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A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

by

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Abstract

The climate belongs to the class of non-equilibrium forced and dissipative systems, for which most results of quasi-equilibrium statistical mechanics, including the fluctuation-dissipation theorem, do not apply. In this paper we show for the first time how the Ruelle linear response theory, developed for studying rigorously the impact of perturbations on general observables of non-equilibrium statistical mechanical systems, can be applied with great success to analyze the climatic response to general forcings. The crucial value of the Ruelle theory lies in the fact that it allows to compute the response of the system in terms of expectation values of explicit and computable functions of the phase space averaged over the invariant measure of the unperturbed state. We choose as test bed a classical version of the Lorenz 96 model, which, in spite of its simplicity, has a well-recognized prototypical value as it is a spatially extended one-dimensional

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1 Introduction

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[illegible] fi [illegible]

[illegible] fi [illegible]

[illegible]

[illegible] fi [illegible] CO_2 [illegible]

[illegible] ffi [illegible]

[illegible] fl [illegible]

fi fi

$$\langle\Phi\rangle^{(1)} \; ! \quad \int_{-\infty}^{+\infty} \; !_1 \chi_\Phi^{(1)} \; !_1 \; f \; !_1 \times \delta \; ! - !_1 \quad \chi_\Phi^{(1)} \; ! \; f \; ! \; ;$$

! fi

$$\chi_\Phi^{(1)} \; ! \quad \int_{-\infty}^{+\infty} tG_\Phi^{(1)} \; t \quad ! \, t \, :$$

f_α t

X x

$\langle\Phi_\alpha\rangle^{(1)}$ t

fi

$\chi_\Phi^{(1)}$ $!$ ff f_β t

f_β $!$

f_β $!$ $\chi_{\bar\Phi}^{(1)}$ $!$ $\langle\Phi_\beta\rangle^{(1)}$ $!$

Φ X x

$\langle\Phi_\beta\rangle^{(1)}$ t

$G_\Phi^{(1)}$ t

f t δ t

$\langle\Phi\rangle^{(1)}$ t $G_\Phi^{(1)}$ t

2.2 Kramer -Kronig rela ion and Dm rDle

$G_\Phi^{(1)}$ t $\in L^2$

$\chi_\Phi^{(1)}$ $!$

$!$ χ $!$

$G_{\Phi}^{(1)}(t)$ … $\chi_{\Phi}^{(1)}(\omega)$

$$\int_{-\infty}^{+\infty} dt\,\Theta(t)\,t^{k} \, i\omega t \quad -i^{k}\frac{d^{k}}{d\omega^{k}}\left(P\frac{}{\omega} \quad \pi\delta(\omega)\right) \approx k \, \frac{(k+1)}{\omega^{(k+1)}}$$

ω / … $t \to +$

$$G_{\Phi}^{(1)}(t) \approx \bar{\alpha}\,\Theta(t)\,t^{\beta} \quad o(t^{\beta})$$

$\omega \to \infty$

$$\chi_{\Phi}^{(1)}(\omega) \approx \alpha\,\omega^{-\beta-1} \quad o(\omega^{-\beta-1})$$

α $\bar{\alpha} i^{(\beta+1)}\beta$ … β … $\bar{\alpha}$ … Φ … fi … ω … $\chi_{\Phi}^{(1)}(\omega)$ $\chi_{\Phi}^{(1)}(-\omega)^{*}$ … χ … χ … ω … α α_R … β … α α_I … β

$\chi_{\Phi}^{(1)}(\omega)$ $\chi_{\Phi}^{(1)}(-\omega)^{*}$

$$-\frac{\pi}{}\omega^{2p-1} \quad \chi_{\Phi}^{(1)}(\omega) \quad P\int_{0}^{\infty} d\nu\,\frac{\nu^{2p} \quad \chi_{\Phi}^{(1)}(\nu)}{\nu^{2}-\omega^{2}}$$

$$\frac{\pi}{}\omega^{2p} \quad \chi_{\Phi}^{(1)}(\omega) \quad P\int_{0}^{\infty} d\nu\,\frac{\nu^{2p+1} \quad \chi_{\Phi}^{(1)}(\nu)}{\nu^{2}-\omega^{2}}$$

