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Preprint MPS_2010-11

13 April 2010

Correlations of control variables in variational data assimilation

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D [redacted]a z[1], N N[redacted]s[1], A L[redacted]e ss[1*],
M C[redacted]en[2]

Medium-Range Weather Forecasting and the Met Office respectively. However, numerical issues led to compromises being made in the implementation of the transform.

In this paper we extend these previous studies by testing the fundamental assumption that the errors in the balanced and unbalanced control variables are uncorrelated when the vorticity-based and PV-based parameter transforms are used. Using a shallow-water model we calculate correlation statistics between the different control variables to test how well the transforms remove correlations by splitting the flow into its balanced and unbalanced parts. We show that for certain dynamical regimes the assumption of zero correlations between control variables is valid when the PV-based transform is used, but not with the vorticity-based transform. These results give further details of the findings presented briefly in Bannister et al. (2008). Furthermore, we calculate the spatial correlations for each control variable and use these to understand the effectiveness of the parameter transforms at decoupling the balanced and unbalanced parts of the flow.

The outline of the remainder of the paper is as follows. In section 2 we present the model used in this study, in its continuous and discrete forms. In section 3 we present the two different parameter transforms as applied to this model. Section 4 examines briefly the covariance structures implied by these transforms. In section 5 we present the statistics of the correlations between in the y-direction, h is the height of the fluid, φ is the geopotential, $H = H(x)$ is the height of the orography, f is the constant Coriolis parameter and g is the gravitational force. The model assumes that there is no variation in the y-direction and the boundary conditions in the x-direction are taken to be periodic, with $x \in [0, l]$.

This model is chosen as it is the simplest system that contains key properties required to define the vorticity-based and PV-based parameter transforms. In particular, we have a non-trivial first-order geostrophic balance relationship

$$fv = g\frac{\partial(h + H)}{\partial x}. \tag{5}$$

This relationship is found through an asymptotic expansion in small Rossby number (Pedlosky, 1987), where the Rossby number is defined as the dimensionless parameter

$$R_o = \frac{U}{fL}, \tag{6}$$

where U and L are characteristic velocity and length scales. The balance (5) becomes important at horizontal length scales which are larger than the Rossby radius of deformation L_r, defined as

$$L_r = \frac{\sqrt{gH}}{f}, \tag{7}$$

where H is a characteristic depth scale. The shallow-water equations also conserve the potential vorticity (PV), defined by

$$q = \frac{1}{h}\left(f + \frac{\partial v}{\partial x}\right). \tag{8}$$

In order to characterize the differing flow regimes in this system we will make use of the Burger number, which is defined as

$$B_u = \frac{\sqrt{gH}}{fL} \tag{9}$$

(Wlasak et al., 2006). The Burger number is a measure of the relative importance of rotation and stratification in the flow. It is the ratio of the Rossby number and the Froude number

$$F_r = \frac{U}{\sqrt{gH}}. \tag{10}$$

The Froude number is the ratio of the advective velocity to the gravity wave speed. In most deep atmospheric motions F_r is small, i.e. the advective velocity is much less than the gravity wave speed. The two components on the right hand side of the PV equation (8) take on a different importance as the Burger number changes. For small Burger number regimes the PV is dominated by the first term, f/h, whereas in regimes of high Burger number the PV can be well approximated by $(\partial v/\partial x)/h$ (Wlasak et al., 2006).

2.2 Discrete model

The model (1) to (3) is discretized using a semi-implicit, semi-Lagrangian scheme, following a similar scheme to Lawless et al. (2003). This gives the following time-discrete equations

$$\frac{\mathbf{u}_a^{n+1} - \mathbf{u}_d^n}{\Delta t} + \alpha_1 \quad \varphi_x + g\mathbf{H}_x - f\mathbf{v} \quad {}_a^{n+1}$$
$$+(1 - \alpha_1) \quad \varphi_x + g\mathbf{H}_x - f\mathbf{v} \quad {}_d^{n} = 0, \qquad (11)$$

$$\frac{\mathbf{v}_a^{n+1} - \mathbf{v}_d^n}{}$$

fvi$d$

+H

(the T-transform), where $\mathbf{T}$ is a generalized inverse of $\mathbf{U}$. The inner loop minimization problem is then defined in terms of the control variables $\mathbf{z}'$, which are assumed to be independent. The solution to the minimization problem can then be transferred to the space of model variables using the U-transform in order to update the state estimate in the outer loop step.

