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Preprint MPS_2010-33

15 November 2010

A collection ideally combines problems artificially constructed to reflect a wide range of possible properties with problems representative of real applications. Problems for which something is

2 Identifiers

We give in Table 1 a list of identifiers for the types of problems available in the collection and in

Table 1: Problems available in the collection and their identifiers.

qep	quadratic eigenvalue problem
pep	polynomial eigenvalue problem
rep	rational eigenvalue problem
nep	other nonlinear eigenvalue problem

Table 2: List of identifiers for the problem properties.

nonregular	symmetric	hyperbolic
real	hermitian	elliptic
nonsquare	T-even	overdamped
sparse	*-even	proportionally-damped
scalable	T-odd	gyroscopic
parameter-dependent	*-odd	
solution	T-palindromic	
	*-palindromic	
	T-anti-palindromic	
	*-anti-palindromic	

2.2 Some De nitions and Properties

Nonlinear eigenproblems are said to be egu r if m = n and det(F(λ)) 0, and non egu r otherwise. Recall that a regular PEP possesses nk (not necessarily distinct) eigenvalues [31], including infinite eigenvalues. As the majority of problems in the collection are regular we identify only nonregular problems, for which the identifier is nonregular.

The identifiers real, hermitian, and symmetric are defined in Table 3. For PEPs, the real identifier corresponds to P having real coe□cient matrices, while hermitian corresponds to Hermitian (but not all real) coe□cient matrices. Similarly, symmetric indicates (complex) symmetric coe□cient matrices, and the real identifier is added if the coe□cient matrices are real symmetric. For problems that are parameter-dependent the identifiers real and hermitian are used if the problem is real or Hermitian for real values of the parameter.

Definitions of identifiers for odd-even and palindromic-like square matrix polynomials, together with the special symmetry properties of their spectra (see [63]) are given in Table 4.

Gy oscopic systems of the form $Q(\lambda) = \lambda^2 M + \lambda G + K$ with M, K Hermitian, $M > 0$, and $G = \; G^*$ skew-Hermitian are a subset of -even (T-even when the coe□cient matrices are real) QEPs ew-.345(o)0.43374782(p)-0.575882(r)-1.06354(o)0.43374,.433A69(0706(n)4(s)-372.9560.4656078s,784.45.2378(3

Table 5: Quadratic eigenvalue problems.

Table 11: Problems in NLEVP.

acoustic_wave_1d	Acoustic wave problem in 1 dimension.
acoustic_wave_2d	Acoustic wave problem in 2 dimensions.
bicycle	2-by-2 QEP from the Whipple bicycle model.
bilby	5-by-5 QEP from bilby population model.
butterfly	Quartic matrix polynomial with T-even structure.
cd_player	QEP from model of CD player.
closed_loop	2-by-2 QEP associated with closed-loop control system.
concrete	Sparse QEP from model of a concrete structure.
damped_beam	QEP from simply supported beam damped in the middle.
dirac	QEP from Dirac operator.
fiber	NEP from fiber optic design.
foundation	Sparse QEP from model of machine foundations.
gen_hyper2	Hyperbolic QEP constructed from prescribed eigenpairs.
gun	NEP from model of a radio-frequency gun cavity.
hadeler	NEP due to Hadeler.
intersection	10-by-10 QEP from intersection of three surfaces.
hospital	QEP from model of Los Angeles Hospital building.
loaded_string	REP from finite element model of a loaded vibrating string.
metal_strip	QEP related to stability of electronic model of metal strip.
mobile_manipulator	QEP from model of 2-dimensional 3-link mobile manipulator.
omnicam1	9-by-9 QEP from model of omnidirectional camera.
omnicam2	15-by-15 QEP from model of omnidirectional camera.
orr_sommerfeld	Quartic PEP arising from Orr-Sommerfeld equation.
pdde_stability	QEP from stability analysis of discretized PDDE.
plasma_drift	Cubic PEP arising in Tokamak reactor design.
power_plant	8-by-8 QEP from simplified nuclear power plant problem.
qep1	3-by-3 QEP with known eigensystem.
qep2	3-by-3 QEP with known, nontrivial Jordan structure.
qep3	3-by-3 parametrized QEP with known eigensystem.
qep4	3-by-4 QEP with known, nontrivial Jordan structure.
railtrack	QEP from study of vibration of rail tracks.
railtrack2	Palindromic QEP from model of rail tracks.
relative_pose_5pt	Cubic PEP from relative pose problem in computer vision.
relative_pose_6pt	QEP from relative pose problem in computer vision.
schrodinger	QEP from Schrodinger operator.
shaft	QEP from model of a shaft on bearing supports with a damper.
sign1	QEP from rank-h2(h)-5529R3992t49(f)n706(r)-335.318(v)-0.573431(i)-0.333276(s)-0.77437

