School of Mathematics, Meteorology
and Physics

Department of Mathematics

Preprint MPS_2010_17

21 April 2010

A hybrid data assimilation scheme for model parameter estimation: application to morphodynamic modelling

by

Polly J. Smith, Sarah L. Dance, Nancy K. Nichols

A hybrid data assimilation scheme for model parameter estimation: application to morphodynamic modelling

Polly J. Smith*, Sarah L. Dance, Nancy K. Nichols

D t t M t t , t , B 220, t t , , G6 6A , K

Abstract

We present a novel algorithm for joint state-parameter estimation using sequential three dimensional variational data assimilation (3D-Var) and demonstrate its application in the context of morphodynamic modelling using an idealised two parameter 1D sediment transport model. The new scheme combines a static representation of the state background error covariances with a flow dependent approximation of the state-parameter cross covariances. For the case presented here, this involves calculating a local finite di erence approximation of the gradient of the model with respect to the parameters. The new method is easy to implement and computationally inexpensive to run. Experimental results are positive with the scheme able to recover the model parameters to a high level of accuracy. We expect that there is potential for successful application of this new methodology to larger, more realistic models with more complex parameterisations.

K : Data assimilation, morphodynamics, parameter estimation, state augmentation

---

1. Introduction

A numerical model can never completely describe the complex physical processes underlying the behaviour of a real world dynamical system. State of the art computational models are becoming increasingly sophisticated but in practice these models su er from uncertainty in their initial conditions

---

E : p.j.smith@reading.ac.uk (.)

that transfer information from the observations to the parameter estimates and therefore play a crucial role in the parameter updating. A good a priori specification of these covariances is therefore vital for accurate parameter updating [13, 14].

By combining ideas from 3D Var and the extended Kalman filter (EKF) we have developed a novel hybrid sequential data assimilation algorithm that provides a flow dependent approximation of the state-parameter cross covariances without explicitly propagating the full system covariance matrix. The technique involves calculating a local finite di erence approximation of the gradient of the model with respect to the parameters; it is simple to code and computationally inexpensive. In this paper we give details of this new method and demonstrate its application in the context of morphodynamic modelling

In the example described in this paper, the model state vector $\mathbf{z}$ is a 1D vector representing bathymetry or bed height, the operator $\mathbf{f}$ represents the equations describing the evolution of the bed-form over time and the vector $\mathbf{p}$ contains parameters arising from the parameterisation of the sediment transport flux. We assume that the system can be represented on a discrete grid and that the system model is `perfect', i.e. it gives an exact description of the true behaviour of the system on the grid.

In this work, the model parameters are assumed to be constants and so are not altered by the forecast model from one time step to the next. The equation for the evolution of the parameters therefore has the simple form

$$\mathbf{p}_{k+1} = \mathbf{p}_k. \tag{2}$$

The augmented system model is derived by appending the parameters to the model state vector, and combining the evolution equation for the parameters

The aim is to combine the measured observations $\mathbf{y}_k$ with the model predictions $\mathbf{w}_k^b$ to produce an updated model state that most accurately describes the true augmented system state $\mathbf{w}_k^t$ at time t_k. This optimal estimate is called the and is denoted $\mathbf{w}_k^a$.

The analysis $\mathbf{w}_k^a = (\mathbf{z}_k^a, \mathbf{p}_k^a)^T$ is found by minimising a cost function penalising the misfit between the state, $\mathbf{w}_k$, the observations, $\mathbf{y}_k$ and the background forecast, $\mathbf{w}_k^b$

$$J(\mathbf{w}_k) = (\mathbf{w}_k - \mathbf{w}_k^b)^{\mathbf{T}} \mathbf{B}_k{}^{-1}(\mathbf{w}_k - \mathbf{w}_k^b) + (\mathbf{y}_k - \tilde{\mathbf{h}}_k(\mathbf{w}_k))^T \mathbf{R}_k^{-1}(\mathbf{y}_k - \tilde{\mathbf{h}}_k(\mathbf{w}_k)). \quad (6)$$

The relative weighting of the background and observations in the analysis are determined by their error statistics, expressed as error covariances, $\mathbf{B}_k$ and $\mathbf{R}_k$, respectively.

