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Boundary value problems for the elliptic sine-Gordon equation in a semi-strip

by

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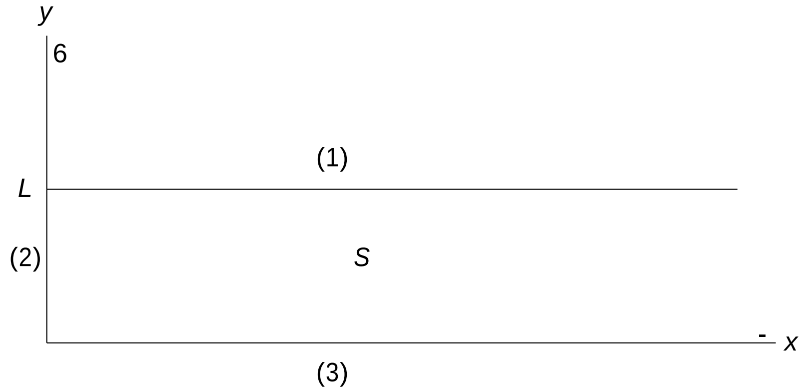


Figure 1: The semistrip S

and we will analyze boundary value problems posed in the semi-infinite strip

$$S = \{0 < x < \infty; \qquad 0 < y < L\};$$

where L is a positive finite constant. The sides

2 Spectral analysis under the assumption of existence

In what follows we assume that (1.1) is supplemented with appropriate boundary conditions on the boundary of the semistrip S so that the existence of a unique solution $q(x,y)$ can be assumed. Furthermore, we assume the following:

$$\begin{aligned} &q(x,L),\ q_y(x,L),\ q(x,0),\ q_y(x,0) \in L^1(\mathbb{R}^+),\\ &xq(x,L),\ xq_y(x,L),\ xq(x,0),\ xq_y(x,0) \in L^1(\mathbb{R}^+),\\ &q(0,y),\ q_x(0,y),\ yq(0,y),\ yq_x(0,y) \in L^1([0,L]). \end{aligned} \tag{2.1}$$

The sine-Gordon equation is the compatibility condition of the following Lax pair for the 2×2 matrix-valued function $\mu(x,y,k)$, $k \in \mathbb{C}$:

$$\mu_x + \frac{\lambda(k)}{2}[\sigma_3, \mu] = Q(x,y,k)\mu, \tag{2.2}$$

$$\mu_y + \frac{\omega(k)}{2}[\sigma_3, \mu] = iQ(x,y,-k)\mu, \tag{2.3}$$

where

$$\lambda(k) = \frac{1}{2i}\left(k - \frac{1}{k}\right), \qquad \omega(k) = \frac{1}{2}\left(k + \frac{1}{k}\right), \tag{2.4}$$

$$Q(x,y,k) = \frac{i}{4}\begin{pmatrix} \frac{1}{k}(1-\cos q) & q_x - iq_y + \frac{i\sin q}{k} \\ -q_x - iq_y - \frac{i\sin q}{k} & -\frac{1}{k}(1-\cos q) \end{pmatrix}, \qquad q = q(x,y). \tag{2.5}$$

Equations (2.2) and (2.3) can be written as the single equation

$$d\left[e^{(\lambda(k)x+\omega(k)y)\frac{\hat\sigma_3}{2}}\mu(x,y,k)\right] = W(x,y,k), \tag{2.6}$$

where the differential form W is given by

$$W(x,y,k) = e^{(\lambda(k)x+\omega(k)y)\frac{\hat\sigma_3}{2}}\left(Q(x,y,k)\mu(x,y,k)dx + iQ(x,y,-k)\mu(x,y,k)dy\right), \tag{2.7}$$

and $\hat\sigma_3$ acts on a 2×2 matrix A by

$$\hat\sigma_3 A = [\sigma_3, A].$$

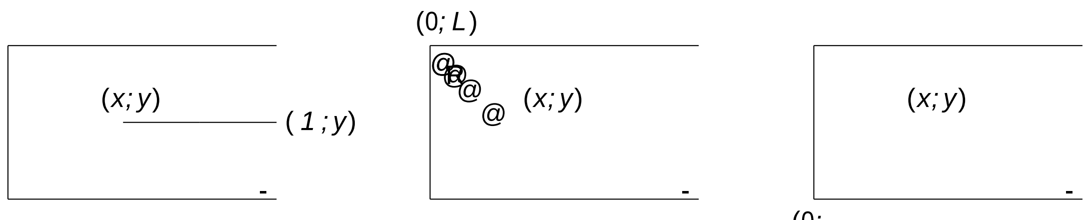
Remark 2.1 *Note that*

$$\overline{\lambda(\bar k)} = -\lambda(k) = \lambda\left(\frac{1}{k}\right), \qquad \overline{\omega(\bar k)} = \omega(k) = \omega\left(\frac{1}{k}\right).$$

2.1 Bounded and analytic eigenfunctions

We define three solutions $\mu_j(x,y,k)$ $j = 1,2,3,$ of (2.6) by

$$\mu_j(x,y,k) = I + \int_{(x_j,y_j)}^{(x,y)} e^{-(\lambda(k)x+\omega(k)y)\frac{\hat\sigma_3}{2}} W(\xi,\eta,k), \tag{2.8}$$