P … p $;:::;$ $\beta -$ = β

p ; : : : ; β = β

$$\chi^{(1)} \quad \frac{}{\pi} P \int d\nu \, \frac{\chi^{(1)}(\nu)}{\nu} ;$$

$\chi^{(1)}$

$\chi^{(1)}$

$\omega \rightarrow$

$\omega \sim$

$c_1 \quad c_2 \, \omega^2 \quad o(\omega^3)$ c_1 c_2

c_1

$\omega \rightarrow \infty$

2.3 A prac ical formDla for he linear D cep ibili and con- i enc rela ion be een D cep ibili ie of differen ob- er able

fi f t fi f t f t ϵ $! t$ Φ x fi

$$\delta\Phi_\epsilon\ t; t_0; x_0 \quad \Phi_\epsilon\ t; t_0; x_0 - \Phi_0\ t; t_0; x_0$$

x_0 t_0 ϵ $\delta\Phi_\epsilon\ t; t_0; x_0$ $O\ \epsilon$

$$\chi^{(1)}_\Phi\ ! \equiv_{\epsilon\to 0\ T\to\infty} \chi^{(1)}_\phi\ !; x_0; \epsilon; T$$

$$\chi^{(1)}_\Phi\ !; x_0; \epsilon; T \quad \frac{}{T}\frac{}{\epsilon}\int_0^T t\,\delta\Phi_\epsilon\ t; x_0 \quad !\,t$$

fi ffi fi fi

$\delta\Phi_\epsilon\ t; x_0$ t $\delta\ \Phi_\epsilon\ t; x_0$ t

$$\chi^{(1)}_{\dot\Phi}\ ! \quad - !\,\chi^{(1)}_\Phi\ ! :$$

fi ffi $\chi^{(1)}_{\dot\Phi}\ !$ Φ

$$\Phi\ x \quad \Gamma\ x ;$$

Γ $F\cdot\nabla\Phi$ Φ

$x \quad F(x) \quad X(x, f(t)) \quad F(x) \quad \epsilon \quad ! \, t(X(x))$

$\Phi(x)$

$\chi^{(1)}_{\Phi}(\omega) \approx$ ... $i\omega\langle\Xi\rangle_0$ ... $\omega\langle\Xi\rangle_0$ ... $\omega\langle\Xi\rangle_0$ / ...

$$G^{(1)}_{\Phi}(t) = \Theta(t)\langle\Xi\rangle_0 \circ t^0;$$

3 Application of the Response Theory to the Lorenz 96 Model

3.1 Statistical properties of the Unperturbed Lorenz 96 Model

... *ab-initio* ...

$$\frac{dx_i}{dt} = x_{i-1}(x_{i+1} - x_{i-2}) - x_i + F$$

... $i = 1, 2, \ldots, N$ ... i ... $x_{i+N} = x_{i-N} = x_i$ ...

... $i \rightarrow i$ ...

... F ... N ...

N F F

N F

6

fi

$E = \sum_i x_i^2$ E

E —

3.2 A mp o ic proper ie of he linear D cep ibili

f t

ϵ ! t

$$\frac{x_i}{t} \quad x_{i-1} \; x_{i+1} - x_{i+2} - x_i \quad F \quad \epsilon \; ! \, t \; X_i$$