Prescription of the matrix $\mathbf{B}_k$ is a key challenge. This matrix plays a crucial role in the filtering and spreading of observational data [13, 14, 15]. In the 3D Var method (e.g. [11]), the background covariances are approximated by a fixed matrix (i.e. $\mathbf{B}_k = \mathbf{B}$ for all k) and the nonlinear optimization problem (6) is solved numerically using a gradient iteration algorithm at each time t_k. In the EKF (e.g. [4]), the background covariances are evolved explicitly according to the linearised model dynamics and the analysis is calculated directly.

For joint state-parameter estimation, it is particularly important that the a priori cross-covariances between the parameters and the state are well specified [13, 15]. Since it is not possible to observe the parameters themselves, the parameter estimates depend on the observations of the state variables. It is the state-parameter cross covariances that pass information from the observed variables to update the estimates of the unobserved parameters. Success of the state augmentation approach therefore relies strongly on the relationships between the parameters and state components being well defined. Previous work [15] indicated that whilst the assumption of static covariances made by the 3D Var algorithm is su cient for state estimation it is insu cient for parameter estimation as it does not provide an adequate representation of the state-parameter cross covariances required by the augmented system. In order to yield reliable estimates of the true parameters these covariances need to evolve with the model. However, updating the background error covariance matrix at every time step is computationally expensive and impractical when the system of interest is of high dimension.

To overcome this problem we have combined ideas from 3D Var and the EKF to produce a new hybrid assimilation scheme that captures the flow dependent nature of the state-parameter cross covariances without explicitly propagating the full system covariance matrix. A simplified version of the EKF forecast step is used to estimate the state-parameter forecast error cross covariances and this is then combined with an empirical, static approximation of the state background error covariances. We outline this new approach in the next section.

3. A hybrid approach

We can partition the background error covariance matrix $\mathbf{B}$ as follows

$$\mathbf{B}_k = \begin{pmatrix} \mathbf{B}_{\mathbf{zz}_k} & \mathbf{B}_{\mathbf{zp}_k} \\ (\mathbf{B}_{\mathbf{zp}_k})^T & \mathbf{B}_{\mathbf{pp}_k} \end{pmatrix}. \tag{7}$$

Here $\mathbf{B}_{\mathbf{zz}_k} \in \mathbb{R}^{m\times m}$ is the background error covariance matrix for the state vector $\mathbf{z}_k$ at time t_k, $\mathbf{B}_{\mathbf{pp}_k} \in \mathbb{R}^{q\times q}$ is the covariance matrix of the errors in the parameter vector $\mathbf{p}_k$ and $\mathbf{B}_{\mathbf{zp}_k} \in \mathbb{R}^{m\times q}$ is the covariance matrix for the cross correlations between the forecast errors in the state and parameter vectors.

In the EKF, the background covariance at t_{k+1} is determined by propagating the analysis covariance forward in time from t_k using a linearisation of the forecast model. We want to avoid updating the whole matrix (7) at every time step. For the state and parameter background error covariances we adopt a 3D Var approach; these matrices are prescribed at the start of the assimilation and held fixed throughout as if the forecast errors were statistically stationary. For the state-parameter cross covariances, we require a flow dependent approximation. If we assume that the state-parameter cross covariances are initially zero, and take a single step of the EKF we find that the state-parameter cross covariance can be approximated as $\mathbf{B}_{\mathbf{zp}_{k+1}} = \mathbf{N}_k \mathbf{P}^a_{\mathbf{pp}_k}$ [14], where $\mathbf{N}_k = \left.\frac{\partial \mathbf{f}(\mathbf{z},\mathbf{p})}{\partial \mathbf{p}}\right|_{\mathbf{z}^a_k,\mathbf{p}^a_k} \in \mathbb{R}^{m\times q}$ is the Jacobian of the forecast model with respect to the parameters and $\mathbf{P}^a_{\mathbf{pp}_k}$ is the parameter analysis error covariance. This leads us to propose the following approximation for the augmented forecast error covariance matrix