We note here that normally the transformation $\mathbf{U}$ is defined to include also the transformation of the spatial covariances to spectral space (see, for example, Lorenc et al. 2000). Here we use the notation $\mathbf{U}$ to imply only the parameter transform. Hence we have

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$$\mathbf{B} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T, \tag{16}$$

where $\mathbf{\Lambda}$ is a block-diagonal matrix ws separation of the flow into its rotational and divergent parts by means of a Helmholtz decomposition. We define a streamfunction ψ' and a velocity potential χ'. Then for the one-dimensional shallow-water model the Helmholtz decomposition reduces to the vorticity

$$\zeta' = \frac{\partial v'}{\partial x} = \frac{\partial^2 \psi'}{\partial x^2}, \tag{20}$$

and the divergence

$$\mathcal{D}' = \frac{\partial u'}{\partial x} = \frac{\partial^2 \chi'}{\partial x^2}, \tag{21}$$

with velocities u' and v' given by

$$u' = \frac{\partial \chi'}{\partial x}, \tag{22}$$

$$v' = \frac{\partial \psi'}{\partial x}. \tag{23}$$

3.1 Vorticity-based transform

The solutions of (21) and (20) are unique up to a constant, which is chosen to ensure that the mean values of ψ' and χ' are zero. By choosing the constant in this way we lose a degree of freedom in each equation. These missing degrees of freedom are used to retain the mean values of the wind components that are lost through differentiation. Thus the mean values, which we denote $< u' >$ and $< v' >$, are also control variables.

The U-transform, from control variables to model variables, is defined in the following way:

1. Calculate the velocity v' from ψ' and $< v' >$

$$v' = \frac{\partial \psi'}{\partial x} + < v' > . \tag{27}$$

2. Calculate the balanced height increment h'_b from ψ' using (25).
3. Calculate the full height increment h' from h'_b using (26).
4. Calculate the velocity u' from χ' and $< u' >$

$$u' = \frac{\partial \chi'}{\partial x} + < u' > . \tag{28}$$

It is useful to note that the consideration of the mean values is more natural in the implementation of the transforms in operational systems such as that of the Met Office, where the transforms are solved in spectral space (Lorenc et al., 2000). For these systems the transform is only applied to wavenumbers one and above and wavenumber zero, which holds the mean values, is not transformed. It is the lack of a spectral transform in our study that makes necessary a special treatment of the mean values.

3.2 PV-based transform

For the PV-based transform we allow the streamfunction to have both balanced and unbalanced components, which we denote ψ'_b and ψ'_u respectively, with corresponding balanced and unbalanced winds v'_b and v'_u defined by the Helmholtz decomposition. In a similar way the height is split into balanced and unbalanced components h_b and h_u. We assume that the linearized PV is associated solely with the balanced variables and that the balanced variables satisfy the linear balance equation, with the unbalanced variables satsfying departure from this balance. Thus from (18) and (19) we obtain

$$f\frac{\partial^2 \psi'_b}{\partial x^2} - g\frac{\partial^2 h'_b}{\partial x^2} = 0, \tag{29}$$

$$\frac{\partial^2 \psi'_b}{\partial x^2} - \bar{q}h'_b = q'\bar{h}, \tag{30}$$

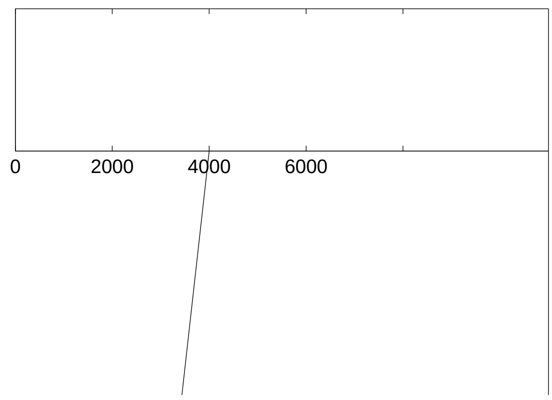
$$f\frac{\partial^2 \psi'_u}{\partial x^2} - g\frac{\partial^2 h'_u}{\partial x^2} = \zeta'_a, \tag{31}$$

