Velocity-based moving mesh methods for nonlinear partial differential equations

by

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Abstract

This report describes several velocity-based moving mesh numerical methods for multidimensional nonlinear time-dependent partial differential equations (PDEs). It consists of a short historical review followed by a detailed description of a recently developed multidimensional moving mesh finite element method based on conservation.

Finite element algorithms are derived for both mass-conserving

Contents

1 Introduction 5

Ti e dependent PDEs

nd nc n d

c n nc nd y

o o d on

nd n n o n

e ocity a sed oving esh ethods 1

d dyn c

AL A y L n n n od

Mo n n n

on od

L od

con on od

4 The consera tion ethod for a ss conserving pro e s

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A o n n n od

M

o n oc y

co n o on

M con n nc on

A con n o

n y

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n d

n d n on d n on

n d n d n on

o d on

o d non n d on

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[illegible] d... on

5 The conservation method for non mass conserving processes 5
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6 Extensions and Further Applications 6
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Summary

Appendices 1

A Propagation of self similar solutions 1

B Propagation of L_2 projections of self similar solutions 4

1 Introduction

n on c d n on o on

$(t; x) = ()$

where f satis es the ODE

$$f\,()\ ()\ f = (f)\,g \qquad (16)$$

A particular property of self-similar solutions for mass-conserving problems (for which (14) holds) is the invariance of the integral (12) via

$$\int x\, dF \qquad \int f(\,)\, d$$

moves with the self-similar velocity w

v n ny c n y oo oc y d con n
no oc y o o nd y

o o o n o **R** **t** o oc y **v** nd
nc on **w** oz n n **R** **t** o **w** con c d oc y
v n n

$$\mathbf{w}_t \quad \mathbf{v}\cdot \square \mathbf{w} \quad .$$

on n d c o

$$\frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathcal{R}(t)} \mathbf{wud} \qquad \int_{\mathcal{R}(t)} \mathbf{w}\{\mathbf{u}_t \quad \square \cdot \mathbf{u}\mathbf{v}\}\mathbf{d}$$

nc on **w** con c d oc y **v**

o[illegible] n[illegible] d[illegible] n[illegible] z[illegible] on [illegible] con[illegible] d[illegible] on [illegible] y[illegible] c [illegible] o [illegible] y [illegible] y on L [illegible] n [illegible] n [illegible] d dyn[illegible] c [illegible] n[illegible] c [illegible] n [illegible] o [illegible] M [illegible] o [illegible] n [illegible] y [illegible] o [illegible] nd [illegible] c d n[illegible] y y [illegible] n[illegible] oc[illegible] o[illegible] o[illegible] nd[illegible]

[illegible] n [illegible] c[illegible] on n[illegible] o[illegible] oc[illegible] y d[illegible] od [illegible] d [illegible] d [illegible] n n[illegible] n [illegible] d o [illegible] d dyn[illegible] c [illegible] c[illegible] on nd con[illegible] n [illegible] n [illegible] o [illegible] y[illegible] n on [illegible] o c [illegible] d AL [illegible] A [illegible] y L [illegible] n [illegible] n [illegible] n [illegible] on M ny AL [illegible] oc[illegible] n[illegible] d y[illegible] n[illegible] c [illegible] n [illegible] o[illegible] nd [illegible] d[illegible] c [illegible] d[illegible] n c[illegible] on

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[illegible]

[illegible]

Moving nite e e ents

[illegible]

n o n z $\mathbf{L}_2$ no o d

c o oc y o nod o $\mathbf{dX}_i/\mathbf{dt}$ i y

$\mathbf{dU}_i/\mathbf{dt}$ nod $\frac{dU}{dt}$ o n d y d n n y

m o o n

$$\frac{\mathbf{dU}}{\mathbf{dt}} \qquad \frac{\mathbf{dU}_j}{\mathbf{dt}}\mathbf{W}_j \quad \mathbf{t} \qquad \mathbf{U}_j \ \mathbf{t} \ \frac{\mathbf{dW}_j \quad \mathbf{t}}{\mathbf{dt} \quad j \quad \mathbf{t}}$$

c d n on y co on n
no n y co c o o nd
o on n y co d nn on A
y L n n n AL nd o n y
c n

4 The Defor a tion ethod

od n y of Mo nd co o n nd
Mo conc n nc o c c o d
n o do n n $\mathbb{R}^n$ n c conc n d
nc o d ϕ →

d ϕ f
ϕ ∂ ,

f n on o nd on c n
y on
n c nc o c nc
oo n con c n n o n oc d n
c od d c d o c c y co o n
nd Mo n z y nd n ϕ on d n y-
ϕ c n od n v n d o