1 2

2.2 Spectral functions

Any two solutions $\ $, $\tilde{\ }$ of (2.6) are related by an equation of the form

$$(x;y;\ ) = \tilde{\ }(x;y;\ )e^{-(\ (\ )x+!(\ )y)\frac{\sigma_3}{2}}C(\ ): \tag{2.12}$$

We introduce the notations

$$S_1(\ ) = \ _1(0;L;\ ); \quad S_2(\ ) = \ _2(0;0;\ ); \quad S_3(\ ) = \ _1(0;0;\ ): \tag{2.13}$$

Then equation (2.12) implies the following equations:

$$_1(x;y;\ ) = \ _2(x;y;\ )e^{-(\ (\ )x+!(\ )y)\frac{\sigma_3}{2}}e^{\frac{\omega(\lambda)}{2}L\sigma_3}S_1(y)$$

$a_1($ (

where the matrices $_{\pm}$ and J are de ned as follows:

$$_{+} = \quad {}_1^{(12)}; \frac{1}{a_3(\ )} \ {}_3^{(1)} \quad ; \quad arg(\ ) \ 2 \ [0; \frac{}{2}];$$

$$_{-} = \quad {}_1^{(12)}; \frac{1}{a_1(\ )} \ {}_2^{(2)} \quad ; \quad arg(\ ) \ 2 \ [\frac{}{2}; \ ];$$

$$_{+} = \quad \frac{1}{a_3(\ \ )} \ {}_3^{(3)}; \ {}_1^{(34)} \quad ; \quad arg(\ ) \ 2 \ [\ ; \frac{3}{2}];$$

$$_{-} = \quad \frac{1}{a_1(- \ )} \ {}_2^{(4)}; \ {}_1^{(34)} \quad ; \quad arg(\ ) \ 2 \ [\frac{3}{2}; 2 \ ];$$

$$J(x; y; \ ) = J \ (x; y; \ ); \ \text{ if } \ arg(\ ) = \ ; \qquad = 0; \frac{}{2}; \ ; \frac{3}{2}; \tag{2.24}$$

where, using the global relation, we nd

$$J^0 = \begin{pmatrix} \frac{a_2(\ )}{a_1(- \ )a_3(\ )} & \frac{b_3(- \ )}{a_3(\ )} e^{- \ (x;y; \ )} \\ \frac{e^{\ \omega(\lambda)L} b_1(\ )}{a_1(- \ )} e^{\ (x;y; \ )} & 1 \end{pmatrix};$$

$$J^{\ =2} = \begin{pmatrix} 1 & \frac{b_2(- \ )}{a_1(\ )a_3(\ )} e^{- \ (x;y; \ )} \\ 0 & 1 \end{pmatrix}; \qquad J^{3 \ =2} = \begin{pmatrix} 1 & 0 \\ \frac{b_2(\ )}{a_1(- \ )a_3(- \ )} e^{\ (x;y; \ )} & 1 \end{pmatrix}$$

and

$$J \ = J^{3 \ =2} (J^0)^{-1} J^{\ =2}; \tag{2.25}$$

where

$$(x; y; \ ) = \ (\ )x + !(\ )y: \tag{2.26}$$

All the matrices J have unit determinant: for $J^{\ =2}$ and $J^{3 \ =2}$ this is immediate, whereas for J^0 we nd

$$\det(J^0) = \frac{a_2(\ ) + e^{-!(\ )L} b_1(\ ) b_3(\ \ )}{a_1(\ \ ) a_3(\ )} = \frac{a_1(\ \ ) a_3(\ )}{a_1(\ \ ) a_3(\ )} = 1;$$

where we have used the equation

$$a_2(\ ) = a_1(\ \ ) a_3(\ ) \quad b_3(\ \ ) b_1(\ ) e^{-!(\ )L}; \qquad 2 \ \mathbb{R}: \tag{2.27}$$

Equation (2.27) is a consequence of equations (2.21) and (2.22) (see also equation (4.19) below).

The function $(x; y;$

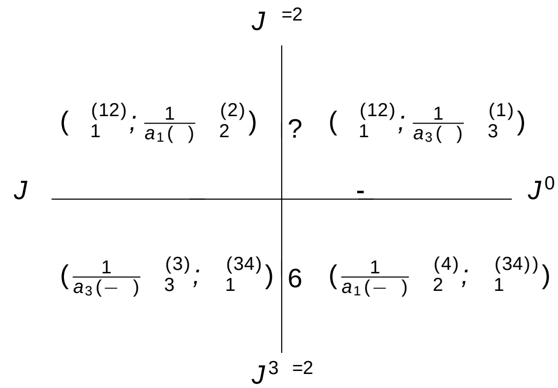


Figure 3: Bounded eigenfunctions and the Riemann-Hilbert problem

The possible zeros of a_1 in the region $\{arg(\ ) \in [\frac{}{2},\ ]\}$ are simple; these zeros are denoted $\ _j$, $j = 1, \ldots, N_1$

(2.28)

The possible zeros of a_3 in the region $\{arg(\ ) \in [0, \frac{}{2}]\}$ are simple; these zeros are denoted $\ _j$, $j = 1, \ldots, N_3$