X_i X_i $x_1; : : : ; x_N$ x_i

3.2.1 Global perturbation

X_i $\forall i$ E $G^{(1)}_{E,a}$ t

$$G^{(1)}_{E,N}(t) \quad \int \rho_0 \, dx \, \Theta(t) \, \Lambda \Pi_0(t) \, E(x)$$
$$\int \rho_0 \, dx \, \Theta(t) \, \tilde{\ } \cdot \nabla E(x(t))$$
$$\int \rho_0 \, dx \, \Theta(t) \sum_i \partial_i E(x(t)))$$

x t E x t t ρ_0 E

$$G^{(1)}_{E,N}(t) \quad \int \rho_0 \, dx \, \Theta(t) \sum_i \partial_i \Big(E|_{t=0} \quad tE|_{t=0} \quad o(t) \Big)$$
$$\int \rho_0 \, dx \, \Theta(t) \Big[\sum_i x_i - \Big(\sum_i x_i - NF \Big) t \quad o(t) \Big] .$$

$$\chi^{(1)}_{E,N}(\omega) \quad i \Big(\sum_i \langle x_i \rangle_0 \Big) / \omega \quad \Big(\sum_i \langle x_i \rangle_0 - NF \Big) / \omega^2 \quad o(\omega^{-2})$$
$$iN \frac{m}{\omega} - N \frac{F - m}{\omega^2} \quad o(\omega^{-2})$$

i

″ = x_i^2 x_i

fi $\chi^{(1)}_{\varepsilon,N}$ $= N\chi^{(1)}_{E,N}$ $\chi^{(1)}_{\varepsilon,N}$

$! \to \infty$

$$\chi^{(1)}_{\varepsilon,N} \; ! \quad i\,\frac{m}{!} - \frac{F - \; m}{!^{2}} \quad o \; !^{-2} \; :$$

π =

χ

$$\chi^{(1)}_{E,N} \; ! \; \frac{F}{- \; !}$$

$$\int_0^\infty \chi^{(1)}_{E,1} \; ! \; ! \quad \frac{\pi}{}:$$

$\chi^{(1)}_{M,\epsilon,1}$

$$\chi^{(1)}_{M,1} \; ! \qquad \frac{}{!} \qquad \frac{}{!^2};$$

$$\int_0^\infty \chi^{(1)}_{M,1} \; ! \; ! \quad \frac{\pi}{};$$

E_j x_j

$$G^{(1)}_{E_j,1} \; t \qquad \int \rho_0 \, x \; \Theta \; t \; @_j \left[-x_1^2 \quad \left(x_j \;\; x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j \quad F\right) t \quad o \; t\right]$$
$$\Theta \; t \left[\langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j \quad F \rangle_0 \, t \quad o \; t\right]:$$

$$\langle x_j \rangle_0 \qquad \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j \quad F \rangle_0$$

ρ_0

$$\chi^{(1)}_{E_j,1} \qquad i \frac{m}{!} \qquad \frac{m}{!^2} \qquad o \;\; !^{-2}:$$

$\chi^{(1)}_{E,1} \qquad \sum_k \chi^{(1)}_{E_k,1}$

E_j E

$\chi^{(1)}_{E_j,1} \approx \chi^{(1)}_{E,1}$

$$\int_0^{\infty} \chi^{(1)}_{E_j,1}(!) \; ! \quad \frac{\pi}{\;} m$$

$\propto \; !^{-2}$

$!^{-2}$

$\chi^{(1)}_{E,1}$ $\chi^{(1)}_{E_j,1}$

x_j

$= E_{j+1} \quad E_{j+2} \quad E_{j-1}$

$$\chi^{(1)}_{\psi,1}(!) \quad -\frac{\langle x_{j-1} \; x_{j+1} - x_{j-2} \rangle_0}{!^2} \quad -\frac{F - m}{!^2} \quad o(!^{-2}) :$$