· f −
∂ .

nc o o on o o d con c y y n
w nd nd n w

^{2}w f −
w · n ∂ .

n o oo o d on d n ϕ
n y ϕ , od n y d n on

d/dt , t (, t) / (t − t f , t)
,

o d y n od c n nc on

h , t d t − t f , t

nd o n $\frac{}{t}$ fo o h, h,
d
A nd c d o conc f o n d o d o
n o fo n fo on y L o nd co o
o no d f co o n o d n fo on

[illegible] n [illegible] d [illegible] n n o [illegible] co [illegible] n o [illegible] n [illegible] on n
[illegible] d [illegible] nc do [illegible] n nd [illegible] y [illegible] c do [illegible] n [illegible] L [illegible] n
[illegible] d y [illegible] d con [illegible] n o [illegible] d y n [illegible] c [illegible] d [illegible] n
[illegible] o [illegible] n [illegible] o [illegible] nc nd [illegible] n [illegible] c [illegible] n [illegible] nc [illegible] n [illegible] on
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[illegible] d [illegible] o [illegible] y L [illegible] n [illegible] o [illegible]

... *a posteriori* ...

4 The consera tion ethod for a ss conserving pro e s

4.1 A oa conservation princip e

$$\int_{\mathcal{R}(t)} \theta \, \mathrm{d}u$$

fo n y o n on t of R t A n
on t fo non o n co n of R t

nd on c n n

inc **u >** in [illegible] in [illegible] [illegible] of **R**

o on A oc y n d on y o y y oc y n n n o y o nd n **Lu** nd n oc y **v**

$$\int_{\mathcal{R}(t)} \mathbf{w}_i \{\mathbf{L}\mathbf{u} \;\; \cdot \; \mathbf{u}\mathbf{v} \} d$$

cf. nc $\mathbf{w}_i$ o oc y **v**. $\mathbf{w}_i \; \mathbf{H}^1\{\mathbf{R} \; \mathbf{t}\}$ n n n on o n

$$\int_{\mathcal{R}(t)} \mathbf{u} \; \mathbf{w}_i \cdot \mathbf{v} d \qquad \int_{\mathcal{R}(t)} \mathbf{w}_i \mathbf{L}\mathbf{u} d \qquad \int_{\mathcal{R}(t)} \mathbf{w}_i \mathbf{u}\mathbf{v} \cdot \mathbf{n} d$$

o

no n o on o on **v** n n n oc y **q** nd oc y o n $\boldsymbol{\varphi}$

$$\int_{\mathcal{R}(t)} \mathbf{u} \; \mathbf{w}_i \cdot \mathbf{v} d \qquad \int_{\mathcal{R}(t)} \mathbf{u} \; \mathbf{w}_i \cdot \; \mathbf{q} \;\; \boldsymbol{\varphi} \; d$$

V

nc **U >** on n o on $\mathbf{S}^1\{\mathbf{R}\ \mathbf{t}\}$ o d d

W i **U** · **n**

4 So ving for the ve ocity

A con n o c n oc y

$\mathbf{k}^{+}$

n y o foc on d AL on
n fo

$$\mu_i \quad \frac{d}{dt} \quad_{\overline{\mathcal{R}}(t)} W_i U d \quad_{\overline{\mathcal{R}}(t)}$$

n o n nod d c n o ondary nod j c n o nd n.

o n nd oc y o n d c y o n nd d $\mathbf{W}_i$ nc on no o on $\mathbf{U}$, on condition o nd o od d o on.

$$\mathbf{M}\underline{\mathbf{U}} \quad \underline{\boldsymbol{\mu}},$$

on ondary condition o d od d n d

co $\mathbf{U}_i$ n $\boldsymbol{\mu}_i$ n y

o o nd d c n c o o o n y on o c o nd

y cond on on $\mathbf{U}$ o o $\boldsymbol{\mu}_i$ n y n

od of c on . n odd n of nod y

on of ... o ... zn ... n ... nc on nd ... d ... n on on ... n ... d ... o $\mathbf{u}_0(\mathbf{x})$ ... y o ... n d y ... n ... zn ... $\mathbf{L}_2$ ... n ... o ... on ... $\mathbf{U}(\mathbf{x})$ ... $\mathbf{L}_2$ ... o c on o $\mathbf{u}_0(\mathbf{x})$ n o ... $\mathrm{span}\{\mathbf{W}_i\}$ ... n ... on

$$\int_a^b \mathbf{W}_i(\mathbf{U}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}))\,\mathbf{dx}$$

... d ... n on $\mathbf{U}(\mathbf{x})$ ... $_j\,\mathbf{U}_j\mathbf{W}_j$ o ...

