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Abstract

Implementations of incremental variational data assimilation require the iterative minimization of a series of linear cost functions. The accuracy and speed with which these linear minimization problems can be solved is determined by the condition number of the Hessian of the problem. In this study we examine how different components of the assimilation system influence this condition number. Theoretical bounds on the condition number for a single parameter system are presented and used to predict how the condition number is affected by the observation distribution and accuracy and by the specified lengthscales in the background error covariance matrix. The theoretical results are verified in the Met Office variational data assimilation system, using both pseudo-observations and real data.

1 Introduction

An important component of numerical weather prediction (NWP) systems is the determination of an appropriate set of initial conditions from observations by means of techniques of data assimilation. In general observations of the atmosphere are only indirectly related to model variables and are many fewer in number than the number of model states that need to be initialized. Data assimilation techniques aim to combine these measurements with a previous forecast,

Variational data assimilation is the method of choice for many current NWP

fice assimilation scheme. In particular we show, using both theory and the operational system, that the conditioning of variational data assimilation is de-

In the case of incremental 4DVar the linearized minimization problem takes the form

$$
\begin{aligned}
\tilde{J}(\delta\mathbf{x}_0) &= \frac{1}{2}[\delta\mathbf{x}_0 - (\mathbf{x}_0 - \mathbf{x}_0)]^T\mathbf{B}^{-} \ [\delta\mathbf{x}_0 - (\mathbf{x}_0 - \mathbf{x}_0)] \\
&+ \frac{1}{2}\sum_{i \quad 0}^{n}(\mathbf{H}_i\delta\mathbf{x}_i - \mathbf{d}_i)^T\mathbf{R}_i^{-} \ (\mathbf{H}_i\delta\mathbf{x}_i - \mathbf{d}_i), \qquad (3)
\end{aligned}
$$

subject to ii

For example, the rate of convergence of the conjugate gradient method can be bounded in terms of the condition number of the Hessian (Golub and Van Loan, 1996, Theorem 10.2.6).

For the preconditioned variational cost function (5) the Hessian is given by the expression

$$\mathrm{S} = \mathrm{I} + \sum_{i=0}^{n} \mathrm{B}^{1/2}\mathrm{M}(\mathbf{t}_i,\mathbf{t}_0)^T\mathrm{H}_i^T\mathrm{R}_i^{-1}\mathrm{H}_i\mathrm{M}(\mathbf{t}_i,\mathbf{t}_0)\mathrm{B}^{1/2}, \qquad (6)$$

where $\mathrm{M}(\mathbf{t}_i,\mathbf{t}_0) = \mathrm{M}(\mathbf{t}_i,\mathbf{t}_{i-1})\mathrm{M}(\mathbf{t}_{i-1},\mathbf{t}_{i-2})\ldots\mathrm{M}(\mathbf{t}_1,\mathbf{t}_0)$. In general we have fewer observations than variables we are trying to estimate and so the second term in the expression (6) is not full rank. In this case the smallest eigenvalue of S is one and the condition number is equal to the largest eigenvalue. We now investigate the different factors that affect the conditioning of the preconditioned Hessian. We first present the conditioning theory using a simple system.

3 Conditioning for a single parameter system

3.1 Theory

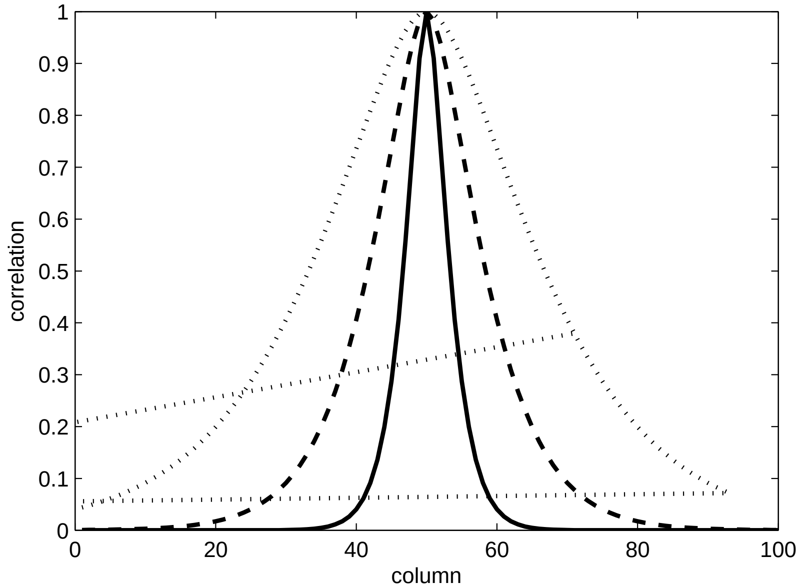
We consider the case of 3DVar applied to a single parameter system on a one-dimensional periodic domain, discretized using N grid points $\xi_i, i = 1,\ldots,N$. For this case the preconditioned Hessian (6) reduces to

$$\mathrm{S} = \mathrm{I}_N + \mathrm{B}^{1/2}\mathrm{H}^T\mathrm{R}^{-1}\mathrm{H}\mathrm{B}^{1/2}. \qquad (7)$$