$\chi^{(1)}_{\psi,1}$ $\chi^{(1)}_{E_j,1}$

$\chi^{(1)}_{E,1}$

$\chi^{(1)}_{E,1}$ $\chi^{(1)}_{E_{loc},1}$ E_{loc} E_j $= x_j^2$ $= x_{j+1}^2$ $= x_{j+2}^2$ $= x_{j-1}^2$

x_j

$$\chi^{(1)}_{x_j,1}(!) \quad \overline{!} \quad \overline{!^2}$$

4 Re Dl

4.1 SimDla ion and Da a Proce ing

$$\chi_{\Phi}^{(1)} \; ! \sim_{K\to\infty} \overline{K} \sum_{i=1}^{K} \chi_{\Phi}^{(1)} \; !; x_i; \epsilon; T \; ;$$

F [illegible]

[illegible] Φ [illegible] $!$ [illegible]

[illegible] K [illegible] ff [illegible]

[illegible] $!$ [illegible]

$!$ [illegible] $!_l$: π $!_h$ π [illegible] : π [illegible]

[illegible] $!$ [illegible] t [illegible] t T [illegible] π=$!$

[illegible]

$\epsilon \approx$ [illegible] ff [illegible]

[illegible] fi [illegible]

[illegible] ϵ : [illegible] K [illegible]

[illegible] ϵ [illegible] K

[illegible] f t [illegible]

[illegible]

K [illegible] T [illegible]

4.2 Global Per Drba ion

fi $\chi^{(1)}_{\varepsilon,N}$ $=N\chi^{(1)}_{E,N}$ " $E=N$

$! \approx$ $! \approx$

$!$

$\sim m=!$

$! \approx$

$\sim - F - m = !^2$

$\chi^{(1)}_{\mu,N}$ $=N\chi^{(1)}_{M,N}$ μ $M=N$

$! \approx$

[illegible] $\chi^{(1)}_{\mu,N}$ [illegible] $! \lesssim$ [illegible] $!$ [illegible] $!_l$ [illegible] fi [illegible] fi [illegible] fi [illegible] fi [illegible] $\chi^{(1)}_{\varepsilon,N}$ [illegible]

$$\chi^{(1)}_{\varepsilon,N}\ ! \ [illegible] \begin{cases} \frac{\omega}{\omega_l}\ [illegible]\ \chi^{(1)}_{\varepsilon,N}\ !_l\ [illegible]; & [illegible] \le ! \le !_l; \\ [illegible]\ \chi^{(1)}_{\varepsilon,N}\ !\ [illegible]; & !_l \le ! \le !_h; \\ \frac{m}{\omega}; & ! \ge !_h; \end{cases}$$

[illegible] $! \to$ [illegible] $\chi^{(1)}_{\varepsilon,N}$ [illegible] fi [illegible]

$$[illegible]\ \chi^{(1)}_{\varepsilon,N}\ ! \ [illegible] \begin{cases} [illegible]\ \chi^{(1)}_{\varepsilon,N}\ !_l\ [illegible]; & [illegible] \le ! \le !_l; \\ [illegible]\ \chi^{(1)}_{\varepsilon,N}\ !\ [illegible]; & !_l \le ! \le !_h; \\ -\frac{F-2m}{\omega^2}; & ! \ge !_h; \end{cases}$$

[illegible] $!$ [illegible] $\chi^{(1)}_{\mu,N}\ !$ [illegible]

[illegible] fi [illegible] fi [illegible] ! [illegible] π [illegible] $e(F)$ [illegible] $m(F)$ [illegible]

$$\frac{d\,m(F)}{dF} = \lambda\gamma F^{\gamma-1}; \quad (6)$$

$$\frac{d\,e(F)}{dF} = \lambda [illegible] \gamma) F^{\gamma} \quad m(F) [illegible] \gamma) : \quad (6)$$

[illegible] fi [illegible] e [illegible] m [illegible] X [illegible]

X_i x

δ t

4.4 FDr her implica ion of Kramer -Kronig rela ion and Dm rDle

$\mathsf{X}^{(1)}_{x_j,1}$ $\mathsf{X}^{(1)}_{M,1}$

$\mathsf{X}^{(1)}_{E_j,1}$ $\mathsf{X}^{(1)}_{E,1}$

E_{loc}

$$\int_0^\infty d\nu \; \chi^{(1)}_{\varepsilon,N} \; \nu \; ^2 \; ; \qquad (65)$$

$$\int_0^\infty d\nu\nu \; \chi^{(1)}_{\varepsilon,N} \; \nu \; ^2 \; \frac{\pi}{} m^2; \qquad (66)$$

