School of Mathematics, Meteorology

Model order reduction for discrete unstable control systems using a balanced truncation approach

C. Boess, N.K. Nichols and A. Bunse-Gerstner†

Abstract

Mathematical modeling of problems occurring in natural and engineering sciences often results in a very large dynamical system. E cient techniques for model order reduction are required, therefore, to reduce the complexity of the system. Almost all such techniques require the dynamical system to be asymptotically stable. Balanced reduction methods, including rational interpolation and Kyrlov subspace methods, see e.g. [12].

Originally, the balanced truncation method was proposed for asymptotically stable continuous-time systems by Moore in 1981 [14]. Pernebo and Silverman [16] extended the method to discrete-time systems in 1982. There already exist some extensions of the standard method to unstable systems. Most of these methods are based on an additive decomposition separating the asymptotically stable from the unstable part of the system. These techniques assume that unstable poles cannot be neglected when modeling the dynamics of a system, see e.g. [7, pp. 1177-1178], [15, 10, 19] and the references therein.

 Z -transform to the system (1):

$$
ZX(z) = AX(z) + BU(z),
$$

\n
$$
Y(z) = CX(z),
$$
\n(2)

where $\mathsf{X}(z)$, $\mathsf{U}(z)$, $\mathsf{Y}(z)$ are the \mathcal{Z} -transforms of $\mathsf{x}_i, \mathsf{u}_i, \mathsf{y}_i$, respectively. Rewriting (2) we obtain

$$
Y(z) = C(zI - A)^{-} B U(z).
$$
 (3)

2.1 Definition

.

For a discrete linear system S of the form (1) the function

$$
G(z) := C(zI - A)^{-} B \tag{4}
$$

is known as the transfer function.

Equation (3) shows that the transfer function relates inputs to outputs in frequency do-

2.4 Theorem:

Let S be a system of the form (1) with corresponding transfer function G. Moreover, let $\hat{\mathcal{S}}$ with corresponding transfer function \hat{G} be a reduced system of the form (5) with order $k < n$ that is computed using balanced truncation. Then the following bound for the error system holds: 2.4 Theorems

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$$
\|\mathbf{G}-\hat{\mathbf{G}}\|_{\mathsf{h}_{\infty}}\leq 2(\mathsf{r}_{+}+\ldots+\mathsf{h}).\tag{10}
$$

where _i are the H4114&e)-O233**10JIR9009Tf31.9%8)-O253310JIR47&e)-9Gg)-214(b)O311@o)-953(cl)-2853(f)69681189dl** $[931096]$

3.1 Definition $(h_{p, -norms})$ Let be a real positive number. For any element

 $\mathsf{F}\in\mathcal{M}^{(\mathsf{p},\mathsf{m})}:=\{\mathsf{F}:\bar{\mathcal{D}}^\mathsf{C}\rightarrow\mathbb{C}^{\mathsf{p}\times\mathsf{m}}|\mathsf{F}\ \text{is holomorphic in}\ \bar{\mathcal{D}}^\mathsf{C}\},$

3.4 Algorithm (-bounded balanced truncation)

(I) Determine a suitable real positive

where G, \hat{G} are the transfer functions of the -shifted systems S , \hat{S} , respectively. Because the systems S and \hat{S} are asymptotically stable and \hat{S} is the result of applying balanced truncation to S the error bound (10) holds. Therefore:

$$
\|\mathbf{G} - \hat{\mathbf{G}}\|_{h_{\infty}} \leq 2 \quad \mathbf{C} + \dots + \mathbf{C}.
$$

where $\left(\begin{smallmatrix} 1\cr r_+ \end{smallmatrix}\right)$, \ldots , $\left(\begin{smallmatrix} 1\cr \end{smallmatrix}\right)$ are the Hankel singular values of G $\,$.

Then the statement of the theorem follows with $\|\mathbf{G} - \hat{\mathbf{G}}\|_{\mathbf{h}_{\infty}} = \|\mathbf{G} - \hat{\mathbf{G}}\|_{\mathbf{h}_{\infty,\alpha}}.$ To summarize, we state that our new technique for balanced truncation of unstable

 $\begin{picture}(220,20) \put(0,0){\line(1,0){10}} \put(15,0){\line(1,0){10}} \put(15,0){\line($

for = 12 (solid line with stars). We note that the solid line with stars is nearly invisible in the latter case because it lies on top of the solid line. This shows that the output of the -reduced system approximates the output of the full order system so well that the two balanced truncation (Figure 4(b), circles). We see that the -bounded approach matches eigenvalues outside as well as inside the unit circle while the standard approach only keeps some of the eigenvalues outside the unit circle, but none inside.