$$\mathbf{B}_{k+1} = \begin{pmatrix} \mathbf{B}_{\mathbf{zz}} & \mathbf{N}_k \mathbf{B}_{\mathbf{pp}} \\ \mathbf{B}_{\mathbf{pp}} \mathbf{N}_k^T & \mathbf{B}_{\mathbf{pp}} \end{pmatrix}. \tag{8}$$

In other words, all elements of the augmented background error covariance matrix are kept fixed except the state-parameter cross covariance $\mathbf{B}_{\mathbf{zp}_{k+1}}$

which is updated at each new analysis time by recalculating the Jacobian matrix $\mathbf{N}_k$.

Explicitly calculating the Jacobian of complex functions can be a di cult task, requiring complicated derivatives if done analytically or being computationally costly if done numerically. A simple alternative is to use a local finite di erence approximation. Defining $\mathbf{z}^b_{k+1} = \mathbf{f}(\mathbf{z}^a_k, \mathbf{p}^a_k)$ and $\hat{\mathbf{z}}^b_{k+1} = \mathbf{f}(\mathbf{z}^a_k, \hat{\mathbf{p}}^a_k)$, the q columns of $\mathbf{N}_k$ are given by computing

$$\frac{\partial \mathbf{f}(\mathbf{z}^a_k, \mathbf{p}^a_k)}{\partial p_i} \approx \frac{\hat{\mathbf{z}}^b_{k+1} - \mathbf{z}^b_{k+1}}{\delta p_i} \qquad i = 1, \ldots, q, \tag{9}$$

for each parameter p_i. Here $\hat{\mathbf{p}}^a_k$ is the current parameter vector with element p_i replaced with $p_i + \delta p_i$ where δp_i is a small perturbation to the current approximation of p_i.

4. The model

We apply our hybrid scheme to a simplified sediment transport model based on the 1D sediment conservation equation [1]

$$\frac{\partial z}{\partial t} = -\left(\frac{1}{1-\varepsilon}\right)\frac{\partial q}{\partial x}, \tag{10}$$

where zn

are known constant values but that the values of A and n are uncertain. To prevent unphysical solutions a small di usive term is added to to the right hand side of (11). This equation is then solved numerically using a combined semi-Lagrangian Crank-Nicolson scheme based on that presented in [17].

5. Experiments & results

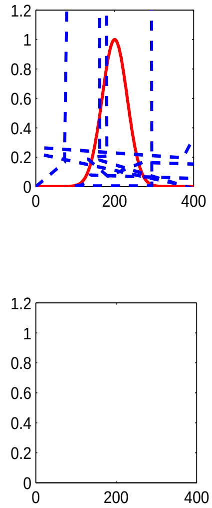
We have tested our scheme by running a series of identical twin experiments using an initially symmetric, isolated bedform, with initial profile given as a Gaussian hump. We assum17.563Td[(n)]TJ/F211.955edformlues of cycle the model parameters are updated and the state analysis is integrated forward using the model (with the new parameter values) to become the background state for the next analysis time.

Figure 1 illustrates the impact incorrect parameter estimates can have on the modelled bathymetry by comparing model runs performed with and without data assimilation over a 24 h period. For this example, the parameter A is initially over estimated (A_0 = 0.02ms^{-1}) and n under estimated (n_0 = 2.4). With no data assimilation (top), the model bathymetry rapidly diverges away from the truth. After 24 hours it has moved beyond the model domain. Running the model with the joint state-parameter assimilation scheme greatly improves the model predictions (bottom). At 24 hours it is almost impossible to distinguish between the predicted model bathymetry and the true bathymetry. The corresponding parameter updates are shown in figures 2(a) and (b). The scheme successfully recovers the true values of A and n to a high level of accuracy.

