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A moving mesh approach to ice sheet modelling

by

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or geothermal heat below the surface.

In order for glaciers to form they first need enough snow over the winter period to be able to survive through the summer, i.e. more accumulation of snow than is lost through melting and evaporation. This needs to be repeated over a number of succesive years, and as more snow builds up, the weight

is the idea of mass balance, and where on the glacier mass is gained or lost. Generally, near the source of the glacier, the accumulation of snow is greater than the ablation (melting/evaporation), so the mass increases. Further away the ablation becomes greater than the accumulation, and the mass decreases. However ice can build up in the lower zone due to ice flow coming from the glacier's upper zone. The front-most end of the glacier is known as the snout, which rarely moves straight away; it waits until the velocity behind it is great enough to push it down the mountain. It is this feature which is of special interest.

2. Model Description

Consider a glacier on a flat bed occupying the region $x \in [0, b(t)]$ as shown in Fig.1. Let $H(x,t)$ represent the thickness of the ice. At the ends of this domain we have two boundary conditions, $H = 0$ at the moving bounary $x = b(t)$, and $\frac{\partial H}{\partial x} = 0$ at the fixed point $x = 0$.

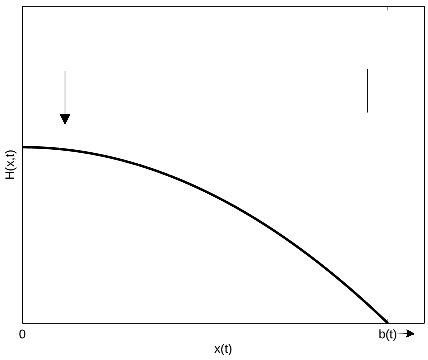
We consider a simple PDE model for glaciers proposed by Oerlemans [3] in 1984.

2.1. Model Derivation

In one dimension the continuity equation for ice can be written as

$$\frac{\partial H}{\partial t} = -\frac{\partial (Hu)}{\partial x} + s(x), \qquad (1)$$

where H is the ice thickness, $s(x) = s\ (x) - s\ (x)$, with s the accumulation rate of snow and s the basal melting rate. Also u is defined as the mean



H(x,t)
0
b(t)
x(t)

where c is a single positive constant parameter.

From (1), using Leibniz's integral rule, and applying the boundary conditions

$$
\begin{aligned}
\frac{d}{dt}\int_0^{(t)} H(x,t)dx &= \int_0^{(t)} \frac{\partial H}{\partial t}dx + H(b(t),t)\frac{db(t)}{dt} \\
&= -\int_0^{(t)} \frac{\partial}{\partial x}[Hu]dx + \int_0^{(t)} s(x)dx \\
&= -[Hu]_0^{(t)} + \int_0^{(t)} s(x)dx \\
&= \int_0^{(t)} s(x)dx, \qquad (8)
\end{aligned}
$$

the physical equivalent of which states that any change in the integral of ice thickness over the whole glacier, or equivalently any change in the ice volume, is due only to the snow term, which represents the net accumulation/ablation of snow over the whole glacier.

3. Snout Behaviour

From (5), with n = 3 we derive the useful form

$$
u = -c(H^{4/3}H\ )^3 = -\frac{27}{343}c\left[(H^{7/3})\ \right]^3, \qquad (9)
$$

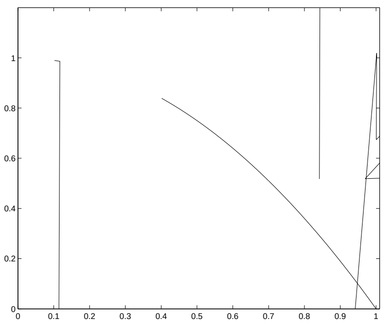
with n = 3.

When expressing the velocity in this manner it is interesting to substitute an expression for H that has the right general shape and satisfies the boundary conditions, i.e.

$$
H = (1-x^2)^{\alpha} \qquad (10)
$$

where $\alpha > 0$, for which

$$
\begin{aligned}
H^{\frac{7}{3}} &= (1-x^2)^{\frac{7\alpha}{3}} \\
(H^{7/3}) &= -2x.\frac{7\alpha}{3}(1-x^2)^{\frac{7\alpha}{3}-1}. \qquad (11)
\end{aligned}
$$



1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

4. Subdomain mass balance (SDMB)

Since the problem involves a moving boundary, a natural description is to use a moving framework, for which we define a velocity $v(x,t)$ at any arbitrary point $\hat{x}$ [1]. To define this velocity we assume that equation (8) holds in any moving subdomain $[0,\hat{x}(t)]$ of $[0,b(t)]$. In physical terms this velocity v is such that the ice volume changes only due to the local accumulation/ablation of snow. In equation form we assume that

$$\frac{d}{dt}\int_0^{\hat{}(t)} H(x,t)dx = \int_0^{\hat{}(t)} s(x)dx \tag{15}$$

for each subdomain $(0,\hat{x}(t))$.