The residues of the function at the corresponding poles can be computed using equations (2.14)-(2.16). Indeed, equation (2.16) yields

$$\ _1^{(12)} = a_3\ _3^{(3)} + b_3 e^{\ (x,y,\ )}\ _3^{(1)},$$

hence

$$Res_{\ _j} \frac{\ _3^{(1)}}{a_3} = \frac{\ _3^{(1)}(\ _j)}{\dot{a}_3(\ _j)} = \frac{\ _1^{(12)}(\ _j)}{\dot{a}_3(\ _j) b_3(\ _j)} e^{-\ (x,y,\ _j)}, \qquad (2.29)$$

where $\dot{a}_3(\ )$ denotes the derivative of a_3 with respect to .
Similarly, using (2.14),

$$Res_{\ _j} \frac{\ _2^{(2)}}{a_1} = \frac{\ _2^{(2)}(\ _j)}{\dot{a}_1(\ _j)} = \frac{\ _1^{(12)}(\ _j)}{\dot{a}_1(\ _j) b_1(\ _j) e^{-\omega(\ _j)L}} e^{-\ (x,y,\ _j)}. \qquad (2.30)$$

Using the notation $[\ ]_1$ for the rst column, $[\ ]_2$ for the second column for the solution of the RH problem (2.23), at equations (2.29) and (2.30) imply the following residue conditions:

$$Res_{\ _j}[\ (x,y,\ )]_2 = \quad e^{-\ (x,y(x,yat\ equ}$$

The inverse problem

Rewriting the jump condition, we obtain

$$_+ \quad _- = \quad _+ \quad _+J \;=\; _+(I \quad J) \;\Rightarrow\; _+ \quad _- = \; _+\tilde{J}; \tag{2.32}$$

where $\tilde{J} = I \quad J.$ The asymptotic conditions (2.10)-(2.11)) imply

$$(x;y;\;) = I + \frac{^*(x;y)}{} + O\left(\frac{1}{^2}\right); \quad |\;| \to \infty. \tag{2.33}$$

Equations (2.32) and (2.33) de ne a Riemann-Hilbert problem.
The solution of this RH problem is given by

$$(x;y;\;) = I + \frac{1}{2\;i}\int \frac{_+(x;y;\;')\tilde{J}(x;y;\;')}{'}d\;'; \qquad 2\;; \tag{2.34}$$

where

$$= \mathbb{R} \cup i\mathbb{R}.$$

Equations (2.33) and (2.34) imply

$$^* = \quad \frac{1}{2\;i}\int \; _+(x;y;\;)\tilde{J}(x;y;\;)d\;. \tag{2.35}$$

Using (2.33) in the rst ODE in the Lax pair (2.2), we nd

$$\frac{i}{4}[\;_3;\;^*] = i\frac{q_x \quad iq_y}{4}\;_1 \;\Rightarrow\; q_x \quad iq_y = 2(\;^*)_{21} = 2\lim_{\to\infty}(\;_{21}); \tag{2.36}$$

($_1$, $_3$ denote the usual Pauli matrices).
In order to obtain an expression in terms of q rather than its derivatives, we consider the coe cient of the term $^{-1}$. The (1,1) element of this coe cient yields

$$\cos q(x;y) = 1 + 4i(\;^*_x)_{11} \quad 2(\;^*)^2_{21}. \tag{2.37}$$

3 Spectral theory assuming the validity of the global relation

3.1 The spectral functions

The above analysis motivates the following de nitions for the spectral functions.

The spectral functions at the $y = 0$ and $y = L$ boundaries

De nition 3.1 *Given the functions $q(x;L)$, $q_y(x;L)$ satisfying conditions (2.1), de ne the map*

$$\mathbb{S}_1 : \{q(x;L); q_y(x;L)\} \to \{a_1(\;); b_1(\;)\}$$

by

$$\begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} = [\Phi_1(0;L)]_1; \qquad \lambda \in \mathbb{C}^+;$$

where $[\Phi_1(x;L)]_1$ denotes the first column vector of the unique solution $\Phi_1(x;L)$ of the Volterra linear integral equation

$$\Phi(x;\lambda) = I - \int_x^\infty e^{i\omega(\lambda)(\xi - x)\frac{\hat{\sigma}_3}{2}} Q(\xi;L;\lambda)\Phi(\xi;\lambda)d\xi; \tag{3.1}$$

and $Q(x;L;\lambda)$ is given in terms of $q(x;L)$ and $q_y(x;L)$ by equation (2.5).

Proposition 3.1 *The spectral functions $a_1(\lambda)$, $b_1(\lambda)$ have the following properties.*

(i) $a_1(\lambda)$, $b_1(\lambda)$ *are continuous and bounded for* $Im(\lambda) \geq 0$, *and analytic for* $Im(\lambda) > 0$.

(ii) $a_1(\lambda) = 1 + O\left(\frac{1}{\lambda}\right)$, $b_1(\lambda) = O\left(\frac{1}{\lambda}\right)$ *as* $\lambda \to \infty$; $Im(\lambda) \geq 0$.

(iii) $a_1(\lambda) = \cos\frac{q(x;L)}{2} + O(\lambda)$, $b_1(\lambda) = i\sin\frac{q(x;L)}{2} + O(\lambda)$ *as* $\lambda \to 0$; $Im(\lambda) \geq 0$.

(iv) $a_1(\lambda)\overline{a_1(\bar{\lambda})} - b_1(\lambda)\overline{b_1(\bar{\lambda})} = 1$, $Im(\lambda) \geq 0$.

