta\left(\frac{x-x_i}{l}\right)\delta\left(\frac{y-y_i}{l}\right); \tag{2.1}$$

$$\mathbf{q} = -\underline{D}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right)(\nabla p + \rho_0 g\mathbf{k}); \tag{2.2}$$

for all $(x,y,z)\in M$, $t\in(0,\infty)$. Here (x,y,z) are rectangular cartesian coordinates with z pointing vertically upwards. The interior of the porous reservoir is denoted by $M\subset\mathbb{R}^3$ and its impermeable boundary by $\partial M\subset\mathbb{R}^3$, with $\bar{M} = M\cup\partial M$. The region $\bar{M}$ is taken as a finite section of a generalized cylinder which has its axis aligned with the z-axis and its cross section bounded by the simple closed piecewise smooth curve $\partial\Omega_l\subset\mathbb{R}^2$, which has interior $\Omega_l\subset\mathbb{R}^2$, with $\bar{\Omega}_l = \Omega_l\cup\partial\Omega_l$. Here $l>0$ is the horizontal length scale associated with Ω_l. The upper and lower boundary surfaces of the reservoir are described by

$$\left.\begin{array}{l} z = hz_+(x/l,y/l) \\ z = hz_-(x/l,y/l) \end{array}\right\}\ (x,y)\in\bar{\Omega}_l;$$

respectively, with $h(>0)$ being the vertical length scale associated with the reservoir, and $z_+,z_- : \bar{\Omega}_1\mapsto\mathbb{R}$ being such that

$$z_+,z_-\in C^1(\bar{\Omega}_1); \tag{2.3}$$

and

$$z_+(x,y) > z_-(x,y) \text{ for all } (x,y)\in\bar{\Omega}_1. \tag{2.4}$$

The normal fields on the upper and lower surfaces are then given by

$$n_+(x;y) = \left(-\tfrac{h}{l} z_{+x}(x;y);\ -\tfrac{h}{l} z_{+y}(x;y);\ 1\right)$$
$$n_-(x;y) = \left(\tfrac{h}{l} z_{-x}(x;y);\ \tfrac{h}{l} z_{-y}(x;y);\ -1\right)$$

for all $(x;y) \in \Omega_1$, with the normals directed out of M. The situation is illustrated in Figure 2.1. The $N (\in \mathbb{N})$ vertical line sources/sinks embedded within M, which