[19]. Here, f is the frequency, c is the speed of sound in the medium, and ζ is the (possibly complex) impedance. We take c = 1 as in [19]. The eigenvalues of Q are the resonant frequencies of the system, and for the given problem formulation they lie in the upper half of the complex plane.

On the 1D domain [0, 1] the n n matrices are defined by

$$M = 4\pi^2 \frac{1}{n}(I_n \quad \frac{1}{2}e_n e_n^T), \quad C$$

Butte ry `pep,real,parameter-dependent,T-even,scalable` **. This is a quartic matrix polynomial $P(\lambda) = \lambda^4 A_4$ λ**

ound tion pep,qep,symmetric,sparse . **This is a quadratic matrix polynomial**

Lo ded st ing `rep,real,symmetric,parameter-dependent,scalable` **. This rational eigenvalue problem arises in the finite element discretization of a boundary problem describing the**

O. nic . pep,qep,real . **This is a 9 9 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ arising from a model of an omnidirectional camera (one with a**

k	1	2	3	4	5	6
λ_k	0	1	1 + [illegible]	2	3	
$\mathbf{x}_k$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\ \frac{-1}{+1}\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\end{bmatrix}$.

For the default value of the parameter, [illegible] = 1 + $2^{-53/2}$, the first and third eigenvalues are ill conditioned.

Q [illegible] `pep,qep,nonregular,nonsquare,real,solution` . This is the 3 4 quadratic matrix polynomial [16, Ex.2.5]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$ ±0 0i4s i

e tive pose pt `pep,qep,real` **. The quadratic matrix polynomial $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$, where $A_i \in \mathbb{R}^{10\times 10}$, comes from the six point relative pose problem in computer vision [57]. In this problem the images of six unknown scene points taken with a camera of unknown focal length from two distinct unknown camera viewpoints are given and it is required to determine the possible solutions for the relative configuration of the points and cameras. The solutions to the problem are obtained from the last three components of the finite eigenvectors of P.**

Sch ödinge r

Spe ke r o `pep,qep,real,symmetric` **.** The quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M, C, K \in \mathbb{R}^{107\times 107}$, is from a finite element model of a speaker box that includes both structural finite elements, representing the box, and fluid elements, representing the air contained in the box [55, Ex. 5.5]. The matrix coe□cients are highly structured and sparse. There is a large variation in the norms: $\|M\|_2 = 1$, $\|C\|_2 = 5.7 \times 10^{-2}$, $\|K\|_2 = 1.0 \times 10^{7}$.

Sp ing `pep,qep,real,symmetric,proportionally-damped,parameter-dependent,scalable` **.**

i es w pep,qep,real,t-even,gyroscopic,parameter-dependent,scalable . This gyroscopic QEP arises in the vibration analysis of a wiresaw [82]. It takes the form $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$, where the $n \quad n$ coe□cient matrices are defined by

$$M = I_n/2, \quad K = \operatorname*{diag}_{1\le j\le n}\left(j^2\pi^2(1 \quad v^2)/2\right),$$

and

$$C = \quad C^T = (c_{jk}), \quad \text{with} \quad c_{jk} = \begin{cases} \dfrac{4jk}{j^2 \quad k^2}v, & \text{if } j+k \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Here, v is a real nonnegative parameter corresponding to the speed of the wire. Note that for $0 < v < 1$, K is positive definite a.450903(h)-0.575882(e)-4171960796(e)-0...433749(t)-0.448453(i)-0.33817dedu-417.8

```
>> [coeffs,fun] = nlevp('fiber');
>> [f,fp,fpp] = fun(0.5)
f =
  1.0000e+000 -5.0000e-001 -7.0746e-001
fp =
            0 -1.0000e+000 -7.0725e-001
fpp =
            0            0  7.0696e-001
```