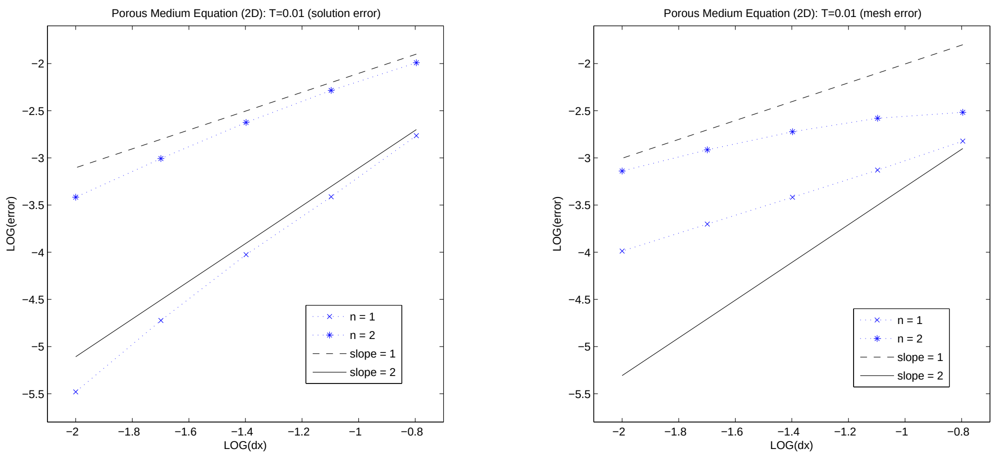
$$\mathbf{M}\underline{\mathbf{U}} \quad \underline{\mathbf{f}}_0,$$

... $\mathbf{M}$ ... nd d ... $\underline{\mathbf{U}}$ ... c ... o co c n $\mathbf{U}_j$ nd ... co on n o ... c ... $\underline{\mathbf{f}}_0$ ...

4 Ea p es

no d n on n n con on od
o con n o on n o o n o

n **N** n o nod $\mathbf{N}_B$ n o o nd y
nod **u** nd **r** n y c o on nd do n d nd $\mathbf{R}_i$
d nc o o nd y nod **i** o o n o on d y
o d n o **n** nc o o **n** ≤ y o on
n d o n **J** ***R*** **J s,** **ae**



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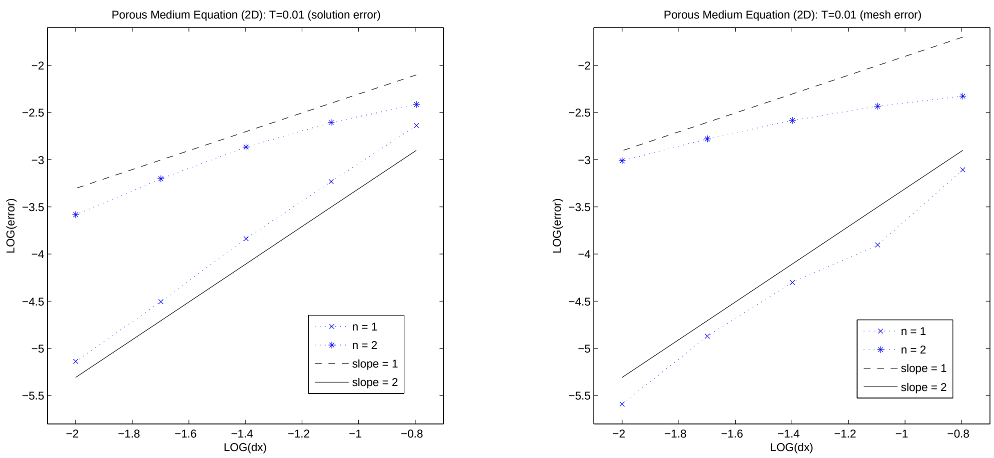