We assume we have a set of $p < N$ direct observations of the parameter at grid points, so that $\mathrm{H}^T\mathrm{H}$ is a diagonal matrix, where the diagonal element is equal to one if that component of the state is observed and zero otherwise. We further assume the observation errors are uncorrelated with error variance σ_o^2, which implies that the observation error covariance matrix $\mathrm{R} = \sigma_o^2\mathrm{I}_p$. We write the background covariance matrix in the form $\mathrm{B} = \sigma^2\mathrm{C}$ where C is the background error correlation matrix with components $c_{i,j}$ and σ^2 denotes the background error variance. We assume that the correlation structures are homogeneous and isotropic, so that the coefficients $c_{i,j}$ depend only on the distance between points ξ_i and ξ_j. Then, under the assumptions given we can show that the condition number of the matrix (7) satisfies

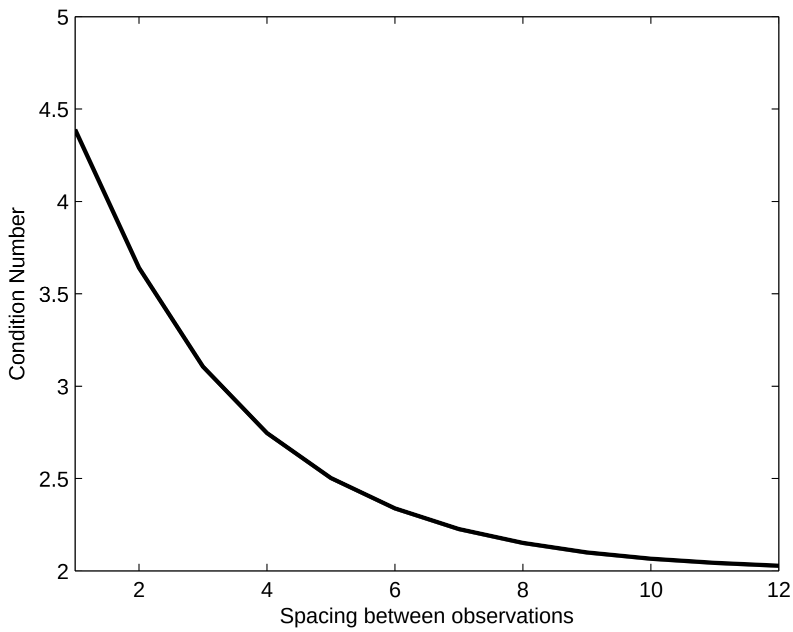
$$\begin{aligned} 1 + \frac{\sigma^2}{\sigma_o^2}\beta &\leq \kappa(\mathrm{I}_N + \mathrm{B}^{1/2}\mathrm{H}^T\mathrm{R}^{-1}\mathrm{H}\mathrm{B}^{1/2}) \\ &\leq 1 + \frac{\sigma^2}{\sigma_o^2}||\mathrm{H}\mathrm{C}\mathrm{H}^T||_\infty \\ &= 1 + \frac{\sigma^2}{\sigma_o^2}\left(\max_{i\in J}\sum_{j\in J}|c_{i,j}|\right), \end{aligned} \qquad (8)$$

where β = $\frac{}{p} \sum_{i\ j \in J} c_{i\ j}$, and J is the set of indices



correlation
column
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
20
40
60
80
100

