

Department of Mathematics

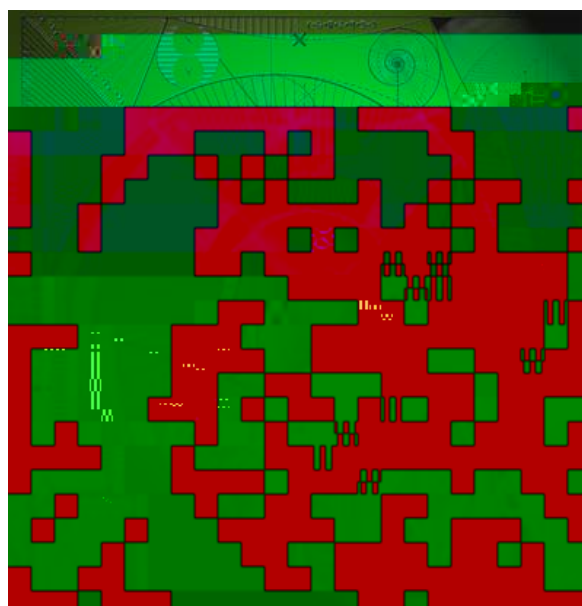
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A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

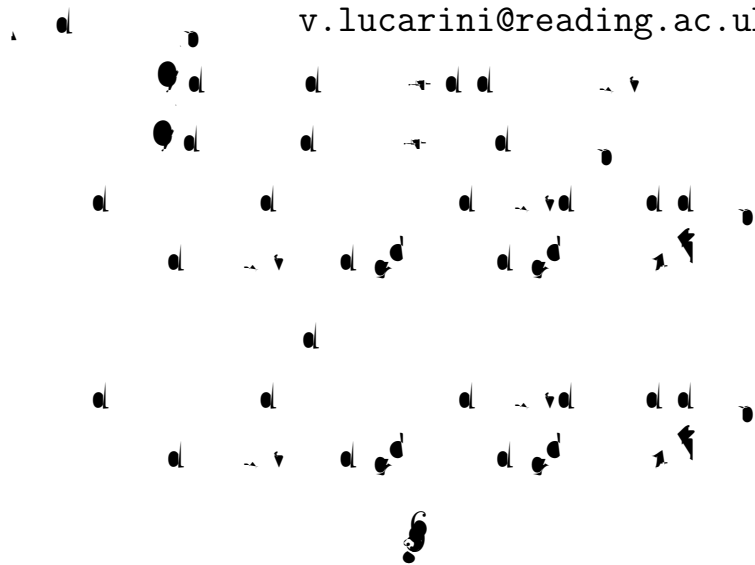
by

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A Statistical Mechanical Approach for the Computation of the Climatic Response to General Forcings

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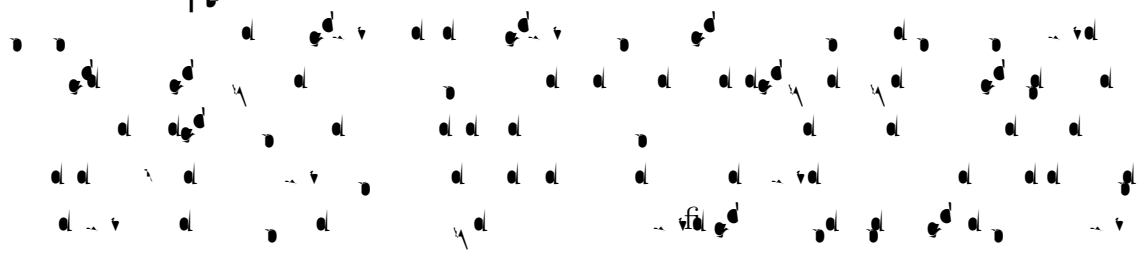
Abstract