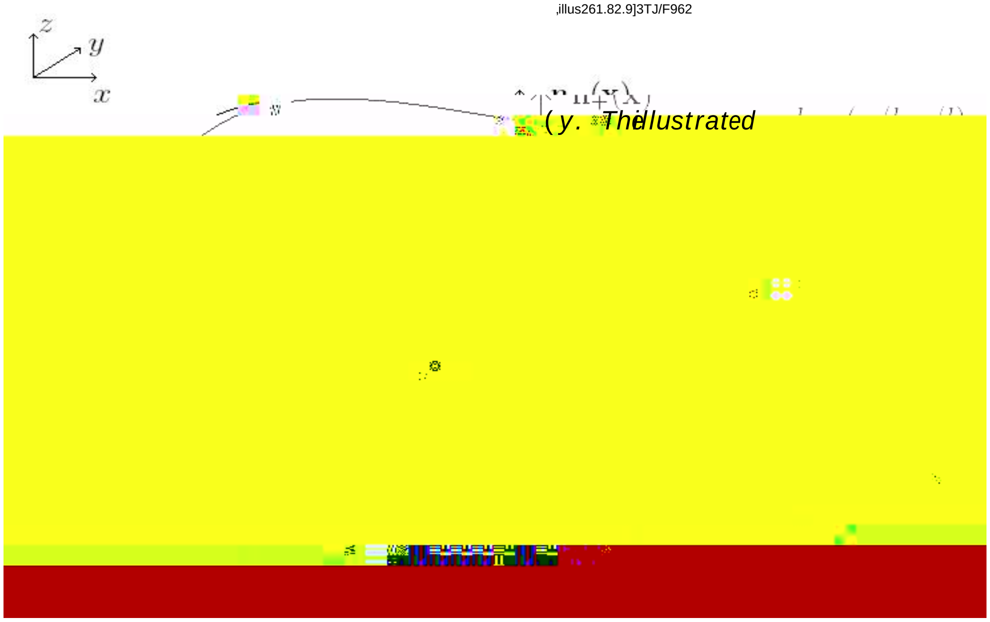

Fig. 2.1. *Porous layer $M \subset \mathbb{R}^3$, with impermeable boundary ∂M*

extend from the lower surface to the upper surface of M, are located at

$$(x_i;y_i) \in \Omega_l; \quad i = 1;\ldots;N.$$

The functions s_i : $z \quad \frac{x_i}{l}; \frac{y_i}{l} \;; z_+^i$

for each $(\mathrm{r}; t)$ 2 M $[0; 1)$. The permeability tensor $\underline{D}(x/l; y/l; z/h)$ has the form

$$\underline{D}$$

Here $\partial M_H \subset \partial M$ is that part of ∂M representing the side walls of the boundary, whilst $\partial M_+, \partial M_- \subset \partial M$ represent the upper and lower surfaces of ∂M respectively, with $\partial M_+ \cup \partial M_- \cup \partial M_H = \partial M$. In addition, $\hat{n}_l(x, y)$ for $(x, y) \in \partial\Omega_l$ represents the outward unit normal field to $\partial\Omega_l$. Finally we have the initial condition,

$$p(\mathbf{r}, 0) = p_0 f\left(\frac{x}{l}, \frac{y}{l}, \frac{z}{h}\right) - \rho_0 g z; \quad \text{for all } (x, y, z) \in M;$$

with $f : M \mapsto \mathbb{R}$ the prescribed initial pressure variation, with

$$f \in PC^1(M) \cap C(M);$$

where $PC^1(M)$ represents the class of piecewise continuously differentiable functions on M. We now set

$$Q = \sum_{i=1}^{N} |Q_i| \quad (> 0).$$

The natural scales for the problem are then $x, y \sim l$ and z Ch]TJ/F8 9.9626 Tf 5.026 Tf 3.872of pJ/F14 9.9626 Tf

0.0)]TJ/F11 9.]T40/510.516 R54 Td 7 Td [(X)]TJ7619534.1136.519 30 -2.90.086]TJ/F11 9.]T40/510.516 54 Td
w100 m1f18.47FTf l SQBT)

with closure $\bar{M}'$ and boundary $\partial M'$. The line source/sink locations are at $(x_i, y_i) \in {}_1$, $i = 1, \ldots, N$. The volume ux conditions (2.5) become,

$$
{}_i = \int_{z_-(x_i,y_i)}^{z_+(x_i,y_i)} s_i(\ ) \, d \ , \quad i = 1, \ldots, N,
$$

where

$$
{}_i = \frac{Q_i}{Q}, \quad i = 1, \ldots, N,
$$

so that

$$
|\ {}_i| = \frac{|Q_i|}{Q} \quad 1, \text{ for } i = 1, \ldots, N, \quad \text{and} \quad \sum_{i=1}^{N} |\ {}_i| = 1.
$$

The boundary conditions (2.11) become, in dimensionless form,

(2.18) $\quad (u(\mathbf{r},t), v(\mathbf{r},t), w(\mathbf{r},t)) \cdot \hat{\mathbf{n}}_1 = 0, \quad$ for all $(\mathbf{r},t) \in \partial M'_H \quad (0, \infty)$;

(2.19) $\quad w(\mathbf{r},t) \quad \{z_{+x}(x,y)u(\mathbf{r},t) + z_{+y}(x,y)v(\mathbf{r},t)\} = 0$; for all $(\mathbf{r},t) \in \partial M'_+ \quad (0, \infty)$;

(2.20) $\quad w(\mathbf{r},t) \quad \{z_{\ x}(x,y)u(\mathbf{r},t) + z_{\ y}(x,y)v(\mathbf{r},t)\} = 0$; for all $(\mathbf{r},t) \in \partial M' \quad (0, \infty)$.