Problems and their properties are stored in a simple database made from cell arrays. The database is accessed with the `query` function in the `private` directory, which is invoked using the `query` argument to `nlevp`. For example, the properties for the Butterfly problem are returned in a cell array by the following call (whose syntax illustrates the command/function duality of MATLAB [39, Sec. 7.5]):

```
>> nlevp query butterfly
ans =
    'pep'
    'real'
    'parameter-dependent'
    'T-even'
    'scalable'
```

A more sophisticated example finds the names of all PEPs of degree 3 or higher:

```
>> pep = nlevp('query','pep'); qep = nlevp('query','qep');
>> pep_cubic_plus = setdiff(pep,qep)
pep_cubic_plus =
    'butterfly'
    'orr_sommerfeld'
    'plasma_drift'
    'relative_pose_5pt'
```

The cell array `pep_cubic_plus` can then easily be used to extract these problems. For example, the first problem in `pep_cubic_plus` can be solved using

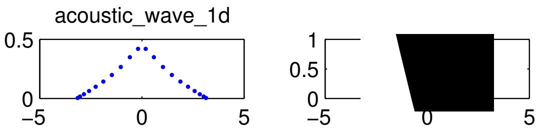
```
coeffs = nlevp(pep_cubic_plus{1}); [X,e] = polyeig(coeffs{:});
```

Table 5–10 were generated automatically in MATLAB using appropriate `nlevp('query',...)` calls.

The toolbox function `nlevp_example.m` provides a test that the toolbox is correctly installed. It solves all the PEPs in the collection of dimension less than 500 using MATLAB's `polyeig` and then plots the eigenvalues. It produces Figure 1 and output to the command window that begins as follows:

```
NLEVP contains 46 problems in total,
of which 42 are polynomial eigenvalue problems (PEPs).
Run POLYEIG on the PEP problems of dimension at most 500:

             Problem   Dim  Max and min9682(a)0.15u82(Eom)0.139682(o)0.139682(s)0.14213139682.(a)
```


acoustic_wave_1d
0.5
0
−5
0
5
1
0.5
0
−5
0
5

The nlevp_example.m function can be used as a template by the user wishing to test a given solver on subsets of the NLEVP problems.

The toolbox function nlevp_test.m automatically tests that the problems in the collection have the claimed properties. It is primarily intended for use by the developers as new problems are added, but it can also be used as a test for correctness of the installation. While many of the tests are straightforward, some are less so. For example, we test for hyperbolicity of a Hermitian matrix polynomial by computing the eigensystem and checking the types of the eigenvalues, using a characterization in [1, Thm. 3.4, P1]. To test for proportional damping we use necessary and su□cient conditions from [60, Thms. 2, 4]. We reproduce part of the output:

```
>> nlevp_test
Testing the NLEVP collection
Testing generation of all problems
Testing T-palindromicity
Testing *-palindromicity
...
Testing proportionally damping
Testing given solutions
NLEVP collection tests completed.
***  Errors: 0
```

Conclusions

The NLEVP collection demonstrates the tremendous variety of applications of nonlinear eigenvalue problems and provides representative problems for testing, provided in the form of a MATLAB toolbox. Version 1.0 of the toolbox was released in 2008 and the current version is 2.0. The toolbox has already proved useful in our own work and that of others [2], [6], [8], [34], [36], [53], [78] and we hope it will find broad use in developing, testing, and comparing new algorithms. By classifying important structural properties of nonlinear eigenvalue problems, and providing examples of these structures, this work should also be useful in guiding theoretical developments.

Acknowledgements

We are grateful to the following people for contributing to the collection:

Maha Al-Ammari, Younes Chahlaoui, and Christopher Munro (The University of Manchester), Zhaojun Bai (University of California, Davis), Lyonell Boulton (Heriot-Watt University), Marlis Hochbruck and Dominik Löchel (Karlsruhe Institute of Technology), Tsung-Ming Huang (National Taiwan Normal University), Xin Huang and Yangfeng Su (Fudan University), Zuzana Kukelova (Czech Technical University), Branislav Mičušík (Austrian Institute of Technology), Valeria Simoncini (University of Bologna), Nils Wagner (University of Stuttgart).

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