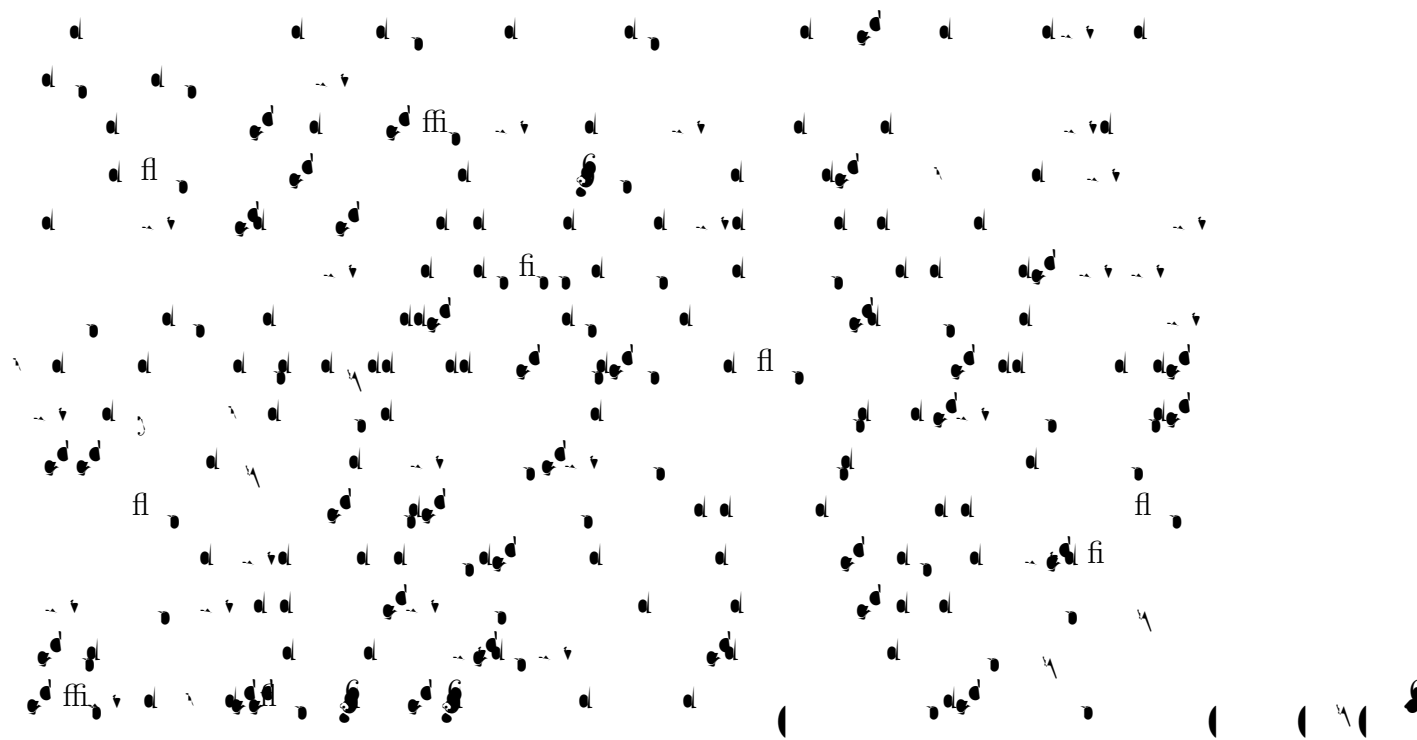
The climate belongs to the class of non-equilibrium forced and dissipative systems, for which most results of quasi-equilibrium statistical mechanics, including the fluctuation-dissipation theorem, do not apply. In this paper we show for the first time how the Ruelle linear response theory, developed for studying rigorously the impact of perturbations on general observables of non-equilibrium statistical mechanical systems, can be applied with great success to analyze the climatic response to general forcings. The crucial value of the Ruelle theory lies in the fact that it allows to compute the response of the system in terms of expectation values of explicit and computable functions of the phase space averaged over the invariant measure of the unperturbed state. We choose as test bed a classical version of the Lorenz 96 model, which, in spite of its simplicity, has a well-recognized prototypical value as it is a spatially extended one-dimensional

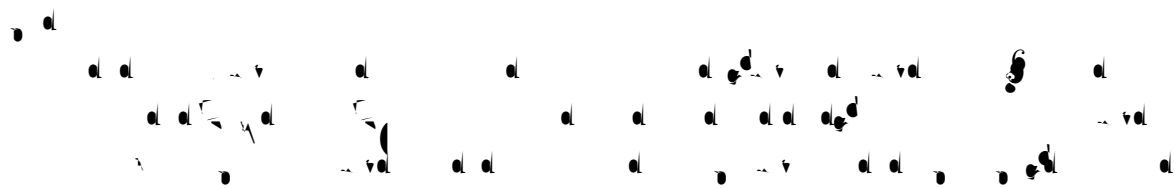
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1 Introduction







fi

66

fi

CO₂

ff

fi

$$\langle \Phi \rangle^{(1)} = \int_{-\infty}^{+\infty} \chi_{\Phi}^{(1)}(f) \times \delta(f - \omega) \chi_{\Phi}^{(1)}(f) ;$$

$$\chi_{\Phi}^{(1)} = \int_{-\infty}^{+\infty} t G_{\Phi}^{(1)}(t) dt :$$

$$\begin{aligned} & \sum_{\alpha} \chi_{\alpha} \langle \Phi_{\alpha} \rangle^{(1)}(t) \\ & \chi_{\Phi}^{(1)}(f_{\beta}) \\ & f_{\beta} \\ & \chi_{\Phi}^{(1)}(\Phi) \\ & \langle \Phi_{\beta} \rangle^{(1)}(t) \\ & G_{\Phi}^{(1)}(t) \\ & \langle \Phi \rangle^{(1)}(t) \\ & G_{\Phi}^{(1)}(t) \end{aligned}$$

2.2 Kramer-Kronig relation and Dm rDle

$$G_{\Phi}^{(1)}(t) \in L^2$$

$$\int_{-\infty}^{+\infty} t \Theta(t) t^k dt = -i \frac{k!}{k!} \left(P - \pi \delta \right) \approx k \frac{(k+1)}{(k+1)}$$

$$G_{\Phi}^{(1)}(t) \approx \bar{\alpha} \Theta(t) t^{\beta} + o(t^{\beta})$$

$$\chi_{\Phi}^{(1)} \approx \alpha \Gamma^{-\beta-1} \quad \text{or} \quad \Gamma^{-\beta-1}$$

$$\frac{\pi}{2^{2p}} \chi_{\Phi}^{(1)}(v) = P \int_0^\infty \frac{v^{2p+1} \chi_{\Phi}^{(1)}(v)}{v^2 - v'^2} dv'$$

$$P \quad p \quad ; :: : \beta - = \beta$$

p ;:::; $\beta = \beta$

$$\chi^{(1)} = \frac{P}{\pi} \int \frac{\chi^{(1)}(v)}{v} dv;$$

$$\begin{aligned} & \chi^{(1)} \quad \text{fi} \quad \chi^{(1)} \quad ! \rightarrow \quad ! \sim \quad c_1 \quad c_2 !:^2 \quad o !^3 \quad c_1 \quad c_2 \\ & ! \rightarrow \infty \end{aligned}$$