Finally we have the initial condition,

(2.21) $\quad p(\mathbf{r},0) = p_0 f(\mathbf{r}), \quad$ for all $\mathbf{r} \in \bar{M}'$,

and $p_0 = p_0 h D_0^H / Q$. The full problem for consideration is now given by (2.13)–(2.16), (2.18)–(2.21), which we refer to as [IBVP]. To proceed it is convenTf 4.718 0 Td [(;)-167(u51,)-390(19(T50)]TJ/F15 9.5 -0.u

$$
(2.23)\qquad \left.\begin{aligned}
\hat u &= -D_x(x,y,z)\hat p_x\\
\hat v &= -D_y(x,y,z)\hat p_y\\
\epsilon^2\hat w &= -D_z(x,y,z)\hat p_z
\end{aligned}\right\}\quad (x,y,z)\in M',
$$

$$
(2.24)\qquad (\hat u(\mathbf{r}),\hat v(\mathbf{r}),\hat w(\mathbf{r}))\cdot\hat{\mathbf{n}}_1 = 0,\quad \text{for all } \mathbf{r}\in\partial M'_H,
$$

$$
(2.25)\qquad \hat w(\mathbf{r}) - \{z_{+x}(x,y)\hat u(\mathbf{r}) + z_{+y}(x,y)\hat v(\mathbf{r})\} = 0,\quad \text{for all } \mathbf{r}\in\partial M'_+,
$$

$$
(2.26)\qquad \hat w(\mathbf{r}) - \{z_{-x}(x,y)\hat u(\mathbf{r}) + z_{-y}(x,y)\hat v(\mathbf{r})\} = 0,\quad \text{for all } \mathbf{r}\in\partial M'_-,
$$

which we will refer to as [PSSP]. Corresponding to (i)–(iii) a solution to [PSSP] has the following regularity:

(Pi) $\hat p\in C^1(M'\setminus d)\cap C^2(M'\setminus d)$;

(Pii) $\lim_{R_i\to 0}[R_i|\underline{D}\,\nabla\hat p|]$ exists uniformly for $z\in[z_-(x_i,y_i)$

$$(2.33)\qquad [\underline{D}(\mathbf{r})\nabla\tilde{p}(\mathbf{r},t)]\cdot(-\varepsilon^2 z_{-x}(x,y), -\varepsilon^2 z_{-y}(x,y), 1) = 0, \quad \text{for all } (\mathbf{r},t)\in\partial M'_-\times(0,\infty);$$

$$(2.34)\qquad \tilde{p}(\mathbf{r},0) = p_0 f(\mathbf{r}) - \hat{p}(\mathbf{r}) = \tilde{p}_0(\mathbf{r}), \quad \text{for all } \mathbf{r}\in M',$$

with regularity

$$(2.35)\qquad \tilde{p}\in C((\overline{M'}\times[0,\infty))\setminus(d\times\{0\}))\cap C^1(\overline{M'}\times(0,\infty))\cap C^2(M'\times(0,\infty)),$$

after which

$$\left.\begin{aligned} \tilde{u} &= -D_x(x,y,z)\tilde{p}_x,\\ \tilde{v} &= -D_y(x,y,z)\tilde{p}_y,\\ \varepsilon^2\tilde{w} &= -D_z(x,y,z)\tilde{p}_z, \end{aligned}\right\} \quad (\mathbf{r},t)\in M'\times(0,\infty).$$

Here

$$(2.36)\qquad \underline{D}(\mathbf{r}) = \begin{pmatrix} D_x(x,y,z) & 0 & 0 \\ 0 & D_y(x,y,z) & 0 \\ 0 & 0 & D_z(x,y,z) \end{pmatrix},$$

for all $\mathbf{r}\in M'$. The strongly parabolic problem (2.30)–(2.35) has a unique solution in $M'\times[0,\infty)$ (see for example [8, Chapter 3]), and we now construct this solution. To this end we first consider the following self-adjoint eigenvalue problem in M',

$$(D_x(x,y,z)\psi_x)_x + (D_y(x,y,z)\psi_y)_y + \varepsilon^{-2}(D_z(x,y,z)\psi_z)_z + \lambda\psi(x,y,z) = 0, \quad \text{for } (x,y,z)\in M';$$

$$[\underline{D}(\mathbf{r})\nabla\psi(\mathbf{r})]\cdot\hat{\mathbf{n}}_1 = 0, \quad \text{for all } \mathbf{r}\in\partial M'_H;$$

$$[\underline{D}(\mathbf{r})\nabla\psi(\mathbf{r})]\cdot(-\varepsilon^2 z_{+x}(x,y), -\varepsilon^2 z_{+y}(x,y), 1) = 0, \quad \text{for all } \mathbf{r}\in\partial M'_+;$$