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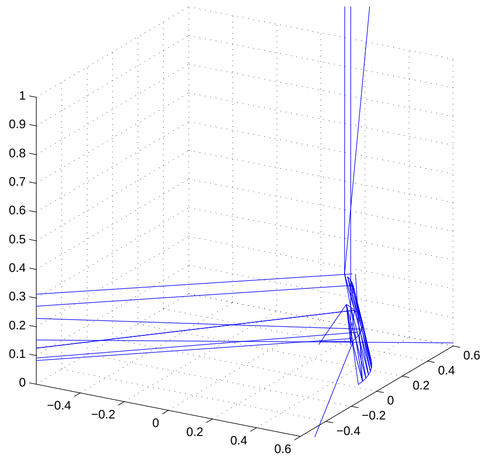
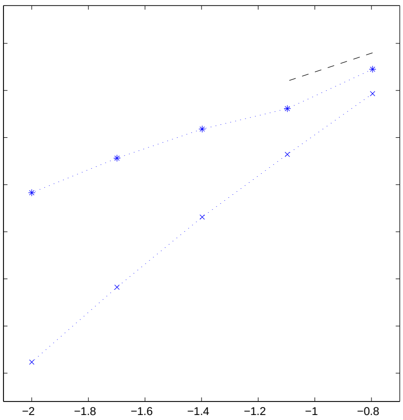
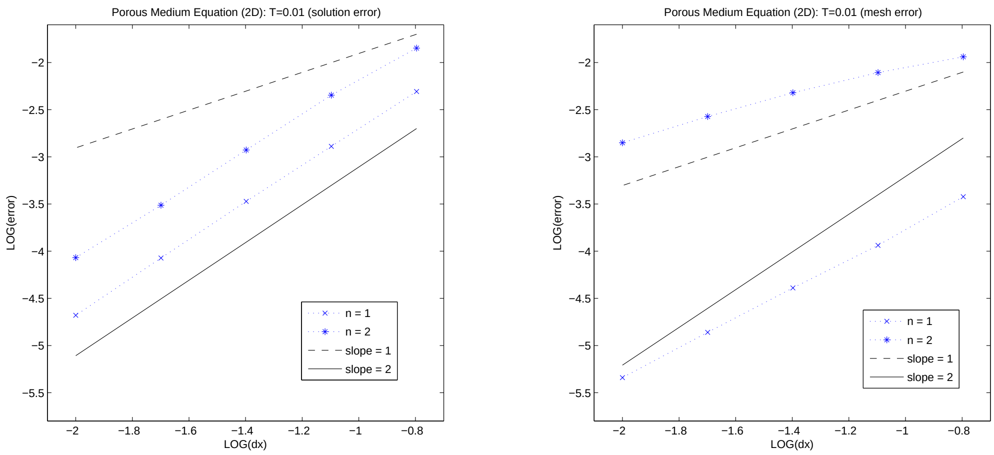


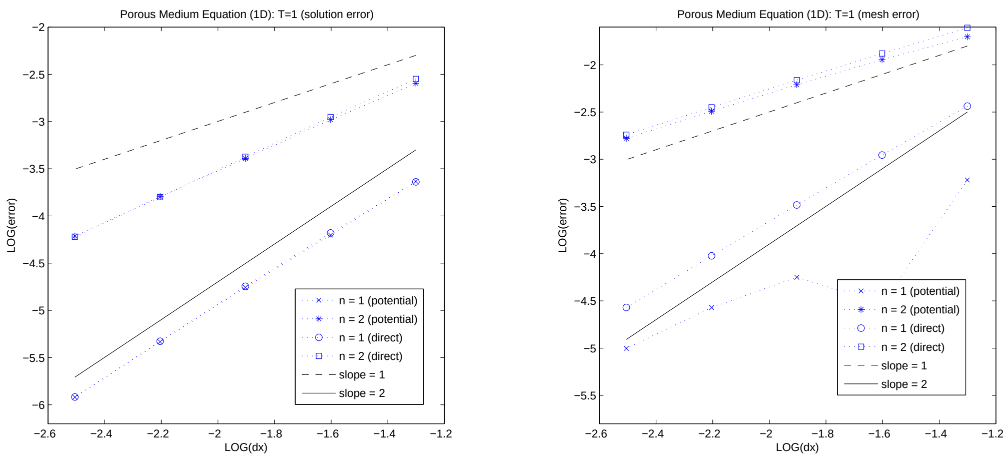
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.4
−0.2
0
0.2
0.4
0.6
−0.4
−0.2
0
0.2
0.4
0.6