(v) The map $\mathbb{Q}_1 : \{a_1, b_1\} \to \{q(x;L), q_y(x;L)\}$, *inverse to* $\mathbb{S}_1$, *is given by*

$$\cos q(x;L) = 1 + 4i \lim_{\lambda\to\infty}(\lambda M_x)_{11} + 2\lim_{\lambda\to\infty}(\lambda M)_{21};$$

$$q_y(x;L) = -iq_x(x;L) + 2\lim_{\lambda\to\infty}(\lambda M)_{21};$$

where M is the solution of the following Riemann-Hilbert problem:

* *The function*

$$M(x;\lambda) = \begin{cases} M_+(x;\lambda) & \lambda \in \mathbb{C}^+ \\ M_-(x;\lambda) & \lambda \in \mathbb{C}^- \end{cases}$$

is a sectionally meromorphic function of $\lambda \in \mathbb{C}$.

* $M = I + O\left(\frac{1}{\lambda}\right)$ *as* $\lambda \to \infty$, *and*

$$M_+(x;\lambda) = M_-(x;\lambda)J_1(x;\lambda); \qquad \lambda \in \mathbb{R};$$

where

$$J_1(x;\lambda) = \begin{pmatrix} 1 & \frac{b_1(-\lambda)}{a_1(\lambda)}e^{-i\omega(\lambda)x} \\ \frac{b_1(\lambda)}{a_1(-\lambda)}e^{i\omega(\lambda)x} & \frac{1}{a_1(\lambda)a_1(-\lambda)} \end{pmatrix}; \qquad \lambda \in \mathbb{R}. \tag{3.2}$$

* *The function $a_1(\lambda)$ may have N_1 simple poles λ_j in* $\mathbb{C}^+$.

* *Let $[M]_i$ denote the i-th column vector of M, $i = 1;2$. The possible poles of M_+ occur at λ_j, and the possible poles of M_- occur at $-\lambda_j$ in $\mathbb{C}^-$, and the associated residues are given by*

$$\begin{aligned} Res_{\lambda_j}[M(x;\lambda)]_2 &= \frac{e^{-i\omega(\lambda_j)x}}{\dot{a}_1(\lambda_j)b_1(\lambda_j)}[M(x;\lambda_j)]_1; \\ Res_{-\lambda_j}[M(x;\lambda)]_1 &= \frac{e^{i\omega(\lambda_j)x}}{\dot{a}_1(-\lambda_j)b_1(-\lambda_j)}[M(x;-\lambda_j)]_2. \end{aligned} \tag{3.3}$$

The spectral functions

(v) The map $Q_2 : \{a_2, b_2\} \to \{q(0,y)\, q_y(0,y)\}$*, inverse to* S_2*, is given by*

$$\cos q(0,y) = 1 + 4i \lim_{\lambda\to\infty} (\lambda M_y)_{11} + 2 \lim_{\lambda\to\infty} (\lambda M)_{21},$$

$$q_x(0,y) = i q_y(0,y) + 2 \lim_{\lambda\to\infty} (\lambda M)_{21},$$

where M *is the solution of the following Riemann-Hilbert problem:*

* *The function*

$$M(y,\lambda) = \begin{cases} M_+(y,\lambda) & \operatorname{Re}\lambda \geq 0 \\ M_-(y,\lambda) & \operatorname{Re}\lambda \leq 0 \end{cases}$$

  is a sectionally meromorphic function of $\lambda \in \mathbb{C}$*.*

* $M = I + O\left(\frac{1}{\lambda}\right)$ *as* $\lambda \to \infty$*, and*

$$M_+(y,\lambda) = M_-(y,\lambda) J_2(y,\lambda), \qquad \lambda \in i\mathbb{R},$$

  where

$$J_2(y,\lambda) = \begin{pmatrix} 1 & \frac{b_2(-\lambda)}{a_2(\lambda)} e^{-\omega(\lambda)x} \\ \frac{b_2(\lambda)}{a_2(-\lambda)} e^{\omega(\lambda)x} & \frac{1}{a_2(\lambda)a_2(-\lambda)} \end{pmatrix}, \qquad \lambda \in i\mathbb{R}.$$

* M *satisfies appropriate residue conditions at the zeros of* $a_2(\lambda)$*.*

Proof of propositions (3.1)-(3.3)

The proof of properties (i)-(iv) follows from the discussion in Section 2.2. In particular, property (iii) follows from the asymptotic behaviour at $\lambda \to 0$,

Regarding the rigorous derivation of the above results, we note the following: If $\{q(x,L), q_y(x,L)\}$, $\{q(x,0), q_y(x,0)\}$ and $\{q(y,0), q_x(y,0)\}$ are in L^1, then the Volterra integral equations (3.1), (3.4) and (3.5) respectively, have a unique solution, and hence the spectral functions $\{a_j, b_j\}$, $j = 1, \ldots, 3$, are well defined. Moreover, under the assumption (2.1) the spectral functions belong to $H^1(\mathbb{R})$, hence the Riemann-Hilbert problems that determine the inverse maps can be characterized through the solutions of a Fredholm integral equation, see [10, 49].
QED