2.3 A practical formula for the linear dependence and consistency relation between dependence of different observable

$$\begin{aligned} & \text{if } f(t) \text{ is a function of } t, \text{ then } f(t) = f(t) \\ & \text{if } f(t) \text{ is a function of } t, \text{ then } f(t) = f(t) \end{aligned}$$

$$\begin{aligned} & \delta \Phi_\epsilon(t; t_0; x_0) = \Phi_\epsilon(t; t_0; x_0) - \Phi_0(t; t_0; x_0) \\ & \delta \Phi_\epsilon(t; t_0; x_0) = O(\epsilon) \end{aligned}$$

$$\chi_\Phi^{(1)} \equiv \lim_{\epsilon \rightarrow 0, T \rightarrow \infty} \chi_\phi^{(1)}; x_0; T$$

$$\begin{aligned} & \chi_\Phi^{(1)}; x_0; T = \frac{1}{T} - \int_0^T dt \delta \Phi_\epsilon(t; x_0) \\ & \delta \Phi_\epsilon(t; x_0) = \delta \Phi_\epsilon(t; x_0) \end{aligned}$$

$$\begin{aligned} & \chi_\Phi^{(1)} - \chi_\Phi^{(1)} : \\ & \chi_\Phi^{(1)} \end{aligned}$$

$$\begin{aligned} & \Phi(x) = \Gamma(x); \\ & \Gamma = F \cdot \nabla \Phi \end{aligned}$$

$$F(x) = X(x) f(t) \quad F(x) = \dots, \quad t \in X(x)$$

$$i = \langle \Xi \rangle_0 \quad \langle \Xi \rangle_0 \quad \langle \Xi \rangle_0 / \quad \chi_\Phi^{(1)} \approx$$

$$G_\Phi^{(1)}(t) = \Theta(t) \langle \Xi \rangle_0 + t^0;$$

3 Application of the Response Theory to the Lorenz-96 Model

3.1 Statistical properties of the Driven Lorenz-96 Model

$$\frac{dx_i}{dt} = x_{i-1}x_{i+1} - x_{i-2} - x_i - F$$

$$i = 1, \dots, N \quad x_{i+N} = x_{i-N} = x_i$$

$$E = \sum_i x_i^2$$

E —

$$\begin{aligned}
 & \gamma \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right) \\
 & \quad \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right) \\
 & \quad \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right)
 \end{aligned}$$

3.2 Asymptotic properties of the linear Dependability

$$\begin{aligned}
 & \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right) \\
 & \quad \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right)
 \end{aligned}$$

$$\frac{x_i}{t} = x_{i-1} - x_{i+1} - x_{i+2} - x_i = F$$

$$\begin{aligned}
 & \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right) \\
 & \quad \left(\frac{1}{t} \left(\sum_{i=1}^N x_i \right) - \frac{1}{t} \left(\sum_{i=1}^N x_i \right) \right)
 \end{aligned}$$

3.2.1 Global per Drba ion

$$\rho_0(x) = \rho_0(x) \Theta(t) \Lambda \Pi_0(t) E(x) \quad \forall i \quad G_{E,a}^{(1)}(t)$$

$$G_{E,N}^{(1)}(t) = \int \rho_0(x) \Theta(t) \Lambda \Pi_0(t) E(x) \int \rho_0(x) \Theta(t) \sim \cdot \nabla E(x) t \int \rho_0(x) \Theta(t) \sum_i @_i E(x) t$$

$$G_{E,N}^{(1)}(t) = \int \rho_0(x) \Theta(t) \sum_i @_i (E|_{t=0} - tE|_{t=0} \circ t) \int \rho_0(x) \Theta(t) \left[\sum_i x_i - \left(\sum_i x_i - NF \right) t \circ t \right]:$$

$$\chi_{E,N}^{(1)}(t) = i \left(\sum_i \langle x_i \rangle_0 \right) = i \left(\sum_i \langle x_i \rangle_0 - NF \right) = i^2 \circ t^{-2} iN \frac{m}{t} - N \frac{F - m}{t^2} \circ t^{-2}$$

$$\begin{aligned}
 & \text{if } \lim_{l \rightarrow \infty} \chi_{\varepsilon, N}^{(1)} = N \chi_{E, N}^{(1)} \chi_{\varepsilon, N}^{(1)} \\
 & \text{!} \rightarrow \infty
 \end{aligned}$$

$$\begin{aligned}
 & \chi_{\varepsilon, N}^{(1)} \left(i \frac{m}{l} - \frac{F - m}{l^2} \right) o \cdot l^{-2} : \\
 & \pi = \chi \\
 & \chi
 \end{aligned}$$

$$\chi_{E,N}^{(1)} = \frac{F}{-}$$

$$\int_0^\infty \chi_{E,1}^{(1)} \frac{\pi}{l} ;$$

$$\chi_{M,\epsilon,1}^{(1)}$$

$$\chi_{M,1}^{(1)} \frac{\pi}{l^2} ;$$

$$\int_0^\infty \chi_{M,1}^{(1)} \frac{\pi}{l} ;$$

$$E_j \quad x_j$$

$$G_{E_j,1}^{(1)}(t) = \int \rho_0(x) \Theta(t) @_j \left[-x_1^2 - (x_j - x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F) t - o(t) \right] \\ \Theta(t) \left[\langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F \rangle_0 t - o(t) \right];$$

$$\langle x_j \rangle_0 - \langle x_{j-1}x_{j+1} - x_{j-1}x_{j-2} - x_j - F \rangle_0$$

$$\rho_0$$

$$\chi_{E_j,1}^{(1)} = i \frac{m}{l} - \frac{m}{l^2} - o(l^{-2}) ;$$

$$E_j \quad \chi_{E,1}^{(1)} = \sum_k \chi_{E_k,1}^{(1)}$$

$$\chi_{E_j,1}^{(1)} \approx \chi_{E,1}^{(1)}$$

$$\begin{aligned}
& \int_0^\infty \chi_{E_j,1}^{(1)} \left(\frac{\pi}{2} m \right) \\
& \propto l^{-2} \\
& \chi_{E,1}^{(1)} \chi_{E_j,1}^{(1)} x_j \\
& = E_{j+1} E_{j+2} E_{j-1}
\end{aligned}$$

$$\chi_{\psi,1}^{(1)} \left(-\frac{\langle x_{j-1} x_{j+1} - x_{j-2} \rangle_0}{l^2} - \frac{F-m}{l^2} \right) o l^{-2} :$$