−2
−2.5
−3
−3.5
−4
−4.5
−5
−5.5
−2
−1.8
−1.6
−1.4
−1.2
−1
−0.8



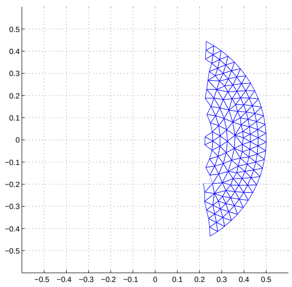
. - Acc cy o o on o o d n on o o d on , **n** nd **n** on nc o **T** . o n d **L_2** o n d - o on o nd o n **L_1** no



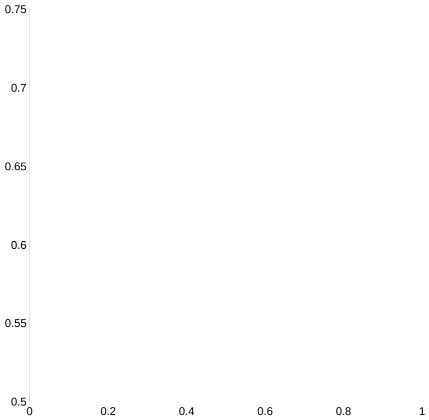
[illegible] A [illegible]

[illegible]

[illegible] p [illegible] n [illegible] $p > \frac{2}{3}$ [illegible] p [illegible]



0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5


0.75
0.7
0.65
0.6
0.55
0.5
0
0.2
0.4
0.6
0.8
1

5 The conserva tion ethod for non a ss conserving pro e s

o n o n n oc d d con on nc nd con n con n o θ n o d n n θ t n o o con n con on nc y con c d y n zn n n n o c y n θ t n

L **u** o o on o n n o o n do n t nd d n **relative** con on nc

$$\overline{\theta\ t} \quad \int_{\Omega(t)} \mathbf{u}\,d \quad \mathbf{c} \quad , \text{ nd nd n o } ,$$

o

$$\theta\ t \quad \int_{\mathcal{R}(t)} \mathbf{u}\,d$$

n zn o nd $\mathbf{c}\ \mathbf{R}$

5 1 e ocity a nd so ution

yno d n no d o nc on $\mathbf{u}/\theta\ t$ d n o on

$$\int_{\Omega(t)} \mathbf{u}_t\,d \quad \int_{\Omega(t)} \mathbf{u}\mathbf{v}.\mathbf{n}\,d \quad \mathbf{c} \quad \theta \quad \dot{\theta}$$

mTJ /R11 11.9551 Tf 91.7633 16.2719816 305 (e)07077.97011 Tf 6.64219 -27.0629 Td[(@)-0.364521]TJ

... on ... $\mathbf{u}_t$ ... on ... y

...

$$\int_{\Omega(t)} \mathbf{L}\mathbf{u}\, d \;\; \int_{\Omega(t)} \mathbf{u}\,\mathbf{v}\cdot\mathbf{n}\, d \;\; \mathbf{c} \;\; \theta \;\; \frac{\theta}{\theta} \int_{\Omega(t)} \mathbf{u}\, d$$

... on ... co ... d ... n ... on

$$\mathbf{L}\mathbf{u} \;\; \cdot \;\; \mathbf{u}\mathbf{v} \;\; \frac{\theta}{\theta}\mathbf{u}.$$

... on ... c n ... d d ... n ... on ... $\mathbf{v}$ nd θ ... n ... c ... θ c n ... o ... n d ... y A ... y n ... o ... o ... d ... n $\mathbf{R}$ $\mathbf{t}$ nd ... n $\mathbf{c}$ $\mathbf{R}$

$$\int_{\mathcal{R}(t)} \mathbf{L}\mathbf{u}\, d \;\; \int_{\mathcal{R}(t)} \mathbf{u}\,\mathbf{v}\cdot\mathbf{n}\, d \;\; \theta,$$

... θ ... y ... $\mathbf{v}\cdot\mathbf{n}$ ... n on ... o nd ... y $\partial\mathbf{R}$ $\mathbf{t}$ ... $\mathbf{u}$ ... d ... c ... on ... c d o ... c ... n ... n ... n

... n ... θ ... no n ... no ... n ... o ... on o ... on ... $\mathbf{v}$ ... n c ... on ... y $\mathbf{curl}\,\mathbf{v}$