By Leibniz's integral rule, making use of (6) and the boundary conditions given in Section 2,

$$\begin{aligned}\frac{d}{dt}\int_0^{\hat{}(t)} H(x,t)dx &= \int_0^{\hat{}(t)} \frac{\partial H}{\partial t}dx + H(\hat{x}(t),t)\frac{d\hat{x}(t)}{dt}\Bigg|_0^{\hat{}} \\ &= cH^5H^3|_{\hat{}} + \int_0^{\hat{}(t)} s(x)dx + H(\hat{x}(t),t)\frac{d\hat{x}(t)}{dt}. \end{aligned} \tag{16}$$

Therefore the assumption (15) is equivalent to

$$\int_0^{\hat{}(t)} \left\{ cH^5H^3 + H(\hat{x}(t),t)\frac{d\hat{x}(t)}{dt} \right\} dx = 0$$

which, since $\hat{x}(t)$ is arbitrary, gives

$$cH^5H^3 + H(\hat{x}(t),t)\frac{d\hat{x}(t)}{dt} = 0.$$

Hence the velocity $v = d\hat{x}/dt$ is driven only by the diffusion term and we obtain

$$v = \frac{d\hat{x}(t)}{dt} = -\frac{cH^5H^3}{H} = -cH^4H^3 = -c_2\left[(H^{7/3})\ \right]$$

where $c_2 = (3/7)^3 c$. Note that this velocity is the same as taking v to be the model velocity (5) at each of the nodes. Reversing the argument implies that the assumption (15) and the velocity (5) are equivalent.

5. Numerical Method

Equation (6) is generally impossible to solve analytically, so we seek a numerical approximation via a mesh. To do this we discretise (15) and (17). The mesh positions are updated at every time step.

Computation is performed with initial conditions (10) with α set to 1. The snow term is approximated for all time by the linear function (as in Van Der Veen [8])

$$s(x) = e(1 - dx), \tag{18}$$

where d and e are the snow parameters, set to be 0.5 and 0.05 respectively. The model is run for a sufficient length of time for the boundary to wait, then move. The initial mesh is chosen to be evenly spaced.

5.1. Explicit time-stepping

To advance the node positions $\hat{x}_i$ from the velocity (17) we use an explicit Euler scheme. Letting k denote the time discretisation level,

$$\frac{\hat{x}_i^{k+1} - \hat{x}_i^k}{\Delta t} = -c_2 \left[\left[(H_i^{7/3}) \right]^3 \right]^k, \tag{19}$$

Therefore, dropping the hat notation for convenience, at each time step we update each mesh point by:

$$\mathbf{x}_i^{k+1} = \mathbf{x}_i^k - c_2 \Delta t \left[\left[(\mathbf{H}_i^{7/3}) \quad \right]^3 \right]^k \tag{20}$$

Since we do not seek high levels of accuracy Euler time-stepping is sufficient, provided the time step is suitably small to ensure stability.

To determine the updated ice thickness we use the same time-stepping scheme on equation (15). Note that the limits have been chosen to give an incremental form.

$$\frac{\left[\int_{j-1}^{j+1} \mathbf{H}\,d\mathbf{x}\right]^{k+1} - \left[\int_{j-1}^{j+1} \mathbf{H}\,d\mathbf{x}\right]^{k}}{\Delta t} = \int_{j-1}^{j+1} s\,d\mathbf{x}.$$

Using the midpoint rule we obtain the approximation

$$(\mathbf{x}_{j+1}^{k+1} - \mathbf{x}_{j-1}^{k+1})\mathbf{H}_j^{k+1} - (\mathbf{x}_{j+1}^{k} - \mathbf{x}_{j-1}^{k})\mathbf{H}_j^{k} = \Delta t(\mathbf{x}_{j+1}^{k} - \mathbf{x}_{j-1}^{k})s_j^k,$$

giving

$$\mathbf{H}_j^{k+1} = \frac{(\mathbf{x}_{j+1}^{k} - \mathbf{x}_{j-1}^{k})}{(\mathbf{x}_{j+1}^{k+1} - \mathbf{x}_{j-1}^{k+1})}(\mathbf{H}_j^k + \Delta t s_j^k). \tag{21}$$

5.2. Results

The model is run with 51 mesh points ($\Delta x = 0.02$), with a time step $\Delta t = 0.005$.