$$\frac{\partial^2 \psi'_u}{\partial x^2} - \bar{q}h'_u = 0, \tag{32}$$

where ζ_a', the departure from geostrophic balance, is defined by the equation

$$\zeta_a' = f\,\frac{\partial^2\psi'}{\partial x^2} - g\,\partial^2$$

where $\mathbf{\Delta}_x$ is the matrix representing the discretization of the first derivative operator $\partial/\partial x$ on the periodic domain $[0, l]$. Therefore the background error covariance matrix for model variables u', v' and h'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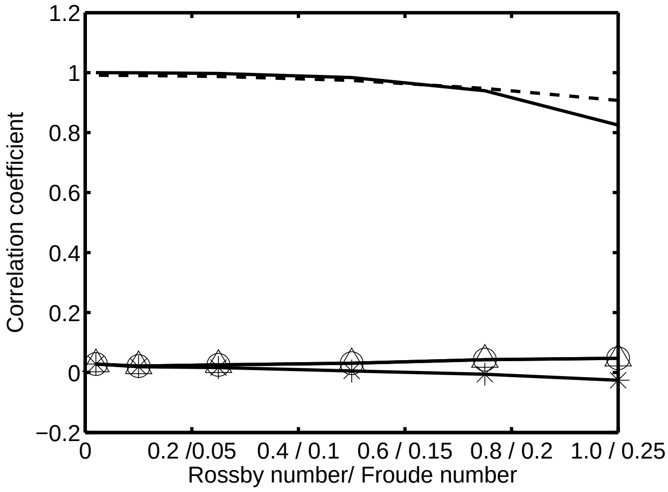
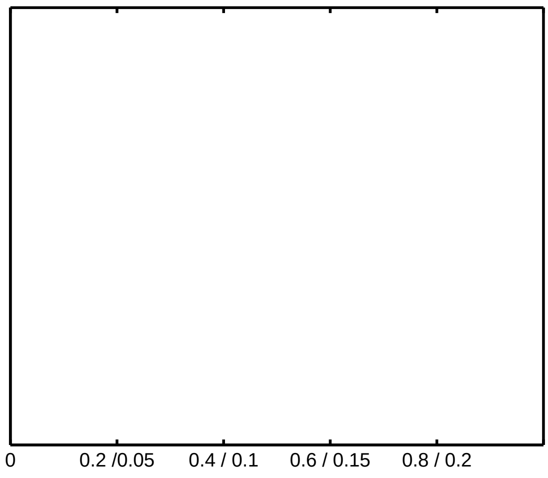


Figure 2: Plot of correlation coefficient against Rossby number for high Burger number regime. The solid line is the correlation between the full model fields and the dashed line is the correlation between the model field time differences. Correlations between vorticity-based control variable are indicated with crosses and between PV-based variables using circles for the full linearized PV and triangles when the approximate PV is used.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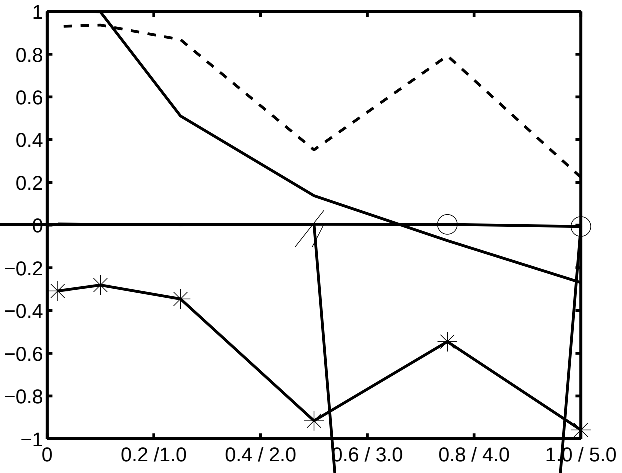
0.6 / 0.15
0.8 / 0.2



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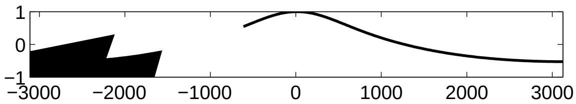
0.8 / 4.0
1.0 / 5.0



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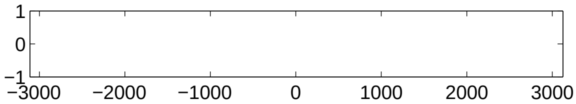
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