5 Distri uted for s

n d d c $\mathbf{w}_i$ on n o o n
nc on d n $\mathbf{R}$ t nd n on o n y con
on nc od d d

$$\overline{\theta\ t} \quad \int_{\mathcal{R}(t)} \mathbf{w}_i \mathbf{u} d \quad \mathbf{c}_i, \text{ nd nd n o } ,$$

$\mathbf{u}$ o o on o n n nd θ t n y
nc $\mathbf{w}_i$ on o n y y n on o i
o n $\mathbf{c}_i$ no n n d c o n
$\mathbf{w}_i$ c c nc on n t
n y n yno d n o nc on $\mathbf{w}_i\mathbf{u}/\theta$ t
nd n on o $\mathbf{R}$ t nd $\mathbf{w}_i$ o
oc y $\mathbf{v}$ no n $\mathbf{v}$ nd θ y

$$\int_{\mathcal{R}(t)} \mathbf{w}_i \{\mathbf{u}_t \quad \cdot \ \mathbf{u}\mathbf{v}\} d \quad \mathbf{c}_i \theta$$

cf d o o
on

$$\int_{\mathcal{R}(t)} \mathbf{w}_i \{\mathbf{L}\mathbf{u} \quad \cdot \ \mathbf{u}\mathbf{v}\} d \quad \mathbf{c}_i \theta .$$

. n $\mathbf{w}_i$ $H^1\{\mathbf{R}\ t\}$ t y n
o n

$$\int_{\mathcal{R}(t)} \mathbf{u} \ \mathbf{w}_i \cdot \mathbf{v} d \quad \int_{\mathcal{R}(t)} \mathbf{w}_i \mathbf{L}\mathbf{u} d \quad \int_{\mathcal{R}(t)} \mathbf{w}_i \mathbf{u}\mathbf{v} \cdot \mathbf{n} d \ - \mathbf{c}_i \theta .$$

n on o i nd n $\mathbf{c}_i$ d o

$$\int_{\mathcal{R}(t)} \mathbf{L}\mathbf{u} \quad \int_{\mathcal{R}(t)} \mathbf{u}\mathbf{v} \cdot \mathbf{n} d \quad \theta$$

y d θ c y $\mathbf{v} \cdot \mathbf{n}$ n on o nd y $\partial\mathbf{R}$ t
$\mathbf{u}$ o d n o n n
no n o on o on $\mathbf{v}$
oc y o n φ y n n c n n

$$\int_{\mathcal{R}(t)} \mathbf{u} \ \mathbf{w}_i \cdot \ \varphi d$$

$$\int_{\mathcal{R}(t)} \{\mathbf{w}_i \mathbf{L}\mathbf{u} - \mathbf{u} \ \mathbf{w}_i \cdot \mathrm{q}\} d \quad \int_{\mathcal{R}(t)} \mathbf{w}_i \mathbf{u}\mathbf{v} \cdot \mathbf{n} d \ - \mathbf{c}_i \theta .$$

nc $\mathbf{uv} \cdot \mathbf{n}$ n on ndary $\partial \mathbf{R}_t$ o n n

o nc n d $\mathbf{U} \cdot \mathbf{n}$ n on o nd y
no n c y

no n o on o n n
n oc y $\mathbf{Q}$ nd oc y o n yn
n on c n n

$$\int_{\overline{\mathcal{R}}(t)} \mathbf{U}\,\mathbf{W}_i \cdot \, d$$

$$\int_{\overline{\mathcal{R}}(t)} \{\mathbf{W}_i \mathbf{L} \mathbf{U} - \mathbf{U}\,\mathbf{W}_i \cdot \mathrm{Q}\}\, d \qquad \int_{\overline{\mathcal{R}}(t)} \mathbf{W}_i \mathbf{U} \cdot \mathbf{n}\, d - \mathrm{C}$$

nc $\mathbf{U} >$ on n on $\mathbf{S}^1\{\mathbf{F}$ nc
$\mathbf{U} \cdot \mathbf{n}$ no n on o nd y o d d n on
o n n $\overline{\mathbf{R}}\ \mathbf{t}$ no c on nc on y d
n on co on n n n
n c d $\mathbf{f}$ co on n n y

$$\mathbf{f}_i \quad \int_{\overline{\mathcal{R}}(t)} \{\mathbf{W}_i \mathbf{L} \mathbf{U} - \mathbf{U}\,\mathbf{W}_i \cdot \mathrm{Q}\}\, d \qquad \int_{\overline{\mathcal{R}}(t)} \mathbf{W}_i \mathbf{U} \cdot \mathbf{n}\, d - \mathrm{C}$$