Varying the parameter α in the initial conditions shows the snout profile behaviour of Section 3. In Fig. 2(a) we see that when $\alpha = 3/7$ the gradient is effectively infinite at the boundary, while a comparison case of $\alpha > 3/7$ shows a finite gradient. Similarly, looking at the initial velocities for each of the two cases we see in Fig. 2(b) that the boundary does not move when $\alpha > 3/7$, while in the comparison case it does. Also of note is that the peak

discontinuity forms. Since the velocity must remain continuous for physical reasons, the limit $\alpha = 3/7$ cannot be attained and the snout must move.

We now consider the transition in time of $v(x,t)$ to $\alpha = 3/7$ from above for small $(b_0 - x)$ under the subdomain mass balance assumption (15). To leading order in $(b_0 - x)$ the time-varying form of (23) is

$$H(x,t) = (b_0 - \hat{x})^{\delta} g(x,t) \tag{26}$$

where $\hat{x} = x(t)$, while that of (24) is

$$v(x,t) = c\alpha^3 (b_0 - \hat{x})^{7\delta-3} \{G(\hat{x},t)\}^3 . \tag{27}$$

We consider the evolution of the velocity under the subdomain mass balance assumption (15) applied to the interval (x, b_0), where $(b_0 - \hat{x})$ is small. Let $\theta(t)$ be the positive mass in the triangle consisting of points $(\hat{x}, H(x,t))$, $(\hat{x}, 0)$ and the fixed point $(b_0, 0)$. To leading order in $(b_0 - \hat{x})$, therefore,

$$\theta(t) = \frac{1}{2}(b_0 - \hat{x}) H(x,t). \tag{28}$$

Hence, from (26), to leading order in $(b_0 - \hat{x})$,

$$\frac{1}{2}(b_0 - \hat{x})^{\delta+1} g(\hat{x},t) = \theta(t)$$

so that on the trajectory given by (28) from (27) the velocity at time t can be written

$$v(t) = c\alpha^3 (b_0 - \hat{x})^{7\delta-3} \left(\frac{2\theta(t)}{(b_0 - x)^{\delta+1}} \right)^7 = c\alpha^3 (2\theta(t))^7 (b_0 - \hat{x})^{-10} \tag{29}$$

and thus on the trajectory given by (28)

$$\alpha^3 = \frac{1}{c} \frac{1}{(2\theta(t))^7} (b_0 - \hat{x})^{10} v(t)$$

7. Discussion

We have seen that a moving mesh method based on local mass balance works well for a simple one-dimensional glacier. We also analysed the conditions required for the snout to move, which required an infinite slope condition to be met asymptotically at the boundary in order for the velocity at this point to be non-zero. For an initial condition in the form of the quadratic function (10) with $\alpha > 3/7$, we were able to simulate the qualitative waiting time behaviour of the glacier as the power of α evolved to $3/7$.

The model (6) works well for glaciers that are advancing, but since the velocity is proportional to the negative slope at the snout the glacier cannot retreat.

However, it is observed that glaciers also break-up, where large sections of top and bottom of the glacier to be moving quicker than the middle section, and the impact of basal sliding could then be analysed more effectively. Alternatively we could keep the depth averaged vertically and consider the domain in the (x, y) plane, as opposed to a cross-section in the x-plane as is modelled currently. This will require additional boundary conditions at the sides of the glacier, the type depending on whether the boundary is a solid wall (no flux condition) or whether the glacier just curves to the ground $(H = 0)$. The model itself then takes the form

$$H_t(x, y, t) = \nabla . \left[H(x, y, t)^5 \nabla H(x, y, t)^3\right] + s(x, y). \tag{37}$$

A viable 2D numerical model using the same moving mesh approach is possible using finite element approximation.

Steering away from $H = 0$ at the snout, Payne et. al. [5] consider the different boundary conditions which occur when a glacier reaches the edge of a cliff or enters the ocean, as well as different representations of u. For cases where the glacier mostly sits on the water (ice shelf) there is the added problem of buoyancy. Payne et. al. propose a maritime boundary condition of the form

$$\left.\frac{\partial v}{\partial x}\right|_{shelffront} = A\left[\frac{1}{4}\rho_i g\left(1 - \frac{\rho_i}{\rho}\right)\right]^n h^n, \tag{38}$$

where ρ is an additional variable introduced for the density of the water.

A further objective in this ongoing work is to introduce the concepts of Data Assimilation. Here the aim is to set up an inverse problem to predict internal variables and model forcing using observations (mostly taken remotely) to further improve the model accuracy.

References