$$[\underline{D}(\mathbf{r})\nabla\psi(\mathbf{r})]\cdot(-\varepsilon^2 z_{-x}(x,y), -\varepsilon^2 z_{-y}(x,y), 1) = 0, \quad \text{for all } \mathbf{r}\in\partial M'_-.$$

with $a_0(\ ) = 0$, via (2.29), and

$$a_j(\ ) = \iiint_{M'} \bar{p}_0(u;v;w)\ \ (u;v;w)\ \ {}_j(u;v;w;\ )\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \tag{2.39}$$

for $j = 1;2;:::$. We observe immediately from (2.38), with (2.37), that

$$\bar{p}(\mathrm{r};t) \to 0 \quad \text{as } t \to \infty;$$

uniformly for $\mathrm{r} \in M'$, and that, in addition,

$$\bar{p}_x(\mathrm{r};t); \bar{p}_y(\mathrm{r};t); \bar{p}_z(\mathrm{r};t) \to 0 \quad \text{as } t \to \infty;$$

uniformly for $\mathrm{r} \in M'$. In fact, we have established:

Theorem 2.2. *For each $\ > 0$, [IBVP] has a unique solution $u;v;w;p : M' \ \ [0;\infty) \mapsto \mathrm{R}$ given by*

$$\begin{aligned} p(\mathrm{r};t) &= \hat{\ }_T t + \hat{p}(\mathrm{r}) + \bar{p}(\mathrm{r};t); \\ u(\mathrm{r};t) &= \hat{u}(\mathrm{r}) \quad D_x(\mathrm{r})\bar{p}_x(\mathrm{r};t); \\ v(\mathrm{r};t) &= \hat{v}(\mathrm{r}) \quad D_y(\mathrm{r})\bar{p}_y(\mathrm{r};t); \\ w(\mathrm{r};t) &= \hat{w}(\mathrm{r}) \quad\ \ ^2 D_z(\mathrm{r})\bar{p}_z(\mathrm{r};t); \end{aligned}$$

for all $(\mathrm{r};t) \in M' \ \ [0;\infty)$. Here $\bar{p} : M' \ \ [0;\infty) \mapsto \mathrm{R}$ is given by (2.38), (2.39), and $\hat{u};\hat{v};\hat{w};\hat{p} : M' \mapsto \mathrm{R}$ is that solution to [PSSP] which satisfies the constraint (2.28). Moreover

$$\begin{aligned} p(\mathrm{r};t) &= \hat{\ }_T t + \hat{p}(\mathrm{r}) + O(\mathrm{e}^{\ \ _1(\ )t}); \\ u(\mathrm{r};t) &= \hat{u}(\mathrm{r}) + O(\mathrm{e}^{\ \ _1(\ )t}); \\ v(\mathrm{r};t) &= \hat{v}(\mathrm{r}) + O(\mathrm{e}^{\ \ _1(\ )t}); \\ w(\mathrm{r};t) &= \hat{w}(\mathrm{r}) + O(\mathrm{e}^{\ \ _1(\ )t}); \end{aligned}$$

as $t \to \infty$, uniformly for $\mathrm{r} \in M'$.

To complete the solution to the problem [IBVP] we must determine $\ _n(\ )\ (> 0)$ and its corresponding eigenfunction $\ _n : M' \mapsto \mathrm{R}$ for each $n = 1;2;:::$, together with the pseudo-steady state $\hat{p};\hat{u};\hat{v};\hat{w} : M' \mapsto \mathrm{R}$ which satisfies the constraint (2.28). In the next two sections we thus focus attention on the study of [PSSP] and [EVP] in turn.

In particular, for a thin porous layer, the parameter $\ $, which measures the aspect ratio of the layer, is small, provided that

$$\frac{h}{l} \qquad \left(\frac{D_0^L}{D_0^H}\right)^{1=2};$$

which we will take to be the case. Thus $0 < \qquad 1$, and in the next two sections we will consider the structure of the solutions to [PSSP] and [EVP] in the asymptotic limit $\ \to 0$, via the method of matched asymptotic expansions.