3.2 The Riemann-Hilbert problem

Theorem 3.1 *Suppose that a subset of the boundary values $\{q(x,L), q_y(x,L)\}$, $\{q(x,0), q_y(x,0)\}$, $0 < x < \infty$, and $\{q(y,0), q_x(y,0)\}$, $0 < y < L$, satisfying (2.1), are prescribed as boundary conditions. Suppose that these prescribed boundary conditions are such that the global relations (2.21) and (2.22) can be used to characterize the remaining boundary values.*
Define the spectral functions $\{a_j, b_j\}$, $j = 1, \ldots, 3$, by definitions (3.1)-(3.3). Assume that the possible zeros $\{\lambda_j\}_{j=1}^{N_1}$ of $a_1(\lambda)$ and $\{\mu_j\}_{j=1}^{N_2}$ of $a_3(\lambda)$ are as in assumption 2.28.
Define $M(x,y,\lambda)$ as the solution of the following 2×2 matrix Riemann-Hilbert problem:

- *The function $M(x,y,\lambda)$ is a sectionally meromorphic function of λ away from $\mathbb{R} \cup i\mathbb{R}$.*
- *The possible poles of the second column of M occur at $\lambda = \mu_j$, $j = 1, \ldots, N_2$, in the first quadrant and at $\lambda = \lambda_j$, $j = 1, \ldots, N_1$, in the second quadrant of the complex λ plane.*

  The possible poles of the first column of M occur at $\lambda = \bar{\lambda}_j$ $(j = 1, \ldots, N_1)$ and $\lambda = \bar{\mu}_j$ $(j = 1, \ldots, N_2)$.

  The associated residue conditions satisfy the relations (2.31).
- *$M = I + O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$, and*

$$M_+(x,y,\lambda) = M_-(x,y,\lambda)J(x,y,\lambda); \qquad \lambda \in \mathbb{R} \cup i\mathbb{R},$$

  where $M = M_+$ for λ in the first or third quadrant, and $M = M_-$ for λ in the second or fourth quadrant of the complex λ plane, and J is defined in terms of $\{a_j, b_j\}$ by equations (2.25).

Then M exists and is unique, provided that the H^1 norm of of the spectral functions is sufficiently small.
Define $q(x,y)$ is terms of $M(x,y,\lambda)$ by

$$q_x - iq_y = 2 \lim_{\lambda\to\infty} (\lambda M)_{21}; \tag{3.7}$$

$$\cos q(x,y) = 1 + 4i(\lim_{\lambda\to\infty} (\lambda M_x)_{11}) - 2(\lim_{\lambda\to\infty} (\lambda M)_{21})^2. \tag{3.8}$$

Then $q(x,y)$ solves (1.1). Furthermore, $q(x,y)$ evaluated at the boundary, yields the functions used for the computation of the spectral functions.

Proof: Under the assumptions (2.1), the spectral functions are in H^1.

In the case when $a_1(\ )$ and $a_3(\ )$ have no zeros, the Riemann-Hilbert problem is regular and it is equivalent to a Fredholm integral equation. However, we have *not* been able to establish a vanishing lemma, hence we require a small norm assumption for solvability.

If $a_1(\ )$ and $a_3(\ )$ have zeros, the singular RH problem can be mapped to a regular one coupled with a system of algebraic equations [22]. Moreover, it follows from standard arguments, using the dressing method [47, 48], that if M solves the above RH problem and $q(x;y)$ is de ned by (3.7)-(3.8), then $q(x;y)$ solves equation (1.1). The proof that q evaluated at the boundary yields the functions used for the computation of the spectral functions follows arguments similar to the ones used in [24].

QED

4 Linearizable boundary conditions

We now concentrate on the particular boundary conditions (1.2).

In this case, equations (2.17)-(2.19) simplify as follows:

$$\begin{pmatrix} A_1(x;\ ) \\ B_1(x;\ ) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{4}\int_x^\infty \begin{pmatrix} q_y(\ ;L)A_1(\ ;\ ) \\ e^{\ (\ )(x-\ )}q_y(\ ;L)B_1(\ ;\ ) \end{pmatrix} d\ ;$$

$$0 < x < 1; \qquad Im(\ ) \quad 0; \tag{4.1}$$

$$\begin{pmatrix} A_2(y;\ ) \\ B_2(y;\ ) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4}\int_y^L \begin{pmatrix} \frac{(1-\cos d)}{\hbar}A_2(\ ;\ ) + [q_x(0;y) \quad i\frac{\sin d}{}]B_2(\ ;\ ) \\ e^{!(\ )(y-\ )}\ q_x(0;y) + i\frac{\sin d}{}]A_2(\ ;\ ) + \frac{(1-\cos d)}{}B_2(\ ;\ ) \end{pmatrix} d\ ;$$

$$0 < y < L; \qquad 2\,\mathbb{C}; \tag{4.2}$$

$$\begin{pmatrix} A_3(x;\ ) \\ B_3(x;\ ) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{4}\int_x^\infty \begin{pmatrix} q_y(\ ;0)A_3(\ ;\ ) \\ e^{\ (\ )(x-\ )}q_y(\ ;0)B_3(\ ;\ ) \end{pmatrix} d\ ;$$

$$0 < x < 1; \qquad Im(\ ) \quad 0: \tag{4.3}$$

In equations (4.1) and (4.3), the only dependence on is through $(\ )$. Thus, since $(\ \frac{1}{}) = (\ )$, it follows that the vector functions $(A_1;B_1)$ and $(A_3;B_3)$ satisfy the same symmetry properties. Hence,

$$a_j(\ \frac{1}{}) = a_j(\ ); \quad b_j(\ \frac{1}{}) = b_j(\ ); \qquad j = 1;3; \quad Im(\ ) \quad 0: \tag{4.4}$$

It turns out that the vector function $(A_2;B_2)$ also satis es a certain symmetry condition, as stated in the following proposition.