$\chi^{(1)}_{\varepsilon,N}$! ^{2} $G^{(1)}_{\varepsilon,N}$ t

5 Prac ical Implica ion for Clima e Change S Dd- ie

ff Φ T_S F x CO_2 fi X x fi CO_2 f t fi

$$\langle T_S\rangle^{(1)}\ t \quad \int_{-\infty}^{+\infty} \sigma_1 G_{T_S}^{(1)}\ \sigma_1\ f\ \ t-\sigma_1\ :$$

t

6 Summary, Discussion and Conclusion

distance

Hasselmann programme

closure theory

[illegible]

fi fi

66 fi N

F

F N

66

fi

fi

spectroscopy of the climate system ffi

CO_2

fi

fi

fi

ffi

CO_2

δ t

6

[illegible] CO_2 [illegible]

http://www-pcmdi.llnl.gov/ climateprediction.net http://cmip-pcmdi.llnl.gov/cmip5/

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Reference

80

86

90

29

36

14

461

61

136

ff

22

131

12

57

206

81

59

131

234

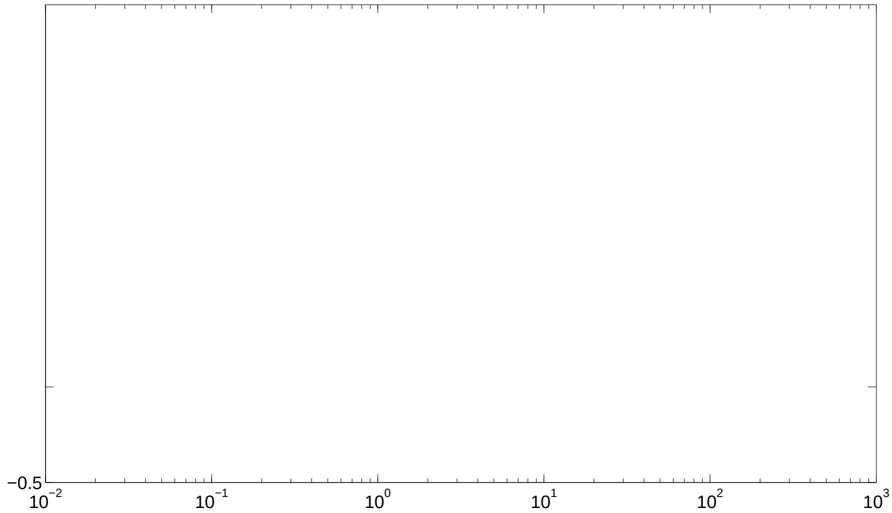
130

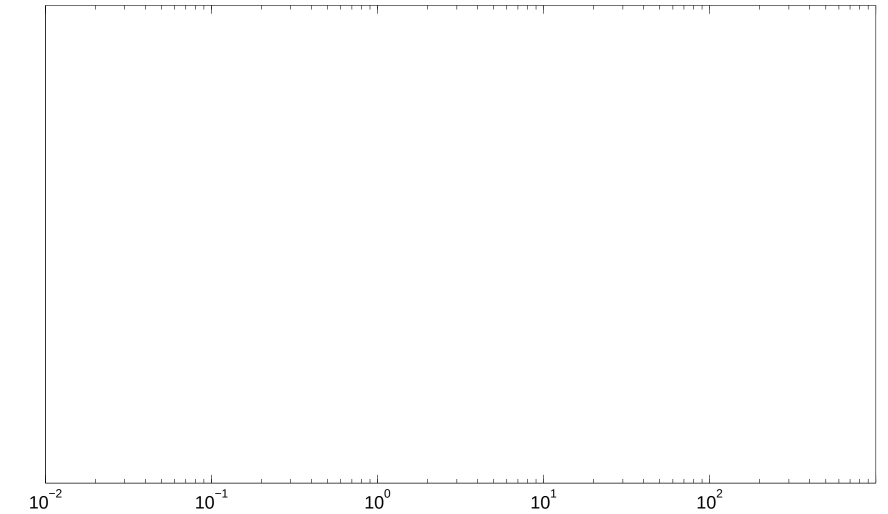
56

41

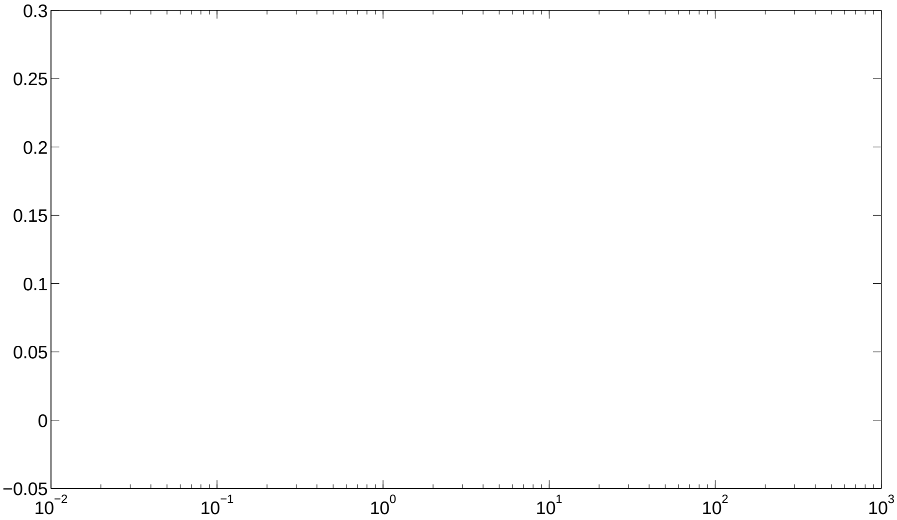
58

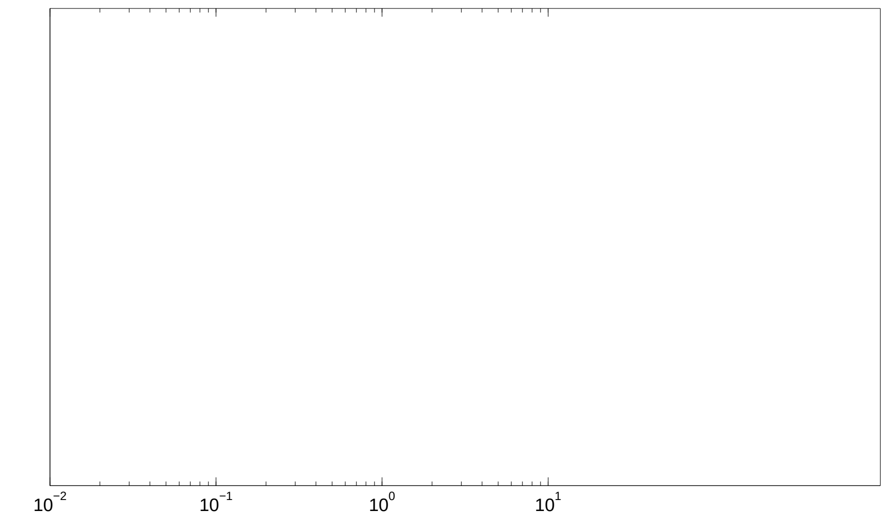
54



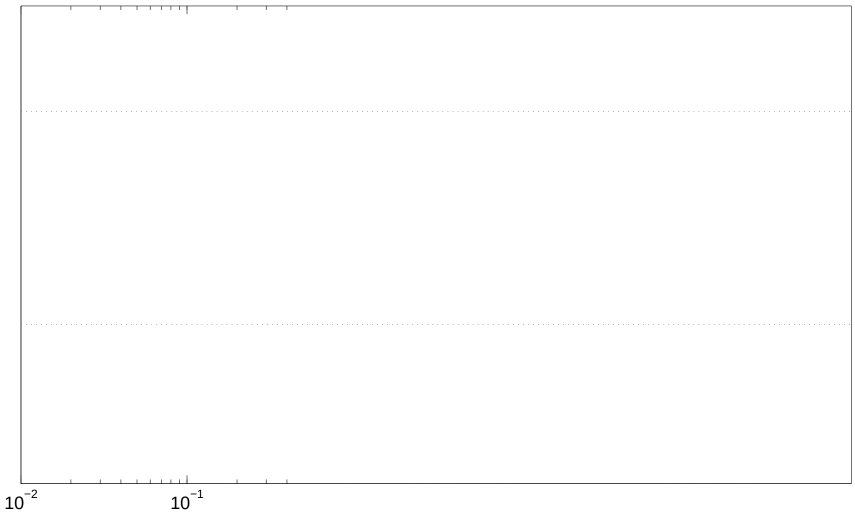


X_i $\forall i$ $E = N$

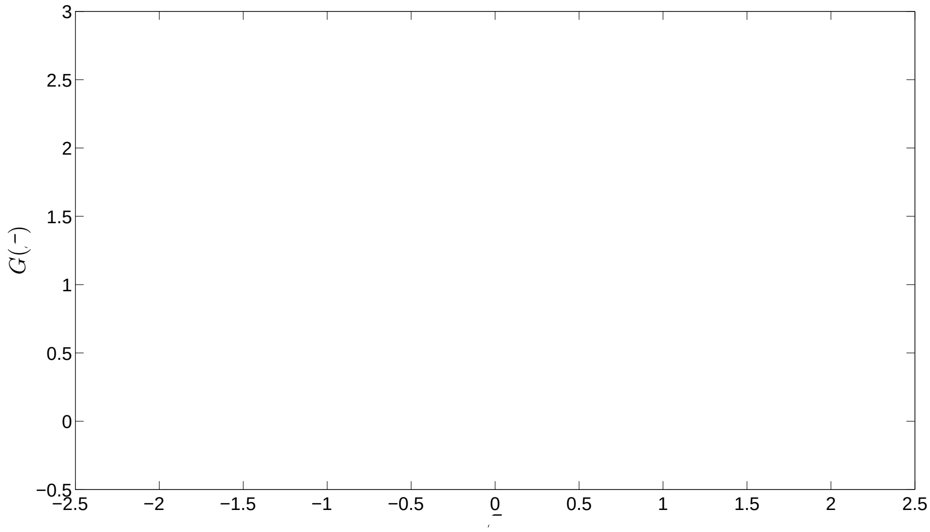




[illegible] $M=N$ [illegible] X_i [illegible] $\forall i$ [illegible]