3. Asymptotic solution to the pseudo-steady state problem [PSSP] as $\ \to 0$. In this section we develop the uniform asymptotic structure to the solution of the pseudo-steady state problem [PSSP] (given by (2.22)–(2.26)) in the limit $\ \to 0$, via the method of matched asymptotic expansions. We recall that existence and uniqueness, for each > 0, follows from Theorem 2.1, and, following Theorem 2.2, we require that solution to [PSSP] which satisfies the constraint

$$\iiint_{M'} \hat{p}(x;y;z)\ \ (x;y;z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = I_0; \tag{3.1}$$

where the constant I_0 is given by

$$I_0 = p_0 \iiint_{M'} f(x;y;z)\ \ (x;y;z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Due to the initial scalings in the nondimensionalization (2.12), we anticipate that $\hat{p};\hat{u};\hat{v};\hat{w} : M' \mapsto \mathbb{R}$ are such that

$$\hat{p};\hat{u};\hat{v};\hat{w} = O(1) \tag{3.2}$$

as $\to 0$, uniformly for,

$$\mathbf{r} \in M' \setminus \bigcup_{i=1}^{N} {}_i = N';$$

where ${}_i$ is an O( ) neighbourhood of d_i, for each $i = 1, \ldots, N$. Thus, following (3.2), we introduce the outer region (N') asymptotic expansions

$$\hat{p}(\mathbf{r}; \ ) = \hat{p}_0(\mathbf{r}) + \ \hat{p}_1(\mathbf{r}) + O(\ ^2);$$
$$\hat{u}(\mathbf{r}; \ ) = \hat{u}_0(\mathbf{r}) + \ \hat{u}_1(\mathbf{r}) + O(\ ^2);$$

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with boundary condition (3.8) requiring

(3.13) $D_x(x,y,z)A_x(x,y)n_x(x,y) + D_y(x,y,z)A_y(x,y)n_y(x,y) = 0, \quad \mathbf{r} \in \partial M'_H,$

where we have written

$$\hat{\mathbf{n}}_1(\mathbf{r}) = (n_x(x,y), n_y(x,y), 0), \quad \mathbf{r} \in \partial M'_H.$$

We next substitute from (3.12) into (3.4) which becomes

$$\hat{w}_{0z} = \sum_{i=1}^{N} s_i(z)\,\delta(x - x_i)\,\delta(y - y_i) - \hat{}_T\,(x,y,z)$$

(3.14) $$+[D_x(x,y,z)A_x(x,y)]_x + [D_y(x,y,z)A_y(x,y)]_y, \quad (x,y,z) \in M'.$$

A direct integration of (3.14), together with an applicationoundary 1.494condi-051 for some positive constant $\wedge_m$

(Bi) $A \in C^1(\bar{\Omega} \setminus \cup_{i=1}^N \mathbf{x}_i) \cap C^2(\Omega \setminus \cup_{i=1}^N \mathbf{x}_i)$;

(Bii) $\lim_{R_i \to 0}[R_i |\underline{\hat{D}} \nabla A|]$ exists uniformly for $\theta \in [0, 2\pi)$; $i = 1, \ldots, N$;

(Biii) $\lim_{R_i \to 0} R_i \int_0^{2\pi} (\underline{\hat{D}} \nabla A) \cdot \underline{\hat{R}}_i \, d\theta = -\Gamma_i$; $i = 1, \ldots, N$.

Here $\mathbf{x}_i = (x_i, y_i) \in \Omega$, $i = 1, \ldots, N$, and (R_i, θ) and $\underline{\hat{R}}_i$ are as defined in §2 (and can now be regarded as plane polar coordinates on Ω based at $(x, y) = (x_i, y_i)$).

Remark 3.1. *It follows from classical theory for strongly elliptic boundary value problems (see for example [8]) that [BVP] has a unique solution.*

In particular, with $A : \bar{\Omega} \mapsto \mathbb{R}$ being the solution to [BVP], we have

$$A(x, y) = -\frac{\Gamma_i}{4\pi (D_x^i D_y^i)^{\frac{1}{2}}} \log\left[\frac{(x - x_i)^2}{D_x^i} + \frac{(y - y_i)^2}{D_y^i}\right] + A_0^i + O\left(([x - x_i]^2 + [y - y_i]^2)^{\frac{1}{2}}\right); \tag{3.18}$$

as $(x, y) \to (x_i, y_i)$, with $A_0^i \in \mathbb{R}$ being a globally determined constant, and $i = 1, \ldots, N$