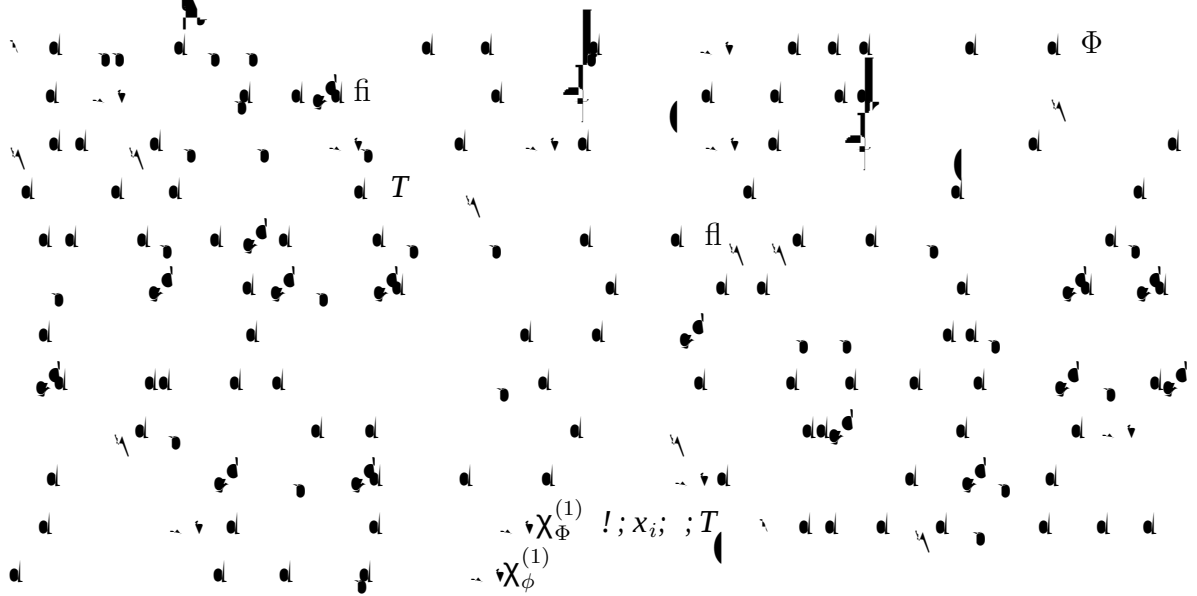
$$\begin{aligned}
& \chi_{\psi,1}^{(1)} \chi_{E_j,1}^{(1)} \chi_{E,1}^{(1)} \\
& \chi_{E,1}^{(1)} \chi_{E_{loc},1}^{(1)} E_{loc} E_j = x_j^2 = x_{j+1}^2 = x_{j+2}^2 = x_{j-1}^2 \\
& x_j
\end{aligned}$$

$$\chi_{x_j,1}^{(1)} \left(\frac{1}{l} - \frac{1}{l^2} \right)$$



4 Re DL

4.1 Simulation and Data Processing



$$\chi_{\Phi}^{(1)} \sim_{K \rightarrow \infty} \frac{1}{K} \sum_{i=1}^K \chi_{\Phi}^{(1)} ; x_i ; T ;$$



F Φ K ff $!$ $!$ $!_l$ $:\pi$ $!_h$ π $:\pi$ t t T $\pi=!$

$\approx$ ff ff ff K K f t

K T

$$\chi_{\varepsilon,N}^{(1)} = N \chi_{E,N}^{(1)} \quad E=N$$

[illegible]

$$\sim -F - m_{\pi}^2$$

$$\chi_{\mu,N}^{(1)} = N \chi_{M,N}^{(1)} \quad \mu \leq M \leq N$$

$$\begin{aligned}
& \chi_{\mu,N}^{(1)} \\
& \leq \\
& \chi_{\varepsilon,N}^{(1)}
\end{aligned}$$

$$\chi_{\varepsilon,N}^{(1)} \leq \begin{cases} \frac{\omega}{\omega_l} \chi_{\varepsilon,N}^{(1)}; & l \leq l; \\ \chi_{\varepsilon,N}^{(1)}; & l_l \leq l \leq l_h; \\ \frac{m}{\omega}; & l \geq l_h; \end{cases}$$

$$\begin{aligned}
& \rightarrow \\
& \chi_{\varepsilon,N}^{(1)}
\end{aligned}$$

$$\chi_{\varepsilon,N}^{(1)} \leq \begin{cases} \chi_{\varepsilon,N}^{(1)}; & l \leq l; \\ \chi_{\varepsilon,N}^{(1)}; & l_l \leq l \leq l_h; \\ -\frac{F-2m}{\omega^2}; & l \geq l_h; \end{cases}$$

$$\begin{aligned}
& \chi_{\mu,N}^{(1)}
\end{aligned}$$

A complex musical score for a string quartet, featuring various musical notations such as notes, rests, and dynamic markings like "fi" and "pi". The score is written for four staves, each representing a different instrument. The notation includes a variety of note values, rests, and articulation marks, creating a dense and intricate musical texture. The dynamic markings "fi" and "pi" are placed at specific points in the score, indicating changes in volume or intensity. The overall style is that of a classical or contemporary string quartet composition.

$$\frac{m}{F} \ll \lambda \gamma F^{\gamma-1};$$

$$\frac{e F}{F} \lambda \frac{Y}{F^\gamma} m F \frac{Y}{F} =$$

$$X_i x$$

$\chi_{x_j,1}^{(1)}$ $\chi_{M,1}^{(1)}$ $\chi_{E_j,1}^{(1)}$ $\chi_{E,1}^{(1)}$ E_{loc}