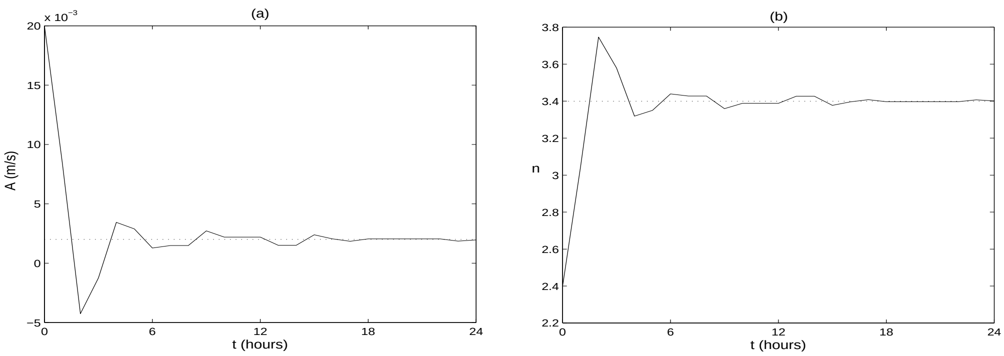
Experiments were repeated for a range of starting combinations of A and n, investigating the sensitivity of the parameter estimates to di erent error ratios, observation combinations and observation noise. The quality of the state and parameter estimates is highly dependent on the accuracy of the information fed into the assimilation algorithm. We do not present the results of these experiments here but refer the reader to [14] for further details and discussion. It was found that various factors can a ect the convergence and accuracy of the parameter estimates, such as the quality of the initial background guesses, the estimated parameter error variances and cross covariances, the location and spatial frequency of the observations, the level of observational noise and the time between successive assimilations.

6. Conclusions

We have presented a novel method for joint state-parameter estimation and demonstrated its e cacy in the context of morphodynamic modelling.



1: **Top**: . **Bottom**: $\mathbf{z}^t$, ( ) $\mathbf{z}^b$, $\mathbf{y}$ $\mathbf{z}^a$.



2: ( ) $A_0 = 0.02$ 1 ( ) $n_0 = 2.4$. ( ) $A = 0.002$ 1 ( ) $n = 3.4$.

level of accuracy. This has a positive impact on the predictive skill of the model. Our findings indicate that there is great potential for the use of 3D Var data assimilation for joint state-parameter estimation. In this paper we have focussed on application of the method to morphodynamic modelling but the versatility of the method has recently been demonstrated via a series of tests with a range of simple dynamical system models [20].

Acknowledgments

This work is funded under the UK Natural Environment Research Council (NERC) Flood Risk From Extreme Events (FREE) programme, with additional funding provided by the UK Environment Agency as part of the CASE (Co-operative Awards in Science and Engineering) scheme.

References

[1] Soulsby R. Dynamics of marine sands. Thomas Telford Publications; 1997.

[2] Scott TR, Mason DC. Data assimilation for a coastal area morphodynamic model: Morecambe Bay. Coastal Engineering 2007;54:91{109.

[3] van Dongeren A, Plant N, Cohen A, Roelvink D, Haller MC, Catalán P.

[18] Gill PE, Murray W, Wright MH. Practical Optimization. Academic Press; 1981.

[19] Rodgers CD. Inverse Methods for Atmospheric Sounding: Theory and Practice; vol. 2 of *S A , O P P* . World Scientific; 2000.

[20] Smith PJ, Dance SL, Nichols NK. A hybrid sequential data assimilation scheme for model state and parameter estimation. Mathematics Report 2/2010; Department of Mathematics, University of Reading*; 2010.

*Available at http://www.reading.ac.uk/maths/research/