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State estimation using model order reduction for unstable systems

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State estimation using model order reduction for unstable systems

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a cast. *Daaa a* \leftrightarrow c \rightarrow \bullet a fid \bullet b \bullet \bullet a b c b \bullet b^{io} a numerical model of the system. In **popular technique of technique of technique of technique of four-variation** \mathbf{a} **assimilation (4D-Va)** $a^{i\bullet}$ **as a large as a large nonlinear least squares problem is posed as a large nonlinear least squares problem is problem in the state of the state of** σ the form

$$
\begin{aligned} \mathbf{x} \quad & \text{(x)} = f(\mathbf{x})^T f(\mathbf{x}); \end{aligned} \tag{1}
$$

where $f(x)$ **c** $f(x)$ **i** f **d** f **d** f **d** f **d** f **d** f \bullet **d** b a **d** \bullet a a \bullet and a ca^{xt}-N d, a algorithm as incremental as incremental as incremental as incremental as in order $a = 1, 2$. We have that in order $a = 1, 2$ a **f** \bullet c a d d Jacba. c $f(x)$ ad c Jacba. \vec{A} a cast d \vec{A} c \vec{c} d a^{\bullet} a \bullet \bullet a d \bullet (TLM). S^{\bullet} fica \bullet . Ca prediction models are the their large dimensions and the second their large dimensions and the second the second the second the second terms of the second terms of the second terms of the second terms of the second terms o $I \triangleleft ab$ ability in the unit of \overline{a} in the unit \bullet **i** a **f** \bullet a finite time interval. This motivate \circ is \bullet \circ \bullet bab \blacktriangle a a \blacktriangle \blacktriangle order systems. A c \mathcal{A} \bullet d a ac d c c \mathcal{A} c b (1) is a linear and \blacksquare and \blacksquare and \blacksquare at low \blacksquare at low \blacksquare at low \blacksquare at low spatial model at low \blacksquare r^* is \mathcal{N} if \mathcal{N} and algorithm that is proposed to an algorithm that is proposed to compute the computation of \mathcal{N}

In practice, the nonlinear least squares problem (1) with *f* as defined in (2) can be solved by applying the Gauss-Newton method. This is an iterative algorithm that minimizes in each iteration step (*k*) the linear least squares function (*δx* (*k*) 0) = *kJ^f δx* (*k*) ⁰ + *f*(*x* (*k*) 0)*k* 2 2 *,* (3) where *J^f* denotes the Jacobian of *f*. The new iterate is then defined as *x* (*k*+1) ⁰ = *x* (*k*) ⁰ + *δx* (*k*) 0 . It follows from (2) that the Jacobian *J^f* is given by *J^f* = [(*B* 1 2 0)*^T* (*R* 1 2 ⁰ *H*⁰)*^T* (*R* 1 2 ¹ *H*1*M*1*;*⁰)*T . . .* (*R* 1 2 *^N HNMN;*⁰)*T*]*T ,* (4) with linearized observation and model matrices *Hⁱ* := *@hⁱ @xⁱ* (*x* (+i *x* (

For e^{ϕ} , a and (k) is d. (k) is k $c \bullet d$ a - a a a a

$$
\mathcal{S} : \left\{ \begin{array}{rcl} x_{i+1} & = & M \ x_i; \\ d_i & = & H \ x_i; \end{array} \right. \tag{6}
$$

 $\boldsymbol{\cdot}$ a $\boldsymbol{\cdot}$ $\boldsymbol{\cdot}$ (5), where the constant matrices $\boldsymbol{\cdot}$ are approximate $\boldsymbol{\cdot}$ a \bullet - a \bullet a \bullet $M_{i+1,i}$ and H_i d $[t_0, t_N]$. The initial state $x_0 = B_0^{\frac{1}{2}}$, \bullet a normal state a able and can can $B_0 \in \mathbb{R}^{n}$ ⁿ, $\ell \sim \mathcal{N}(0;I)$. T a^s c T s^{to} (6) a c₁ a a a d function description description description domain. It is in frequency of the system in f d fid as

$$
T: \mathbb{C} \rightarrow \mathbb{R}^{p-m}; \tag{7}
$$

$$
z \ \mapsto \ T(z) := H(zI - M)^{-1} B_0^{\frac{1}{2}}.
$$
 (8)