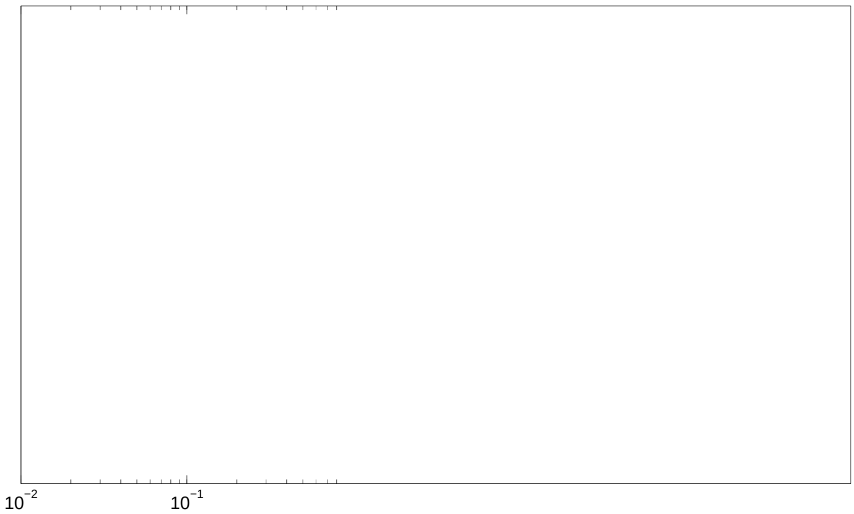
$E = N$ $M = N$ X_i $\forall i$ fi



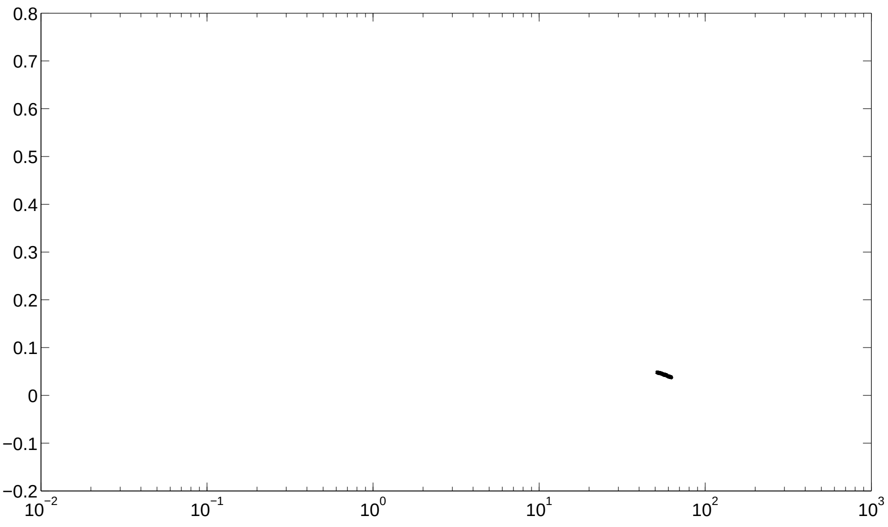
$E = N$ $M = N$

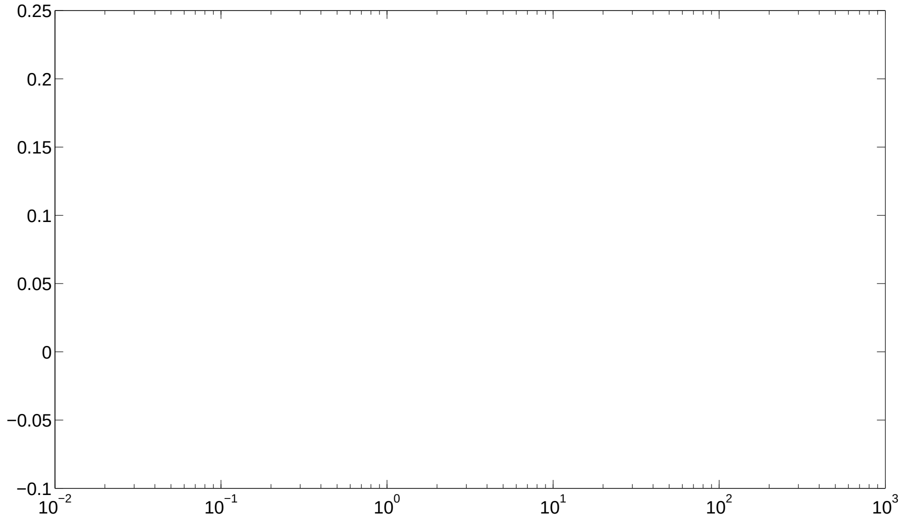
X_i $\forall i$

fi



10−2
10−1


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}


0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}