oc y d d n c on nd o X $\mathbf{t}$ nd $\mathbf{t}$ d
d n

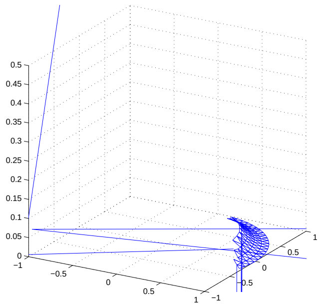
o on $\mathbf{U}$ c n o y co d o o

ind o

Ad nc X, t nd in in ny con m n n d nd

co U_i n y in in AL ion in o o in μ_i o d nc n μ_i in nd in on ond y cond ion c n o o d d d in c ion

cond on $\mathbf{u}_x$ o d y y o nd y
$\mathbf{v}$



0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−1
−0.5
0
0.5
1
−1
−0.5
0
0.5
1

nc on $\partial\overline{\mathbf{R}}_k^F$ A n $\mathbf{U} >$ n n o $\mathbf{R}$ t on

n o on $\mathbf{S}^1\{\overline{\mathbf{R}}$ t$\}$ n o $\_k$ o d d

n c n no n nd n on o n n

o o y cond on on o n n c y c y

o d d o o d n o nd y o

cond on co n n on nd n

nd d o y d

nc oc y o n n o nd n o on

oc c n co d n o n o o do

n onc n d o od c oc y d con n o o

n n o nd nd o n on $\partial\overline{\mathbf{R}}_k^F$ co y

n o n o n c n y cc on n o n

nc on oc d nod on o n n n o nd

c d y o cond on on o n

o n o n c

. n y nod o on nd $\_k$ n d d n

nd d n o on $\mathbf{U}$ co d y n

o on c n don n

d $\mathbf{C}_i$ nd d d $\_k$ on nd d nd d

n d d n AL o

1 Ea pe A T o Ph se Stef n Pro e

od n o n o o

d d nd n od n o n d

nd o d n c n o con on

n c nd c n o o d n n d d

o d on o n od d y on

$$\mathbf{K}_S\,\mathbf{u}_t \qquad \cdot\ \mathbf{k}_S\ \mathbf{u} \qquad \text{n o d on}$$

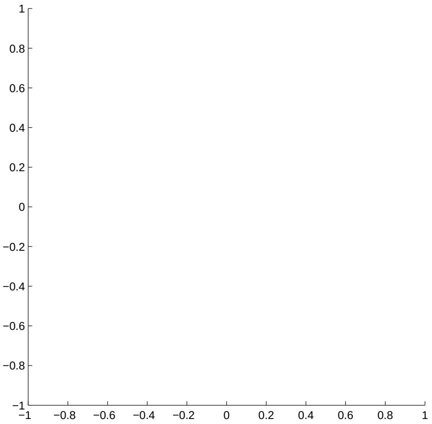
$$\mathbf{K}_L\,\mathbf{u}_t \qquad \cdot\ \mathbf{k}_L\ \mathbf{u} \qquad \text{n d on},$$

n $\mathbf{u}$ nd $\mathbf{K}_{S,L}$ $\mathbf{k}_{S,L}$ n c y

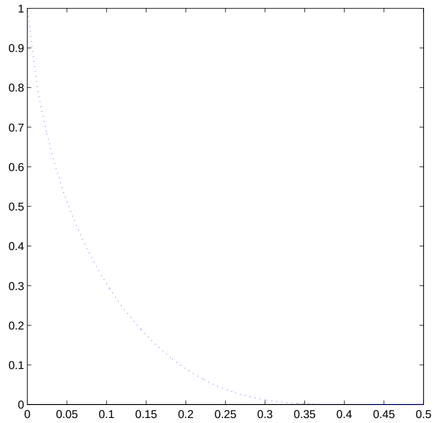
o c c c y nd cond c y o o

no on ny $\mathbf{k}$ d o nd on c n oc d

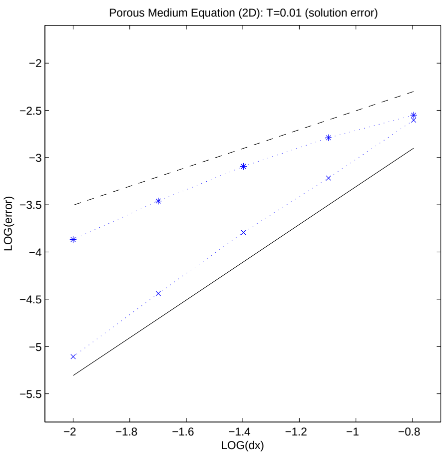
[illegible]