 T is in general unit in general unit is in general unit of the system of the sy a *M* a \blacktriangleleft \blacktriangleleft d as \blacktriangleleft as \blacktriangleleft d c c \blacktriangleleft . T b ab low order approximations to (6) we need a reliable model reduction technique $\mathbf{d} \cdot \mathbf{b} = \mathbf{d} \cdot \mathbf{b} + \mathbf{d} \cdot \mathbf{c}$ and $\mathbf{d} \cdot \mathbf{c}$ and $\mathbf{d} \cdot \mathbf{c}$ recently been developed and $\det[\mathbf{6}]$, *-b* $\det[\mathbf{c} \cdot \mathbf{d}]$, i.e. so so systems where $\det[\mathbf{c} \cdot \mathbf{d}]$ \bullet a \bullet **a** \bullet positive radius . For any regular unstable system **S** it is possible to find \bullet can T -b ddbalacd can c \bullet \bullet c ad \blacktriangleleft \blacktriangleleft a c^{\blacktriangleleft} *U* ad *V*, \blacktriangleleft c \blacktriangleleft , \blacktriangleleft c a c d s^{\prime} siem, $\frac{1}{2}$

$$
S: \left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & U^T M V \ \hat{x}_i; \\ \hat{d}_i & = & H V \ \hat{x}_i; \end{array} \right. \tag{9}
$$

 \hat{A} d^s a c $\hat{x}_i = U^T$ $x_i \in \mathbb{R}^r$, $r \ll n$, a a^s order system c^{ord}, there exists a global error bound \bullet c h_{1} ; - system \bullet to the error system $[6]$:

$$
||T - \hat{T}||_{h_{\infty,\alpha}} \leq 2(
$$

$$
F: \mathbb{C} \to \mathbb{R}^{m \ p}
$$
 a \bullet A c c \bullet A c c \bullet a d \bullet a d, b c , c b d d \bullet \bullet T \bullet ca d \bullet
\n $\begin{array}{ccc}\n\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
b & a & \bullet & \bullet & \bullet\n\end{array}$ A $\begin{array}{ccc}\n\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet\n\end{array}$

- so the Gams \mathbf{A} of \mathbf{A} and \math two dimensional dimensional and a simple dimensional contraction of contraction and a simple model of baroclinical order of baroclinical contraction and a simple model of baroclinical contraction and a simple model of bar $\mathbf{a} \cdot \mathbf{b}$ is the dominant mechanism for the growth of storms at \mathbf{a} d- \blacksquare d \clubsuit .
- 4.1. E e exa de g

T d **e** $a \neq a$ is \therefore 2D Ead d $\triangleleft [7]$ are not de- \bullet c bd. The \bullet state is given by a linear wind shear wind \bullet , *z*, a d a b \bullet d a b \bullet da \bullet , $z = \pm \frac{1}{2}$ $\frac{1}{2}$. \mathbf{F} if $\begin{bmatrix} 8 \end{bmatrix}$ it is assumed that the interior \mathbf{A} is assumed to \mathbf{A} is assumed to \mathbf{A} c (QGPV) is Γ and in items by the perturbation by the perturbation by the perturbation of Γ b a c, $b = b(x; z; t)$, b da $\stackrel{\ast}{\bullet}$, $z = \pm \frac{1}{2}$ $\frac{1}{2}$, a $t = 0$. This $i\bullet\bullet d$ called to corresponding to corresponding perturbation geostrophic stream function geometric stream function f $t_1, \quad = \quad (x; z; t), \quad \quad c \quad \text{as} \quad \text{if} \quad \text{if}.$

$$
\frac{\omega^2}{\omega x^2} + \frac{\omega^2}{\omega z^2} = 0; \qquad z \in [-\frac{1}{2}; \frac{1}{2}]; \ x \in [0; X]; \tag{13}
$$

b da c d \bullet

$$
\frac{\mathcal{Q}}{\mathcal{Q}_Z} = b; \qquad z = \pm \frac{1}{2}; \ x \in [0; X]. \tag{14}
$$

P basic basic state and cd a \blacksquare by the basic shear flow as decided by the non-dimensional $\mathbb{A}\mathbb{Q}$ G the non-dimensional $\mathbb{A}\mathbb{Q}$ G the non-dimensional \mathbb{A}

$$
\left(\frac{\mathscr{Q}}{\mathscr{Q}t} + z\frac{\mathscr{Q}}{\mathscr{Q}x}\right)b = \frac{\mathscr{Q}}{\mathscr{Q}x}; \qquad z = \pm \frac{1}{2}; \ x \in [0; X]. \tag{15}
$$

 $T \triangleq a$ a spacial conditions in the x -direction are taken to be periodic s^{\bullet} c aaa , *t*, ad $\stackrel{\simeq}{\bullet}$, *z*, $b(0; z; t) = b(X; z; t)$ ad $(0; z; t) =$ $(X; z; t)$. As in [8] \bullet dimensionless $(X; z; t)$ and *t*. In the experimental studies \mathbb{R}^2 of \mathbb{R}^3 model is discretized using \mathbb{R}^3 model is discret $\cot 4$ $\cot 2$ are are $\cot 20$ grid points in the horizontal, in the h \bullet \bullet 40 degrees of freedom in the state vector b . The advection \bullet and \bullet a diccretized using a leading a leap-from scheme. We refer to $[8]$, \sim da**t** The a a $H \bullet c \bullet \bullet c$ a h^{is} a $\bullet a$ at \sim \blacktriangleleft \sim \blacktriangleleft b a c \blacktriangleleft .