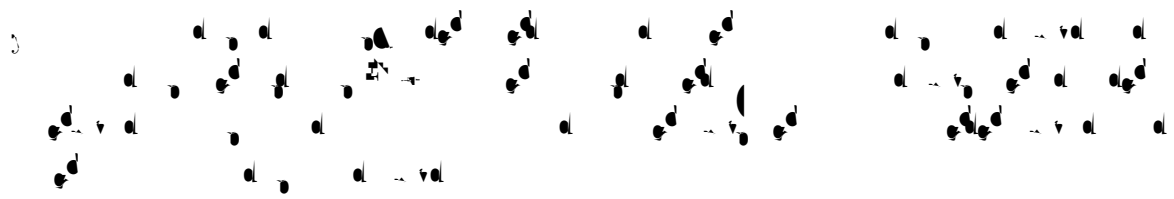
$$\int_0^\infty \langle v | \chi_{\varepsilon,N}^{(1)} | v \rangle^2 dv = \frac{\pi}{2} m^2;$$

$$\int_0^\infty \langle v | \chi_{\varepsilon,N}^{(1)} | v \rangle^2 dv = \frac{\pi}{2} m^2;$$

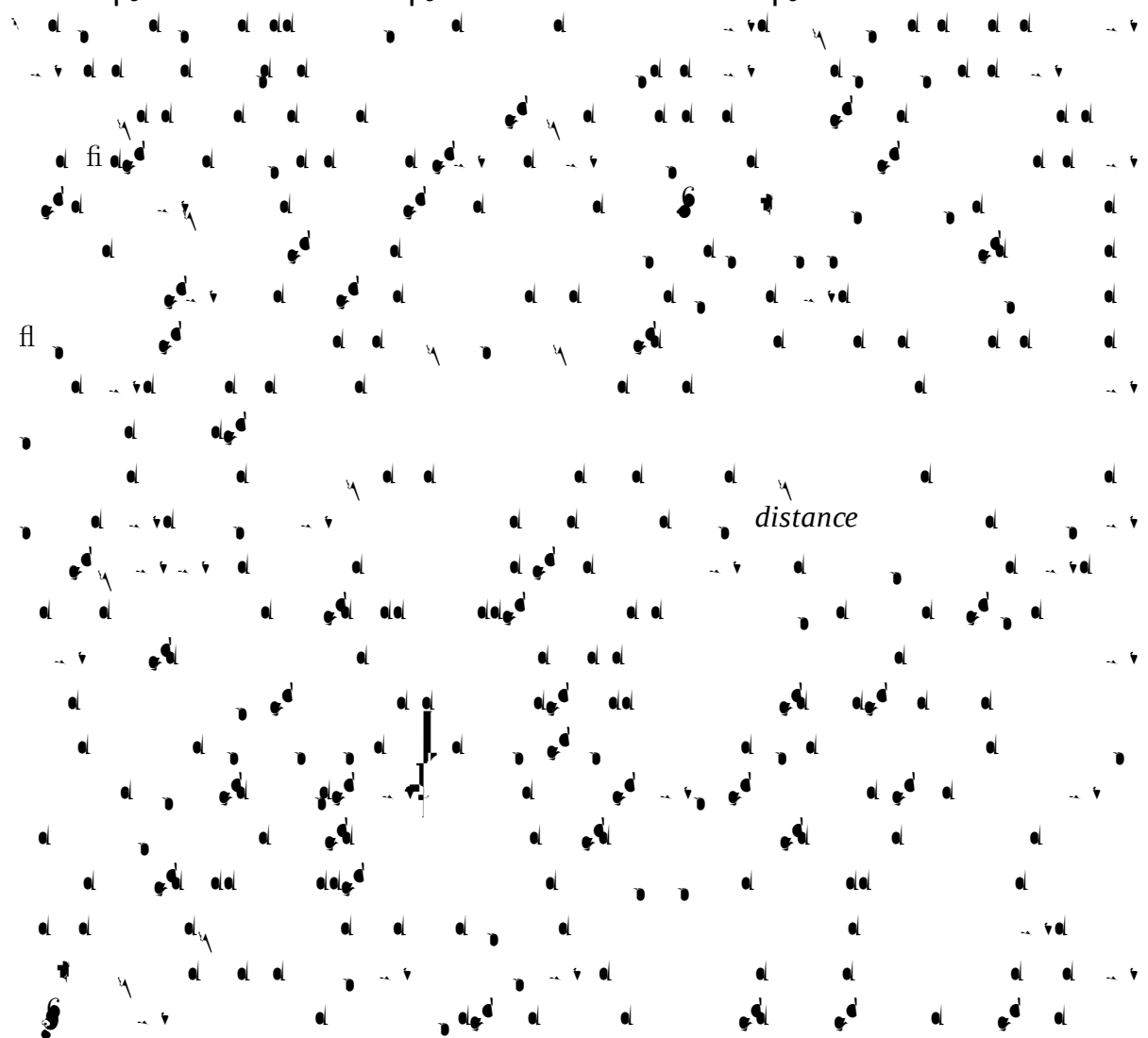
$$G_{\varepsilon,N}^{(1)}(t) = \int_0^\infty \langle v | \chi_{\varepsilon,N}^{(1)} | v \rangle^2 dv$$