1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5



Porous Medium Equation (2D): T=0.01 (solution error)
LOG(error)
LOG(dx)
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
−2
−1.8
−1.6
−1.4
−1.2
−1
−0.8

fo o o nd c $\mathbf{R}$

By d n n c o n y n c on oc y $\mathbf{v}$

$$\int_{\mathcal{R}(t)} \mathbf{m}\,\square\,\mathbf{w}_i \cdot \mathbf{v}\,d \qquad \int_{\mathcal{R}(t)} \mathbf{w}_i \frac{\mathbf{dm}}{\mathbf{du}} \mathbf{u}_t\,d \qquad \int_{\mathcal{R}(t)} \mathbf{w}_i \mathbf{m} \mathbf{v} \cdot \mathbf{n}\,d \quad - \mathbf{c}_i \boldsymbol{\theta}$$

d

$$\mathbf{m}_t \quad \frac{\mathbf{dm}}{\mathbf{du}} \mathbf{u}_t,$$

$\mathbf{dm/du}$ n d o $\mathbf{m}$ c o $\mathbf{u}$ c nno o d c y $\mathbf{u}_t$ n nc nc on $\mathbf{dm/du}\ \mathbf{w}_i$ do no n $\mathrm{span}\{\mathbf{w}_i\}$ n n o y o c $\mathbf{Lu}$ n o $\mathrm{span}\{\ \mathbf{dm/du}\ \mathbf{w}_i\}$ n n o on

cf. **v** **dx/dt**

on c n o d o **v** ny on **x t** o d d **v**
n **x** **a t** L n n co d n **x t** c n n o nd y
n on o

$$\frac{dx}{dt} \quad v$$

cf

o on y co d o y o con n

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o f fo t $\square$ **R** t

ocy **v** **t**, nd c d y o on

u $\mathbf{t}_0$,

Mo or nc **v** o on n od cn oc y o n **φ**
v **φ** on

$$\frac{\partial}{\partial \mathbf{x}} \quad \mathbf{f} \quad {}^{0} \frac{\partial}{\partial \mathbf{x}} \quad - \quad \sum_{\mu} - \frac{\beta_\mu \mathbf{x}_\mu^2}{\mathbf{t}_0} \quad \boldsymbol{\varphi} \quad ,$$

n nc **u** nd **f** o n n o do n nd
n on o nd y on n o on

$$\boldsymbol{\varphi} \quad - \quad \sum_{\mu} \frac{\beta_\mu \mathbf{x}_\mu^2}{\mathbf{t}_0} \quad \text{con} \quad \mathbf{v} \quad \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}} \quad \frac{\boldsymbol{\beta} \ \mathbf{x}}{\mathbf{t}_0} \quad ,$$

o **t** n n $\mathbf{t}_0, \mathbf{t}_0$ **t** con n β γ nd on of $\mathbf{R}$ t conc d

o of $\mathbf{R}$ t_0 o d $O(t^2)$ nd o on

nc t t_0

$$u(t_0, x) = t_0^{\gamma} f\left(\frac{x_1}{t_0^{\beta_1}}, \frac{x_2}{t_0}, \dots, \frac{x_d}{t_0}\right)$$

nd t_0 y o

$$u(t, x) - t^{\gamma} f\left(\frac{x_1}{t^{\beta_1}}, \frac{x_2}{t}, \dots, \frac{x_d}{t}\right) = O(t^2)$$

t $t_0 < t \le t_1$ t t t_1 t_0 t

$$u(t_1, x) - t_1^{\gamma} f\left(\frac{x_1}{t_1^{\beta_1}}, \frac{x_2}{t_1}, \dots, \frac{x_d}{t_1}\right) = O(t^2),$$

o $u(t_1, x)$ o on $u^{ss}(t_1, x)$ t_1 t_0 t

o d $O(t^2)$ n o n t t_1

Theore B-

L

u t, o o on o con n c n o o n d y

U t, y con on nc

oc y oc y **t**

U t$_0$, co nc d **L$_2$** o c on o o on **t t$_0$**

n **U t,** n **L$_2$** o c on o o on **t > t$_0$**

Proof-

oo n n A nd A c **assumed** oc y oc y **t** By on

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