Thus, the failure of the standard method is not surprising. Because of the simple structure of this first test model we know that if the input vector **u_i is chosen as the first** unit impulse, then all state vectors x_i are multiples of the eigenvector e associated with the eigenvalue ≈ 0.8 . The reduced order model computed by the standard method neglects all directions of eigenvectors associated with asymptotically stable eigenvalues. Thus, the output of the standard low order system is not able to approximate the response of the full order system, which is a power of the asymptotically stable eigenvalue .

(a) Eigenvalues using standard method

the error bound (dashed line). Figure 5(b) plots the behavior of the relative error norm e_{rel} of the first impulse output for di erent values of . The relative error norm is defined as

$$
e_{rel} := \frac{\|y - \hat{y}\|_2}{\|y\|_2},
$$

where $y := [y_1, ..., y_r], \hat{y} := [\hat{y}_1, ..., \hat{y}_r]$ are the vectors of outputs of the full and the low order sy110Td [(y)0.-0.250651(l)-34haof 1-0.182261]TJ /R1718(s)0.0906373(y110Td [(y)0.-0.250651(l)-34h)0.3

namely , λ , λ10, λ12, λ14, λ16, λ20, λ22, λ²⁸ and ²⁹. Thus, whenever a component of the impulse response stimulates one of these 10 eigenvalues, then the standard approach will supply a low order system where the output matches the output of the full order system exactly (assuming the absence of rounding errors), see Appendix A.1.2, Table 1. For all remaining components of the impulse response (where none of these 10 eigenvalues is new -bounded balanced truncation method supplies much better approximations to the input-output behavior of the full order system than the standard balanced truncation approach for unstable systems (as long as the time window is not chosen to be too large). The new method enables a reduction up to an order $k = 5$ while still capturing the most important information for all channels of the impulse response.

4.2 Second simple test model

The second test model $\mathcal{S}^{(2)}$ is chosen to be a single-input, single-output (SISO) system of the form (15), i.e. the input and the output matrices are a column and a row vector, respectively. The system matrix $A^{(2)} \in \mathbb{R}^{3 \times 3}$ is a real dense matrix that has real and complex eigenvalues inside as well as outside the unit circle. The input matrix B $^{(2)}\in\mathbb{R}^{+\times}$ is the first canonical unit vector and the output matrix $\textbf{C}^{(2)} \in \mathbb{R}^{\times \times \times}$ is, as in the previous example, a row vector which only contains ones. The distribution of the eigenvalues of $A^{(2)}$ is shown in Figure 10 (see also Appendix A.2).

matrix are kept by the two di erent model reduction techniques. The standard balanced truncation method is capable of matching sorne of the eigenvalues outide the unit circle but none inside (Figure 13(a)) while the -bounded approach also matches (approximately) an eigenvalue inside the unit circle (Figure 13(a)). This explains why the standard method cannot supply very accurate approximations of an output that is composed of a linear combination of both stable and unstable modes.

Figure 11: Comparison of impulse responses of full and reduced systems of order $k = 10$ using standard balanced truncation (a) as well as -bounded balanced truncation for $= 4.0$ (b)

accurately approximate the outputs of the full order systems (see Figures 14(c), 14(d)). We note again that the actual choice of bis not significant as long as it is not too close rotation. The corresponding continuous shallow water equations are given by

$$
\frac{Du}{Dt} + \frac{1}{x} + 9\frac{\frac{u}{N}}{x} - fv = 0,
$$

$$
\frac{Dv}{Dt} + fu = 0,
$$

$$
\frac{D \ln}{Dt} + \frac{u}{x} = 0,
$$

where

$$
\frac{D}{Dt} \equiv \frac{1}{t} + (U_c + u) \frac{1}{x}
$$

 $=$ $ah.$

and

where u denotes the departure of the velocity in the x-direction from a known constant forcing mean flow U_c , $\tilde{H} = \tilde{H}(x)$ is the height of the orography, f is the Coriolis parameter and g is the gravitational force. The model assumes that velocities u and v as well as the depth h do not vary in the y-direction. Moreover, the model states are periodic in the x-direction. The continuous equations are discretized using a two-time-level semi-implicit semi-Lagrangian integration scheme, following [11]. The discrete nonlinear system is then linearized by computing the Jacobian of the nonlinear system equations. The resulting discrete linear system is known as the tangent linear model.