Proposition 4.1 *Let $q_x(0;y)$ be a su ciently smooth function. Then the vector solution of the linear Volterra integral equation (4.2) satis es the following symmetry conditions (where we do not indicate the explicit dependence of A_2, B_2 on y):*

$$A_2(\frac{1}{}) = \frac{1}{1 \quad F(\ )^2}[A_2(\ ) \quad F(\ )B_2(\ ) + F(\ )e^{!(\ )(y-L)}B_2(\ \ ) \quad F(\ )^2e^{!(\ )(y-L)}A_2(\ \ )];$$

$$B_2(\frac{1}{}) = \frac{1}{1 \quad F(\ )^2}[B_2(\ ) \quad F(\ )A_2(\ ) + F(\ )e^{!(\ )(y-L)}A_2(\ \ ) \quad F(\ )^2e^{!(\ )(y-L)}B_2(\ \ )];$$

$$0 < y < L; \qquad 2\,\mathbb{C}; \tag{4.5}$$

where the function $F(\ )$ is given by

$$F(\ ) = i\frac{1 \quad ^2}{1 + \quad ^2}\tan\frac{d}{2}: \qquad (4.6)$$

Proof: Let the $2 \quad 2$ matrix valued function $\ _2(y;\ )$ be de ned by

$$\ _2(y;\ ) = \begin{matrix} A_2(y;\ ) & B_2(y;\ \ ) \\ B_2(y;\ ) & A_2(y;\ \ ) \end{matrix} ; \quad 0 < y < L; \quad 2\ \mathbb{C}: \qquad (4.7)$$

Then $\ _2$ satis es the ODE

$$(\ _2)_y + \frac{!(\ )}{2}[\ _3;\ \ _2] = iQ(0;y;\ \ )\ _2; \quad 0 < y < L;$$
$$\ _2(L;\ ) = I; \qquad (4.8)$$

where $Q(x;y;\ )$ is de ned in (2.5), and $q(0;y) = d$.
Letting

$$\ _2(y;\ ) = \ _2(y;\ )e^{\frac{\omega(\lambda)}{2}\ _3(y-L)} \qquad (4.9)$$

it follows that $\ _2$ satis es the ODE

$$(\ _2)_y = V\ \ _2; \qquad (4.10)$$
$$\ _2(L;\ ) = I; \quad 0 < y < L; \quad Re(\ )\ \ 0;$$

where

$$V(y;\ ) = \frac{1}{4} \begin{matrix} (\ + \frac{\cos d}{}) & q_x(0;y) + \frac{i\sin d}{} \\ q_x(0;y)\ \ \frac{i\sin d}{} & + \frac{\cos d}{} \end{matrix} : \qquad (4.11)$$

We seek a non singular matrix $R(\ )$, independent of y, such that

$$V(y;\frac{1}{\ }) = R(\ )V(y;\ )R(\ )^{-1}: \qquad (4.12)$$

It can be veri ed that such a matrix is given by

$$R(\ ) = \begin{matrix} 1 & F(\ ) \\ F(\ ) & 1 \end{matrix} ; \qquad (4.13)$$

where F is de ned by (4.6).
Replacing in equation (4.10) by $\frac{1}{\ }$, and using (4.12), we nd the following equation: De ned by (4.13)

(1)]TJ]0 d 08 0.398(4.13)]TJ -3(4.10)]TJ/F15 7.9675 Tf 9.962 0 Td [(4)-333(de .98]TJ -3716.391 -26.71 0 Td 0 0 2170.6

This equation and equation (4.9) imply

$$ {}_2(y; \frac{1}{}) = R(\ )\ {}_2(y;\ )\ \ e^{!(\ )\frac{\sigma_3}{2}(y-L)}R(\ )^{-1}\ : \qquad (4.14)$$

The rst column vector of this equation implies (4.5).
QED

Remark 4.1 *Recalling that* $a_2(\ ) = A_2(0;\ )$*, and* $b_2(\ ) = B_2(0;\ )$*, equations (4.5) immediately imply the following important relations:*

$$a_2(\frac{1}{}) = \frac{1}{1\quad F(\ )^2}[a_2(\ )\quad F(\ )b_2(\ ) + F(\ )e^{-!(\ )L}b_2(\quad)\quad F(\ )^2e^{-!(\ )L}a_2(\quad)];$$
$$b_2(\frac{1}{} = \frac{1}{1\quad F(\ )^2}[b_2(\ )\quad F(\ )a_2(\ ) + F(\ )e^{-!(\ )L}a_2(\quad)\quad F(\ )^2e^{-!(\ )L}b_2(\quad)];$$
$$Im(\ )\quad 0: \qquad (4.15)$$

In summary, the basic equations characterizing the spectral functions are:

(a) the symmetry relations (4.4) and (4.15);

(b) the global relations (2.21) and (2.22);

(c) the conditions of unit determinant.