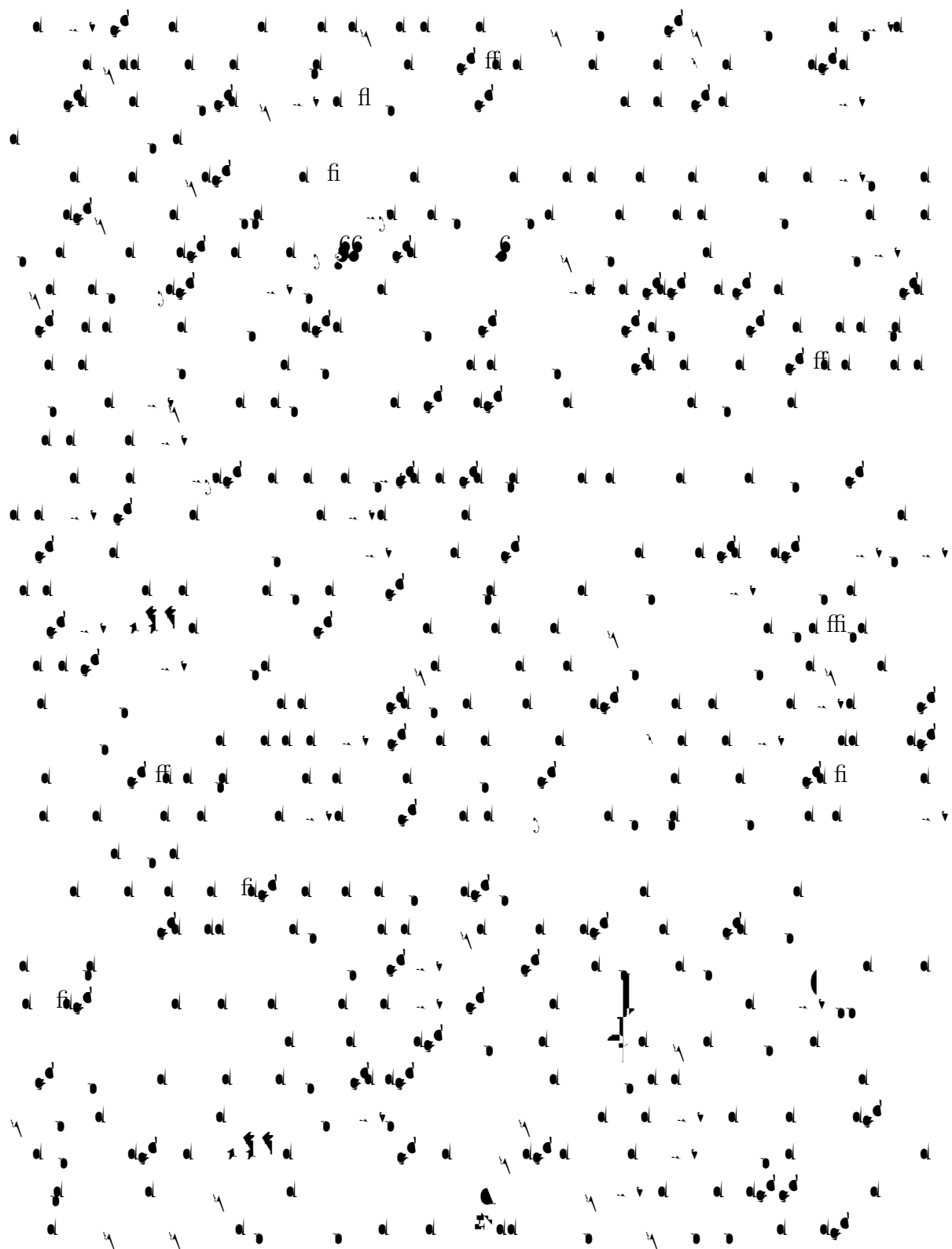
5 Practical Implication for Climate Change Scenarios

$$\begin{aligned} & \text{The following equation is used to calculate the change in temperature } T_S \text{ due to a change in } CO_2 \text{ concentration:} \\ & \Delta T_S = \frac{F_{\Delta CO_2}}{4} \frac{1}{\Phi} \frac{1}{T_S} \frac{1}{X_{CO_2}} \frac{1}{f} \frac{1}{t} \end{aligned}$$



6 Summary, Discussion and Conclusion





The diagram is a complex network of interconnected nodes and lines, representing a system of relationships. Key elements include:

- climate system**: Located on the left side, with a large, bold, italicized font.
- spectroscopy of the**: Located on the right side, with a large, bold, italicized font.
- CO₂**: Appears multiple times, associated with specific nodes and lines.
- delta t**: Located at the bottom right, with a large, bold, italicized font.
- Mathematical symbols and labels**: Various symbols like F , N , f_i , 66 , δ , and t are scattered throughout the diagram, often appearing in bold or italicized fonts.
- Nodes and Lines**: The diagram consists of numerous small, dark, irregular shapes (nodes) connected by thin, light-colored lines, forming a dense, interconnected network.

The image shows a page of a musical score, likely for a symphony or orchestral work. The score is written in a complex, modern style, featuring a large number of notes, rests, and dynamic markings. The notation is dense and spans across multiple staves. Key markings include 'ff' (fortissimo), 'fi' (fistissimo), and a large '66' which could be a measure number or a tempo marking. The score is presented in a black and white format, with the notes and markings clearly visible against the white background.

- <http://www-pcmdi.llnl.gov/>
climateprediction.net <http://cmip-pcmdi.llnl.gov/cmip5/>

VL acknowledges the financial support of the EU-ERC research grant NAMASTE and of the Walker Institute Research Development Fund.

Reference

This image shows a page of musical notation, likely a score for a string ensemble or orchestra. The notation is written on multiple staves, with various musical symbols, notes, and rests. The page includes several measures, with specific measures numbered: 80, 86, 90, and 29. The notation is complex, featuring many beamed notes and dynamic markings such as *ff* (fortissimo) and *f* (forte). The page is oriented vertically, with the music written from top to bottom.