A time-invariant linear model that approximates the tangent linear model of the system is used in the experiments. It is a multiple-input, multiple-output (MIMO) system. Its system matrix $A^{(3)}$ and its input matrix $B^{(3)}$ are both of dimension 1500 \times 1500. The output matrix $C^{(n)} \in \mathbb{R}^{n \times n}$ is chosen such that every other point is observed. We refer to the first, second and third set of 500 components of the state vector as the u-, v- and -field, respectively.

This test model is only slightly unstable, i.e. only 10 of the 1500 eigenvalues lie strictly outside the unit circle and the absolute value of the largest eigenvalue is approximately 1.00013 (see Figure 15 for the distribution of the eigenvalues). However, the system is still an interesting test model because many of the asymptotically stable poles are so close to

a365(p-3425s898333o)852o.6(47.4495o.6(47.4490o.6(47.449183(o)-34254490.2506,89834.360239.764.0.08056983 **a3**65(p-3425s898333o)852o.6(47.4495o.6(47.4490o.6(47.449183(o)-34254490.2506,89834.360239.764.0.08056983

Very similar results hold for the -field vector components of the 250th impulse response as shown in Figure 17. The error of the

4.4 Summary of numerical experiments

All the numerical experiments demonstrated the superiority of the -bounded balanced truncation method over the currently used balanced truncation approach for unstable systems, especially over a short time window. This result is not very surprising. If the system has a considerable number of unstable poles, then the standard approach for unstable systems cannot supply a good approximation to the input-output behavior of the full order system. The reason is that essential or even all information of the asymptotically stable part of the full order system is lost (depending on the chosen reduction order). Thus, at the beginning of the time window (where the asymptotically stable part still influences the behavior of the system) we cannot expect the standard approach to supply good approximations.

Moreover, the shallow water test model showed that the standard balanced truncation method fails not only for systems with large numbers of unstable poles, but also for systems that have only a few unstable poles, but large numbers of asymptotically stable modes that are very close to being unstable.

5 Conclusions

Model order reduction of unstable control systems is an important problem to be considered. However, most of the known and approved model reduction methods are for asymptotically stable systems only. The existing approaches for unstable systems are based on an additive decomposition of the system into its asymptotically stable and its unstable part. The model reduction procedure is then applied to the asymptotically stable subsystem while the unstable part remains unchanged. This procedure may only supply good approximations to the full order system if the number of unstable poles is rather small or if the asymptotically stable part of the system is of minor importance. These assumptions are rather restrictive. At the beginning of the time window, especially, the standard low order approximations are poor because at the initial time steps the asymptotically stable components (which are neglected in the standard approach) still have influence on the behavior of the system.

In this paper we have proposed a novel approach for model reduction for unstable systems using balanced truncation. The new -bounded balanced truncation method is independent of the number of unstable poles. It equally takes into account the asymptotically stable as well as the unstable modes of the full order system within the reduction process. We were able to show that the new method is embedded in a theoretical framework very similar to that of the original balanced truncation method for asymptotically stable systems. While balanced truncation for asymptotically stable systems computes a low order system that is close to being optimal with respect to the h_2 -norm, the -bounded balanced truncation method supplies a low order system close to being optimal in the h_2 -norm. Moreover, we have proved a theoretical error bound for the new -bounded approach based on neglected Hankel singular values.

In numerical experiments with two simple unstable test models we have shown that the new method computes a low order model that approximates the input-output behavior of the full order system very accurately. It is possible even to reduce the order up to a sixth of the order of the original system while still capturing the essential information in the response. Comparison with the standard balanced truncation approach for unstable systems demonstrated the superiority of the new -bounded balanced truncation method, especially at the beginning of the time window.

In addition to the simple models, we have also investigated a more realistic test model

derived from discretized and linearized shallow water equa

A Appendix

A.1 First test model $\mathcal{S}^{(n)}$

A.1.1 Eigenvalues of first test model

[5] C. Boess.