We now proceed to $O(\epsilon)$. The problem for $\hat{p}_1, \hat{u}_1, \hat{v}_1, \hat{w}_1 : M' \mapsto \mathbb{R}$ is similar to the leading order problem and is not repeated here. We obtain

$$
\begin{aligned}
\hat{p}_1(x,y,z) &= B(x,y), \quad (x,y,z) \in M', \\
\hat{u}_1(x,y,z) &= -D_x(x,y,z)B_x(x,y), \quad (x,y,z) \in M', \\
\hat{v}_1(x,y,z) &= -D_y(x,y,z)B_y(x,y), \quad (x,y,z) \in M', \\
\hat{w}_1(x,y,z) &= \int_{z_-(x,y)}^{z} \{(D_x(x,y,\zeta)B_x(x,y))_x + (D_y(x,y,\zeta)B_y(x,y))_y\}\, d\zeta \\
&\quad - z\,\psi_x(x,y)D_x(x,y,z\psi(x,y))B_x(x,y) - z\,\psi_y(x,y)D_y(x,y,z\psi(x,y))B_y(x,y), \\
&\qquad \text{for } (x,y,z) \in M',
\end{aligned}
\tag{3.22}
$$

where $B : \Omega \mapsto \mathbb{R}$ is the solution to the strongly elliptic boundary value problem,

$$
\begin{aligned}
\hat{\nabla}\cdot(\underline{\hat{D}}(\mathbf{r})\hat{\nabla}B) &= 0, \quad \mathbf{r} \in \Omega, \\
(\underline{\hat{D}}(\mathbf{r})\hat{\nabla}B)\cdot\hat{\mathbf{n}}(\mathbf{r}) &= 0, \quad \mathbf{r} \in \partial\Omega, \\
\int\!\!\int_\Omega \hat{\psi}(\mathbf{r})B(\mathbf{r})\, dx\, dy &= 0.
\end{aligned}
$$

The unique solution $B \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ is given by

$$B(x,y) = 0, \quad (x,y) \in \Omega,$$

and so

$$\hat{p}_1(x,y,z) = \hat{u}_1(x,y,z) = \hat{v}_1(x,y,z) = \hat{w}_1(x,y,z) = 0, \quad (x,y,z) \in M',$$

via (3.22). The outer region asymptotic expansions are now complete to $O(\epsilon^2)$, and we have

$$
\begin{aligned}
\hat{p}(\mathbf{r};\epsilon) &= A(x,y) + O(\epsilon^2), \\
\hat{u}(\mathbf{r};\epsilon) &= -D_x(x,y,z)A_x(x,y) + O(\epsilon^2), \\
\hat{v}(\mathbf{r};\epsilon) &= -D_y(x,y,z)A_y(x,y) + O(\epsilon^2), \\
\hat{w}(\mathbf{r};\epsilon) &= \int_{z_-(x,y}^{z}
\end{aligned}
$$

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as $\to 0$. Thus, in the inner region we write,

$$(x, y) = (x_i, y_i) + \ (X, Y), \qquad (3.24)$$

with $(X, Y) \in \mathbb{R}^2$ such that $X, Y = O(1)$ as $\to 0$, together with

$$\hat{p} = \frac{i}{2\ (D_x^i D_y^i)^{1/2}} \log \ + P, \quad \hat{u} = \ ^{1}U, \quad \hat{v} = \ ^{1}V, \quad \hat{w} = \ ^{2}W, \qquad (3.25)$$

)

where $P, U, V, W : \mathbb{R}^2 \ [z\ (x_i, y_i), z_+(x_i, y_i)] \mapsto \mathbb{R}$ are such that $P, U, V, W = O(1)$ as $\to 0$. We now introduce inner region asymptotic expansions as

$$\begin{aligned} P(X, Y, z;\ ) &= P_0(X, Y, z) + O(\ ), \\ U(X, Y, z;\ ) &= U_0(X, Y, z) + O(\ ), \\ V(X, Y, z;\ ) &= V_0(X, Y, z) + O(\ ), \\ W(X, Y, z;\ ) &= W_0(X, Y, z) + O(\ ), \end{aligned} \qquad (3.26)$$

as $\to 0$, $(X, Y, z) \in \mathbb{R}^2 \ [z\ (x_i, y_i), z_+(x_i, y_i)]$. We substitute from (3.24)–(3.26) into the full problem [PSSP], to obtain the leading order problem as) 1) 1599 823• 167[cm []0 d 0.5.398 w[(m 8.52

$$U_{0X} + V_{0Y} + W_{0z} = s_i(z)\ (X)\ (Y), \qquad (3.27)$$

$$\begin{aligned} U_0 &= \ D_x(z) P_{0X}, \\ V_0 &= \ D_y(z) P_{0Y}, \quad (X, Y, z) \in D, \\ W_0 &= \ D_z(z) P_{0z}, \end{aligned} \qquad (3.28)$$

$$W_0(X, Y, z_+^i) = 0, \quad W_0(X, Y, z^i\ ) = 0, \quad (X, Y) \in \mathbb{R}^2. \qquad (3.29)$$