This image shows a page of musical notation, likely a score for a piece. The notation is written on a single staff and includes various musical symbols such as notes, rests, and bar lines. The page is divided into several measures, with measure numbers 36, 14, 66, 461, 61, 136, 22, 131, 12, and 57 clearly visible. The notation is dense and complex, suggesting a high level of musical sophistication. The page is oriented vertically, with the musical staff running from top to bottom. The background is white, and the notation is in black ink.

36

14 6 66

461

61

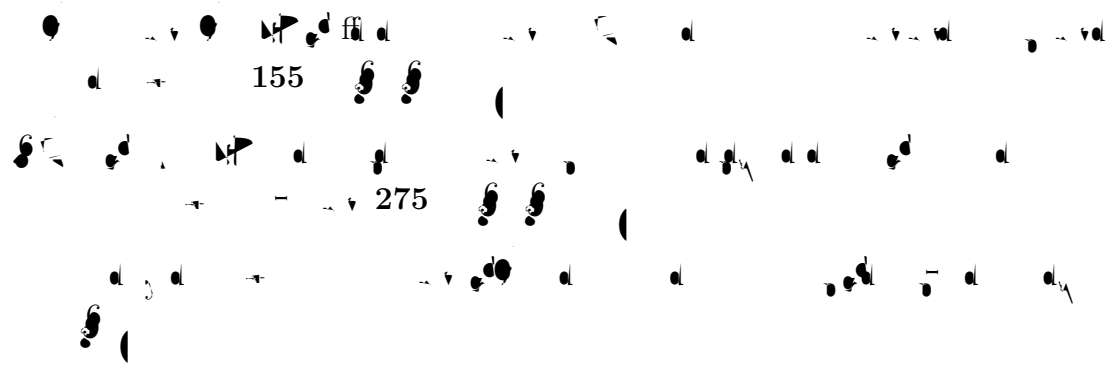
136 66

22

131

12

57



6/8

206

81

59

131

234

130

56

41

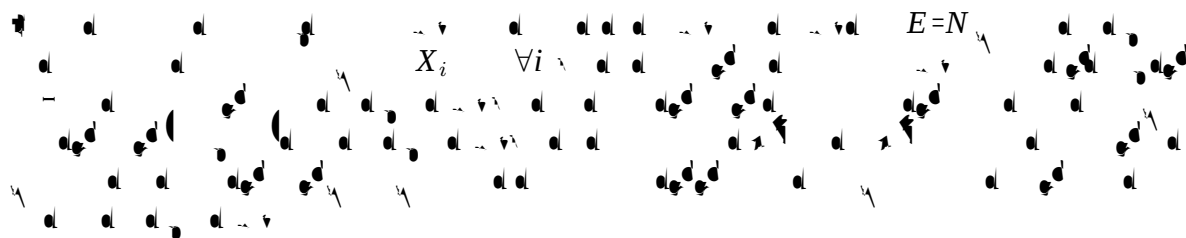
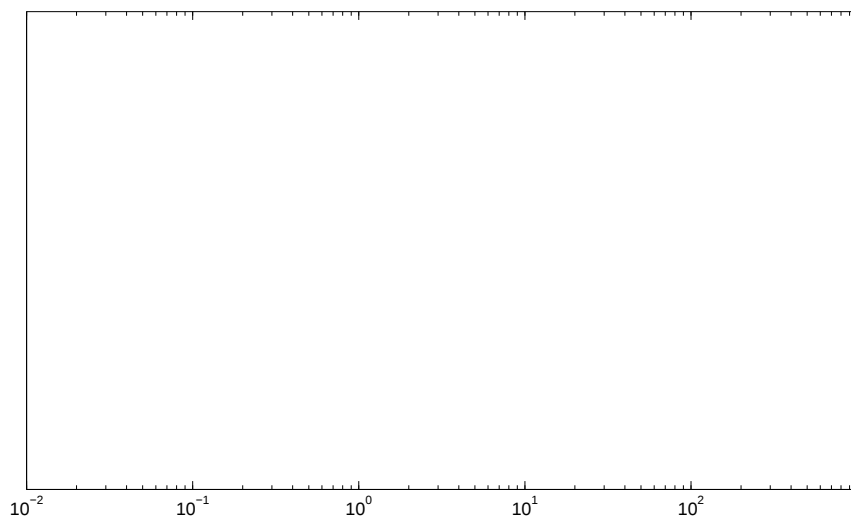
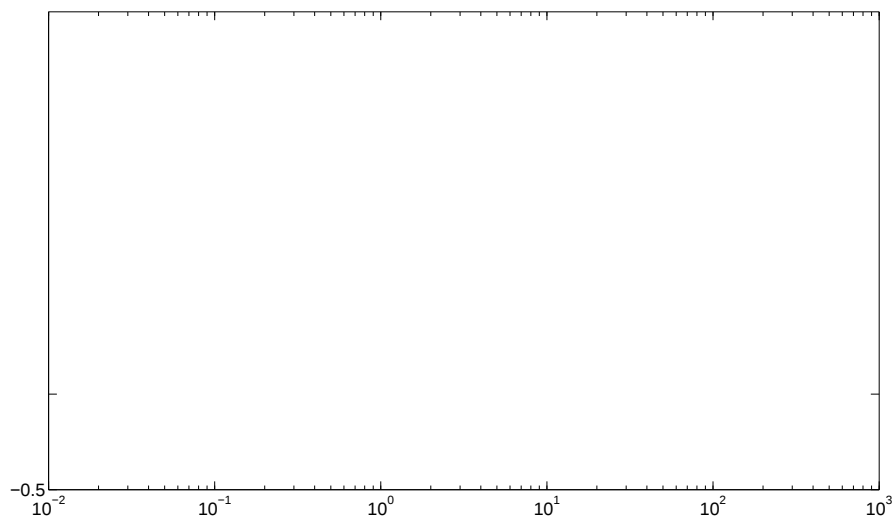
58