6 184.8527.79 011-222(+F11 9.96172 [28 4.113 G()]TJ/F2

Proof: for simplicity, we will use the notations

$$f = f( ); \qquad \hat{f} = f( ): \tag{4.22}$$

Replacing with in (2.21) and solving the resulting equation and equation (2.22) for a_2

The left hand side of (4.29), using (4.16) with $j = 1$, simpli es as follows:

$$\frac{a_1}{b_1} + \hat{a}_1 b_1 \quad \frac{a_1}{b_1}(1 + b_1\hat{b}_1) = \hat{a}_1 b_1 \quad a_1\hat{b}_1:$$

Thus, equation (4.29) becomes

$$b_1\hat{a}_1 \quad \hat{b}_1 a_1 = \mathrm{e}^{-!(\;)L}(b_3\hat{a}_3 \quad \hat{b}_3 a_3) + \mathrm{e}^{-!(\;)L}F \quad F: \tag{4.30}$$

Replacing in this equation by yields

$$(b_1\hat{a}_1 \quad \hat{b}_1 a_1) = \quad \mathrm{e}^{!(\;)L}(b_3\hat{a}_3 \quad \hat{b}_3 a_3) + \mathrm{e}^{!(\;)L}F \quad F: \tag{4.31}$$

Indeed, equation (4.17) yields

$$\hat{b}_1 = \frac{1}{a_1}(G + \hat{a}_1 b_1).$$

Replacing $\hat{b}_1$ in equation (4.16) with $j = 1$ by the above expression, and making use of (4.34), we find the first of equations (4.36). The second of equations (4.36) can be obtained in a similar way by eliminating $\hat{a}_1$ instead of $\hat{b}_1$.

Remark 4.5 The equations satisfied by a_3 and b_3 can be obtained from equations (4.36) by replacing $G(\lambda)$ by $G(-\lambda)$. Hence

$$a_3(-\lambda) = \frac{1}{h(\lambda)}\left(a_3(\lambda) - G(\lambda)b_3(\lambda)\right); \quad b_3(-\lambda) = \frac{1}{h(\lambda)}\left(b_3(\lambda) - G(\lambda)a_3(\lambda)\right); \qquad \lambda \in \mathbb{R}; \tag{4.37}$$

where G is given by (4.21) and $h(\lambda)$ is given by (4.35).

Remark 4.6 The function G is an entire function, thus each of equations (4.36) defines the jump condition of a scalar RH problem. However, it will be shown in section 5 that equations (4.36) and (4.37) are sufficient to determine the jump matrix (2.25).

Remark 4.7 Equations (4.17)-(4.20) imply the following identity:

$$e^{\omega(\lambda)L}[a_1(\lambda)^2 - b_1(\lambda)^2] + e^{-\omega(\lambda)L}[a_1(-\lambda)^2 - b_1(-\lambda)^2] = (e^{\omega(\lambda)L} + e^{-\omega(\lambda)L})(1 - F^2) + 2F^2; \qquad \lambda \in \mathbb{R}. \tag{4.38}$$

Indeed, equations (4.19)-(4.20) imply

$$e^{\omega(\lambda)L}(a_2^2 - b_2^2) = e^{\omega(\lambda)L}\hat{a}_1^2(a_3^2 - b_3^2) - e^{-\omega(\lambda)L}b_1^2(\hat{a}_3^2 - \hat{b}_3^2) - 2\hat{a}_1 b_1(a_3\hat{b}_3 - \hat{a}_3 b_3) \tag{4.39}$$

Replacing in this equation λ by $-\lambda$, adding the resulting equation to equation (4.39) and using equation (4.30) we find

$$e^{\omega(\lambda)L}(a_2^2 - b_2^2) + e^{-\omega(\lambda)L}(\hat{a}_2^2 - \hat{b}_2^2) = (e^{\omega(\lambda)L} + e^{-\omega(\lambda)L})(a_1^2 - b_1^2)(\hat{a}_1^2 - \hat{b}_1^2) + 2(a_1\hat{b}_1 - \hat{a}_1 b_1)(a_3\hat{b}_3 - \hat{a}_3 b_3). \tag{4.40}$$

Using (4.32), the right hand side of (4.40) equals the following expression:

$$(e^{\omega(\lambda)L} + e^{-\omega(\lambda)L}) - (a_1\hat{b}_1 - \hat{a}_1 b_1)\left[(e^{\omega(\lambda)L} + e^{-\omega(\lambda)L})(a_1\hat{b}_1 - \hat{a}_1 b_1) - 2(a_3\hat{b}_3 - \hat{a}_3 b_3)\right].$$

Using equations (4.17) and (4.18) the last expression becomes the right hand side of (4.38).