Here $z^i\ = z\ (x_i, y_i)$, $D = \mathbb{R}^2\ (z^i\ , z_+^i)$ and $D\ (z) = D\ (x_i, y_i, z)$, for $z \in [$

for all $z \in [z^{i}, z^{i}_{+}]$.

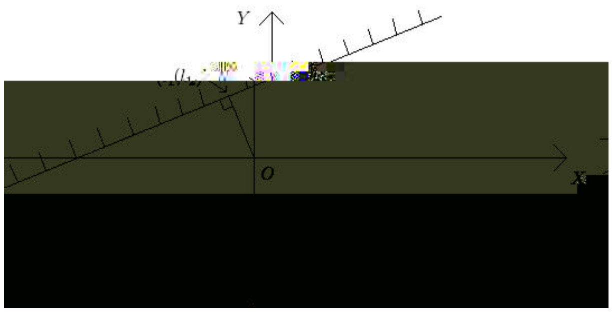
We can now eliminate U_0, V_0 and W_0 via (3.28), and obtain the following strongly elliptic problem for P_0, namely, for $j;k = 0;1;2;\ldots$. The constants B_r, $r = 1;2;\ldots$ are given by

$$B_r = \frac{1}{2} \int_{z_-^i}^{z_+^i} s_i(s) \quad {}_r(s)\,\mathrm{d}s; \qquad r = 1;2;\ldots: \tag{3.36}$$

The functions U_0, V_0 and W_0 are now obtained directly from (3.28) via (3.35) and (3.36). The only remaining question is how to actually compute the eigenvalues and corresponding eigenfunctions of [SL]. This is straightforward and is addressed in [13]. The solution to the leading order problem is now complete. The asymptotic expansion for $\hat{p}$ in the inner region is thus,

$$\hat{p}(R_i;z;\ ) = \frac{i}{2\ D_h^i} \log \quad + F_i(R_i;z) + O(\ )$$

as $\to 0$, with $(R_i;z) \in (0;\infty) \quad [z^i;z_+^i]$, and $F_i(R_i;z)$ given by (3.35). To obtain an approximation to $\hat{p}$ close to the i



Y
O
X

with $F_k(;)$ as defined in (3.35) and $F'_k(;)$ obtained from $F_k(;)$ by replacing $_k$ with $_{k+1}$ and $s_k( )$ with $s_{k+1}( )$, whilst

$$R_k = (X^2 + Y^2)^{1/2}; \quad R'_k = ((X \quad l_1)^2 + (Y \quad l_2)^2)^{1/2}.$$

The asymptotic solution to [PSSP] as $\to 0$ uniformly for $(x, y, z) \in M'$ is now complete. We next turn our attention to the eigenvalue problem [EVP].

4. Asymptotic solution to the eigenvalue problem [EVP] as $\to 0$. In this section we develop the asymptotic solution to the eigenvalue problem [EVP] as $\to 0$. As in [13], we first employ the theory developed by Ramm [18] to establish that the set of eigenvalues to [EVP], (2.37), with > 0, splits into two disjoint subsets as $\to 0^+$, which we denote by

$$S_+ = \quad {}^{+}_{1}( ), \; {}^{+}_{2}( ), \ldots \quad ; \qquad S \quad = \quad {}_{0}$$

with $\bar{A} : {}_1 \mapsto \mathbb{R}$ such that $\bar{A} \in C^1({}_1) \cap C^2({}_1)$. Boundary condition (4.5) then requires, after an integration,

$$\underline{\hat{D}}(\mathbf{r})\hat{\nabla}\bar{A} \cdot \hat{\mathbf{n}}_1(\mathbf{r}) = 0; \quad \mathbf{r} \in \partial{}_1;$$