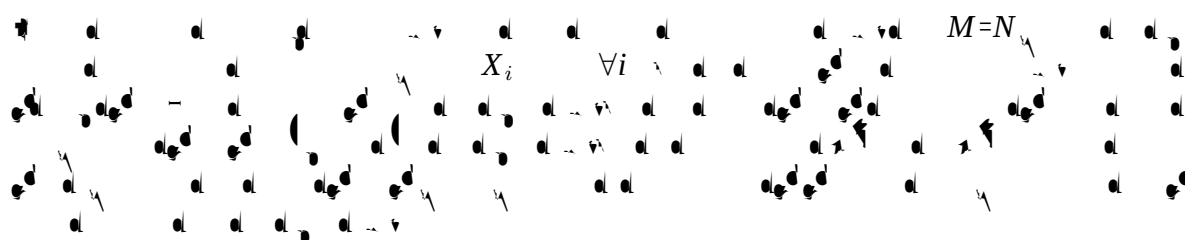
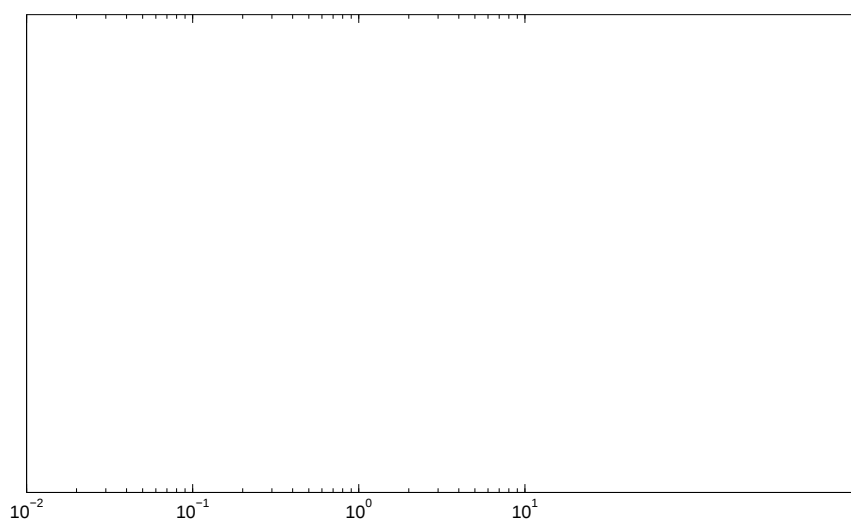
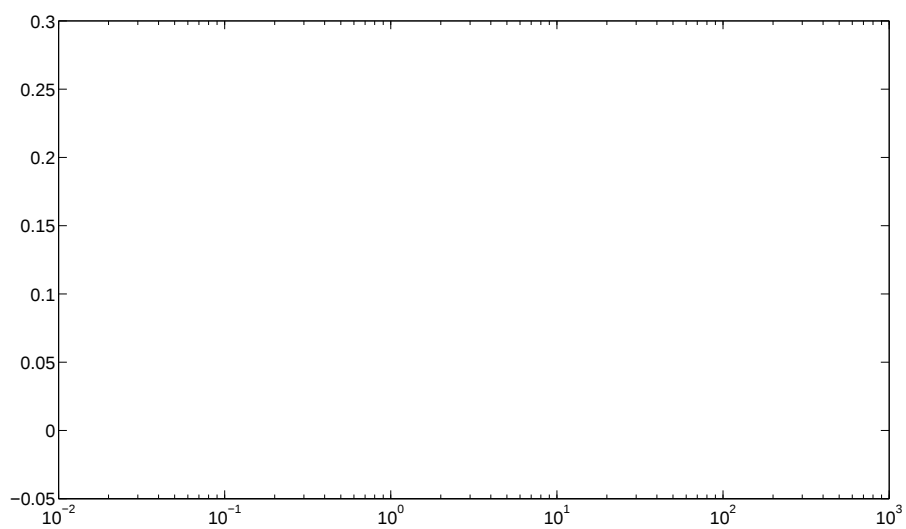
54

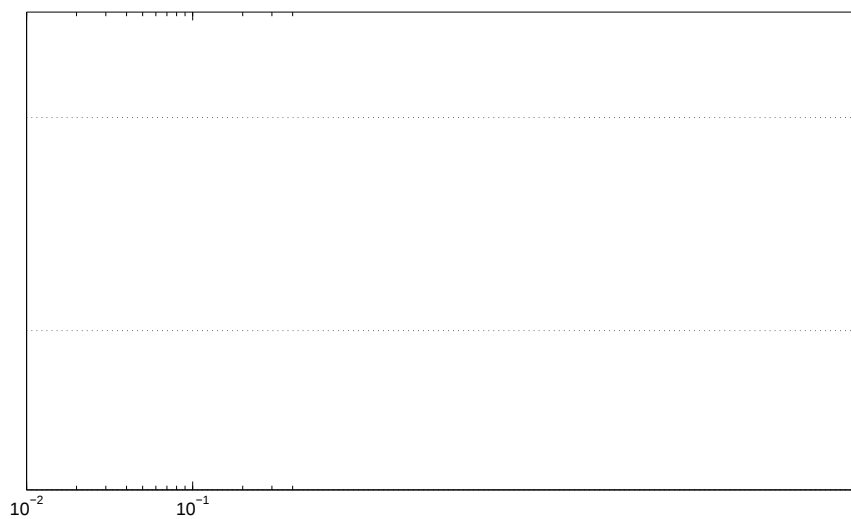
66

ff

p







$$E=N \quad \quad \quad X_i \quad \quad \quad M=N \quad \quad \quad \forall i \quad \quad \quad \text{fi}$$