Variance σ_o^2	


5
4.5
4
3.5
3
2.5
2
Condition Number
2
4
6
8
10
12
Spacing between observations

condition number,

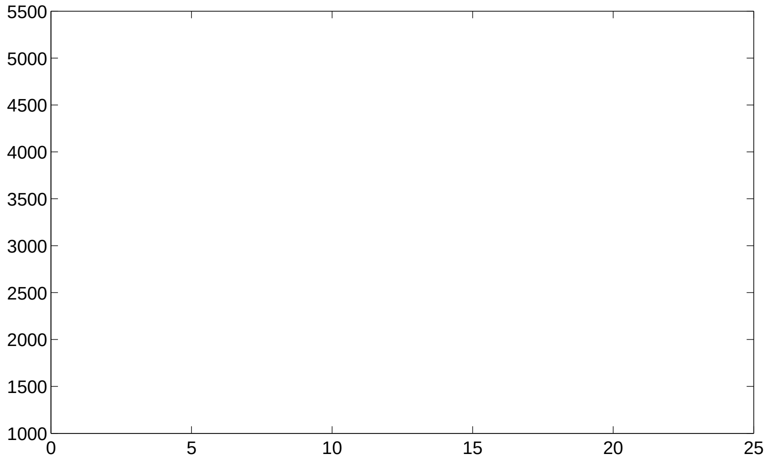
$$
1 + \frac{1}{(n+1)p}\frac{\sigma^2}{\sigma_o^2}\sum_{i\,j}^{(n+1)p}(\hat{\mathrm{H}}\mathrm{C}\hat{\mathrm{H}}^T)_{i\,j}
\leq \kappa(\mathrm{S}) \leq 1 + \frac{\sigma^2}{\sigma_o^2}||\hat{\mathrm{H}}\mathrm{C}\hat{\mathrm{H}}^T||_\infty. \tag{13}
$$

A comparison of this expression with (8) shows that the bounds on the condition number of the 4DVar Hessian for this system are very similar to those of the 3DVar Hessian, with the matrix $\hat{\mathrm{H}}$ taking the place of the linear observation operator H. Hence many of the qualitative conclusions discussed in section 3.1

	Condition number	
Error Variance	3DVar	4DVar
1	152422	180781
10	15243	18078
25	6098	7232
50	3050	3618
75	2033	2412
100	1525	1809

Table 2: Change of condition number with observation error variance in the Met Office 3DVar and 4DVar systems using pseudo-observations.

The largest eigenvalues of the Hessian are calculated using the Lanczos method and, since the Hessian is of the form (6), the condition number is simply equal to the largest eigenvalue. We begin by verifying the theory developed above


5500
5000
4500
4000
3500
3000
2500
2000
1500
1000
0
5
10
15
20
25

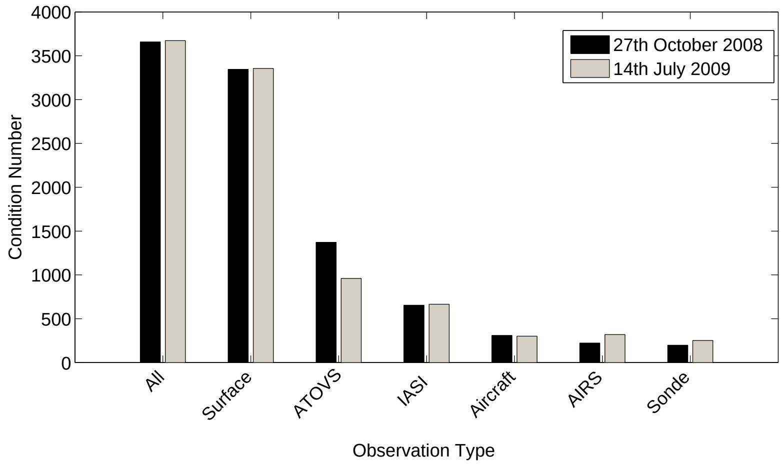


Figure 5: Conditioning of the Met O□ce 3DVar scheme using only selected

Experiment		Condition number	
Scheme	Observations	No Thinning	Thinned Data
3DVar	Only Surface	3395	501

0 0 13672(x)0q ... 1785.79 l S 1788.25 6Q 286 m 1788.25 66Q 286 31 Tm [(E)0.3562018xp681.9

5 Conclusions

The conditioning of the variational data assimilation problem plays an important role in determining how accurately the current state of the atmosphere can be determined in operational NWP. In most operational assimilation systems an initial preconditioning is performed by means of a variable transformation. In this work we have considered the conditioning of this preconditioned problem. Theoretical results have illustrated that the condition number of the problem is likely to increase when the observations are accurate and dense or when the background error correlation lengthscales are large. These results have been confirmed in a simple scalar example and illustrated using the variational data assimilation system of the Met Office.

With advances in observing technology and the move to higher resolution systems, it is clear that many more dense and accurate observations will be used in future variational data assimilation schemes of NWP centres. The results presented here imply that this will worsen the conditioning of the minimization problem. We have shown that thinning of the data can help to improve the conditioning, but a balance must be sought between the loss of accuracy due to solving an ill-conditioned problem and the loss of accuracy caused by removing

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