5 Spectral theory in the linearisable case

In the case of the linearisable boundary conditions (1.2), it is possible to express $q(x,y)$ in terms of the solution of a RH problem whose jump matrices are computed explicitly in terms of the given constant d. Indeed, recall that the jump matrices of the basic RH problem of section 2.4 are defined as follows:

$$J_{\pi/2} = \begin{pmatrix} 1 & -\Gamma(\lambda)e^{-\theta(x,y,\lambda)} \\ 0 & 1 \end{pmatrix}; \quad J_{3\pi/2} = \begin{pmatrix} 1 & 0 \\ \Gamma(\bar{\lambda})e^{\theta(x,y,\lambda)} & 1 \end{pmatrix}; \qquad \Gamma(\lambda) = \frac{b_2(\bar{\lambda})}{a_1(\lambda)a_3(\lambda)};$$

$$J^0 = \begin{pmatrix} R(\ ) & \frac{b_3(-\ )}{a_3(\ )}e^{-\ (x;y;\ )} \\ \frac{e^{\ \omega(\lambda)L} b_1(\ )}{a_1(-\ )}e^{\ (x;y;\ )} & 1 \end{pmatrix}; \qquad R(\ ) = \frac{a_2(\ )}{a_1(\ \ )a_3(\ )};$$

and

$$J \ = J^{3\ =2}(J^0)^{-1}J^{\ =2}; \tag{5.1}$$

Equation (4.19) and (4.20) imply that

$$R(\ ) = 1 \quad e^{-!(\ )L}\frac{\hat{b}_3}{a_3}\frac{b_1}{\hat{a}_1}; \quad I(\ ) = \frac{\hat{b}_3}{a_3} \quad e^{!(\ )L}\frac{\hat{b}_1}{a_1}: \tag{5.2}$$

Thus in the linearisable case, the jump matrices involve only the rations $\frac{\hat{b}_3}{a_3}$ and $\frac{\hat{b}_1}{a_1}$, evaluated at and at . Equations (4.35) and (4.34) imply that these rations are given by

$$\frac{\hat{b}_3}{a_3} = \quad \frac{G}{h} + \frac{b_3}{a_3 h}; \qquad \frac{\hat{b}_1}{a_1} = \frac{G}{h} + \frac{b_1}{a_1 h}: \tag{5.3}$$

Hence the jump matrices depend on the known function $\frac{G}{h}$ as well as on the unknown functions $\frac{b_1}{a_1 h}$ and $\frac{b_3}{a_3 h}$. Using the fact that these unknown functions are bounded and analytic in $\mathbb{C}^+$, it is possible to formulate a RH problem, equivalent to the basic one de ned by (5.1), in terms of the known function $\frac{G}{h}$ only. This new RH problem is therefore de ned by the following jump matrices:

$$\tilde{J}^{\ =2} = \begin{pmatrix} 1 & \tilde{\Gamma}(\ )e^{-\ (x;y;\ )} \\ 0 & 1 \end{pmatrix}; \quad \tilde{J}^{3\ =2} = \begin{pmatrix} 1 & 0 \\ \tilde{\Gamma}(\ \ )e^{\ (x;y;\ )} & 1 \end{pmatrix}; \quad \tilde{\Gamma}(\ ) = \quad \frac{G}{h}(1+e^{-!(\ )L});$$

$$\tilde{J}^0 = \begin{pmatrix} \tilde{R}(\ ) & \frac{G}{h}e^{-\ (x;y;\ )} \\ \frac{e^{\ \omega(\lambda)L}G}{\hat{h}}e^{\ (x;y;\ )} & 1 \end{pmatrix}; \qquad \tilde{R}(\ ) = 1 \quad \frac{G^2}{h\hat{h}}e^{-!(\ )L};$$

and

$$\tilde{J} \ = \tilde{J}^{3\ =2}(\tilde{J}^0)^{-1}\tilde{J}^{\ =2}; \tag{5.4}$$

Theorem 5.1 *Let $q(x;y)$ satisfy equation (1.1) and the boundary conditions (1.2). Then $q(x;y)$ is given by equations (2.36)-(2.37) with replaced by $\tilde{\ }$, where $\tilde{\ }$ is the solution of the Riemann-Hilbert problem (2.23) with the jump matrix J replaced by the matrix $\tilde{J}$ de ned as follows:*

$$\tilde{J}(x;y;\ ) = \tilde{J}\ (x;y;\ ); \text{ if } arg(\ ) = \ ; \qquad = 0; \frac{}{2}; \ ; \frac{3}{2}; \tag{5.5}$$

where

$$\tilde{J}^0 = \begin{pmatrix} 1 \quad \frac{G^2(\ )}{h(\ )h(-\ )}e^{-!(\ )L} & \frac{G(\ )}{h(\ )}e^{-\ (x;y;\ )} \\ e^{-!(\ )L}\frac{G(\ )}{h(-\ )}e^{\ (x;y;\ )} & 1 \end{pmatrix};$$

$\mathcal{J}^{=2}$

Thus the above Riemann-Hilbert problem (5.7) is regular.

We now prove that the Riemann-Hilbert problem defined by (5.7) is uniquely solvable. It can be verified that when $d \in \mathbb{R}$, then $h(\ ) = \overline{h(\ )}$. In this case, the jump matrices $J^{(\ )}$ satisfy the following conditions: the matrices are Schwarz invariant on the imaginary axis and have zero real part on the real axis of the complex   plane. Under these assumptions, it follows from general results (see e.g. [10, 29, 49]) that the so-called \vanishing lemma" holds. This guarantees the existence of a unique solution.
QED

6 Conclusions

We have studied boundary value problems for the elliptic sine-Gordon posed on a semistrip. In particular we have shown that if the prescribed boundary conditions are zero along the unbounded sides of the semistrip and constant on the bounded side, then it is possible to obtain the solution in terms of a Riemann-Hilbert problem which is uniquely defined inob1 9.9626 Tf 3.875

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