where $\mathbf{r} = (x; y) \in \partial{}_1$ and $\underline{\hat{D}}(\mathbf{r})$ is defined in (3.17), whilst $\hat{\nabla} = (\frac{\partial}{\partial x}; \frac{\partial}{\partial y})$. At $O({}^2)$ we obtain an inhomogeneous version of (4.4)–(4.7). As in [13], the solvability requirement on this inhomogeneous boundary value problem provides a strongly elliptic partial differential equation which must be satisfied by $\bar{A}(\mathbf{r})$, $\mathbf{r} \in {}_1$, namely

$$\hat{\nabla} \cdot \left(\underline{\hat{D}}(\mathbf{r})\hat{\nabla}\bar{A}\right) + \tilde{}\,\hat{}(\mathbf{r})\bar{A} = 0; \quad \mathbf{r} \in {}_1.$$

Thus $\bar{A} : \ \mapsto \mathbb{R}$ and $\tilde{} \in \mathbb{R}$ satisfy the regular self-adjoint eigenvalue problem,

$$\hat{\nabla} \cdot \left(\underline{\hat{D}}(\mathbf{r})\hat{\nabla}\bar{A}\right) + \tilde{}\,\hat{}(\mathbf{r})\bar{A} = 0; \quad \mathbf{r} \in \ ;$$

$$\underline{\hat{D}}(\mathbf{r})\hat{\nabla}\bar{A} \cdot \hat{\mathbf{n}}(\mathbf{r}) = 0; \quad \mathbf{r} \in \partial \ ;$$

where the subscripts on ${}_1$, $\partial{}_1$, $\hat{\mathbf{n}}_1(\mathbf{r})$ have been dropped for convenience, and we recall that $\hat{} : \ \mapsto \mathbb{R}$ is given by $\hat{}(x; y) = \int_{z_-(x;y)}^{z_+(x;y)} (x; y; \ )\, d$, for all $(x; y) \in$.

We refer to this eigenvalue problem as [EVP]$'$. Now, established theory (see for example [19]) determines that the set of eigenvalues of [EVP]$'$ is given by $\tilde{} = {}^2$

to time t as $t \to \infty$. Generalisations to cases where a line source or sink is near a boundary wall, or where line sources and sinks are not well spaced, have also been considered, in §3.1 and §3.2 respectively.

For a full description of the entire computational procedure required to obtain numerical approximations to the pressure and flow fields throughout the layer, and examples demonstrating the application of the theory to some simple situations, we refer to [12]. We finally remark that since the initial-boundary value problem is solved for a general C^1 initial condition, the effect of time dependent transient effects due to temporal changes in the well discharge rates can easily be accounted for.

REFERENCES

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, 1972.

[2] A-J. A. Al-Khalifah, K. Aziz, and R. N. Horne, *A new approach to multiphase well test analysis*, Soc. Petroleum Engineers, 16743 (1987).

[3] L. V. Branets, S. S. Ghai, S. L. Lyons, and X.-H. Wu, *Challenges and technologies in reservoir modeling*, Communications in Computational Physics, 6 (2009), pp. 1–23.

[4] G. S. Busswell, R. Banerjee, R. K. M. Thambynayagam, and J. B. Spath, *Generalized analytical solution for reservoir problems with multiple wells and boundary conditions*, Soc. Petroleum Engineers, 99288 (2006).

[5] Z. Chen, *Reservoir Simulation: Mathematical Techniques in Oil Recovery*, SIAM, 2007.

[6] H. Cinco-Ley, F. Samaniego-V, and A. N. Dominguez, *Transient pressure behavior for a well with a finite-conductivity vertical fracture*, Soc. Petroleum Engineers J., 18 (1978), pp. 253–264.

[7] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[8] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[9] J. P. Gilchrist, G. S. Busswell, R. Banerjee, J. B. Spath, and R. K. M. Thambynayagam, *Semi-analytical solution for multiple layer problems with multiple vertical, horizontal, deviated and fractured wells*, in Proceedings of the International Petroleum Technology Conference, Dubai, U.A.E., SPE, Houston, TX, 2007. paper 11718.

[10] E. Gomes and Z. A. Reza, *A new semi-analytic pressure transient model for layered gas reservoir under various reservoir and well conditions*, Soc. Petroleum Engineers, 39526 (1998).

[11] A. Gringarten and H. J. Ramey Jr, *The use of source and Green's functions in solving*