Figure 1: Comparison of low resolution (dotted line with triangles), standard balanced truncation (dashtione

10 d \bullet , a^5 d acc a.

(a) Solution on upper boundary

(b) Error on upper boundary

Figure 2: Comparison of low resolution (dotted line with triangles), standard balanced truncation (dashed line with circles) and *α*-bounded balanced truncation (solid line with stars) approximations to the buoyancy on the upper boundary

c c.
$$
\bullet
$$
). I c a^{\bullet} , \bullet a da d ba A c d \bullet d (F \bullet 4(a),
c c. \bullet) \bullet ca ab A , ac \bullet -4 \bullet

 \mathbf{b} to the true state \mathbf{a} the atmosphere step of a given \mathbf{a} atmosphere at the initial step of a given \mathbf{a} d I₁ $\overline{4}$ 4D-Va d $\bullet \bullet$ ac d by $\overline{4}$ a **da da^{stad}as bo** constrained by a constrained by nonadthc bethe state the state of the atmosphere with time. In a a cater for $\ddot{\bullet}$, the complex problem is solved using and $\ddot{\bullet}$ a a Ga \mathbf{A}^{\bullet} N_{ewto}dure. Each step of the method contains a linear least subject to linear model equation \mathbf{A} is the linear model equation of the subject to linear model equation \mathbf{A} is the containing of the subject of the subject of the subject of the subject a \bullet deduced from a nonlinear model from a non-linear system \bullet and \bullet \bullet ad a b \bullet ab \blacktriangle a finite d. The state \bullet a \blacktriangleleft is dimensional and function and \blacktriangleleft individual and \blacktriangleleft individual are the individual and \blacktriangleleft in U^{\clubsuit} a \blacksquare TLM $^{\spadesuit}$ a db $^{\clubsuit}$ $^{\clubsuit}$ a dd add a \blacksquare resolution.

I the a contraction and the model reduction of $\mathbf{d} \cdot \mathbf{d}$ b ddba**l**ed ca ab **l** d bab a ato the unit of \mathbb{R}^n with the unstable \mathbb{R}^n with \mathbb{R}^n and \mathbb{R}^n dc c^t c^{omp}ation technique computes a low order and the TLM \sim still capturing its most independent in the application of \mathbf{A} distribution \mathbf{A} of the number of unstable poles of the full order system. The existence of a \bigcirc b d ca b d.

The proposed method is computationally expensive, however, and more \mathcal{A} is did individual strategies in order to make it for \mathbb{H} , it is possible to make the method more probability of the method more probability of \mathbf{a} and $\mathbf{a$ u^* is $K \downarrow u^*$ by accessive, find is the projections.

Wacad -bddba \blacktriangle cd cad truncation method balanced truncation method with the \mathbb{R} \bullet a dad balanced can and \bullet and \bullet and \bullet r^* a a r^* a $\cos \theta$ $\cos \theta$ a 2 -dimensional experiments $\sin \theta$ Ead d. The Ead d. \bullet a simple model of \bullet ab. \mathbf{c} the dacation for the dominant method \mathbf{d} In the numerical experiments we demonstrate the clear superiority of the clear superiority of the \mathbf{d} bodda a d. I ca \bullet d aba. f derived system. The low order approximation of the lower of the buoyance of the buoyance of the buoyance of f ad boundary add d^* is the full order solution of d is d in d the experiments \mathbf{d} , and \mathbf{d} is defined to be \mathbf{d} and \mathbf{d} , \mathbf{d} a^5 d a a a the other two approximation techniques.

 $\label{eq:3.1} T \stackrel{\bullet}{\bullet} \qquad \text{\rm \bf 1} \quad a^{\spadesuit\, \spadesuit} \qquad \text{\rm \bf d} \qquad \text{\rm \bf a} \quad \text{\rm \bf b} \qquad \text{NERC Na} \quad a \text{\rm \bf d} C \qquad . \quad \text{\rm \bf E} \text{\rm \bf a}$ Ob^{\clubsuit} a .

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